BULL. AUSTRAL. MATH. SOC. 13A15, 13H05, 14H20
VOL. 31 (1985), 321-324.

NOTES ON IRREDUCIBLE IDEALS

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Every ideal of a Noetherian ring may be represented as a finite intersection of primary ideals. Each primary ideal may be decomposed as an irredundant intersection of irreducible ideals. It is shown that in the case that Q is an M-primary ideal of a local ring (R, M) satisfying the condition that $Q : M = Q + M^{s-1}$ where s is the index of Q, then all irreducible components of Q have index s. (Q is "index-unmixed".) This condition is shown to hold in the case that Q is a power of the maximal ideal of a regular local ring, and also in other cases as illustrated by examples.

Introduction

Let R be a commutative ring with identity. An ideal I of R is *irreducible* if it is not a proper intersection of any two ideals of R. A discussion of some elementary properties of irreducible ideals is found in Zariski and Samuel [*4*] and Gröbner [*1*].

If R is Noetherian then every ideal of R has an irredundant representation as a finite intersection of irreducible ideals, and every irreducible ideal is primary. It is properties of representations of primary ideals as intersections of irreducible ideals which will be discussed here. We assume henceforth that R is Noetherian.

Received 25 October 1984.

If A is an ideal of R, then $(\mathrm{Rad}(A))^r$ is contained in A for some positive integer r, and the smallest such r is called the *index* of A. Noether [2] showed that the number of components in an irredundant representation of an ideal as an intersection of irreducible ideals is independent of the representation chosen, and Gröbner [1] showed that the number of such components for an M-primary ideal Q of a local ring (R, M) is the dimension over R/M of $(Q : M)/Q$. It is easy to see that the index of an intersection of P-primary ideals is the maximum of the indices of the components. In the following discussion, a condition is given under which all of these components will have the same index.

Results and examples

LEMMA. *Let (R, M) be a local ring, and Q an M-primary ideal of R. If $Q = Q_1 \cap \ldots \cap Q_n$ is irredundant, with $Q_1, \ldots, Q_n$ ideals of R, then for all i, $Q : M$ is not contained in Q_i.*

Proof. If $Q : M$ were contained in say Q_n, then

$$Q = Q \cap (Q : M) = (Q_1 \cap \ldots \cap Q_{n-1}) \cap Q : M .$$

If $A = Q_1 \cap \ldots \cap Q_{n-1}$, then let r be the smallest integer such that AM^r is contained in Q. Then r is positive since A is different from Q, but AM^{r-1} is contained in $(Q : M) \cap A = Q$, contradicting the choice of r.

THEOREM. *Let Q be an M-primary ideal of index s of the local ring (R, M). If $Q : M = Q + M^{s-1}$, then all ideals in any irredundant representation of Q as an intersection of ideals of R have index s.* (We may say that Q is *index-unmixed.*)

Proof. Suppose $Q = Q_1 \cap \ldots \cap Q_n$ is irredundant. From the lemma, $Q : M$ is not contained in any Q_i, $i = 1, \ldots, n$. Hence, neither is M^{s-1}. But for $i = 1, \ldots, n$, $M^s \subset Q \subset Q_i$. So each Q_i has index s.

COROLLARY. *If (R, M) is a regular local ring, then all ideals* *appearing in any irredundant representation of* M^s *as an intersection of ideals of* R *have index* s .

Proof. It is well-known (see for example Northcott, [2], p. 70) that if $v(x)$ denotes the largest power of M to which x belongs, then $v(xy) = v(x) + v(y)$. From this it follows that for any positive integers s, t with $s > t$, $M^s : M^t = M^{s-t}$. (In fact suppose x belongs to $M^s : M^t$ and let $r = v(x)$. Let y be any element of R with $v(y) = t$. Then xy belongs to xM^t which is contained in M^s . But xy does not belong to M^{r+t+1} since $v(xy) = r + t$. Thus $s - t \leq r$ and x therefore belongs to M^{s-t} .) Hence $M^s : M = M^{s-1}$ and the theorem applies to show that M^s is index-unmixed.

EXAMPLE 1. Let k be a field and let R be the local ring at the origin of the curve defined by $Y^2 - X^2 - X^3$ in $k[X, Y]$. That is, $R = k[x, y]_P$ where $y^2 = x^2 + x^3$, $P = (x, y)k[x, y]$. Let $M = PR$. R is not regular since the origin is a singular point of the curve, but for all n , $M^n : M = M^{n-1}$, so that all powers of M are index-unmixed.

EXAMPLE 2. Let R be the local ring at the origin of the curve defined by $Y^2 - X^3$ in $k[X, Y]$. As before, $R = k[x, y]_P$ where $y^2 = x^3$, $P = (x, y)k[x, y]$, $M = PR$. Let $Q = x^sR$. Then Q has index $s + 1$. Also $Q : M = Q + M^s$, so that by the theorem, Q is index-unmixed.

EXAMPLE 3. Let R be a polynomial ring $k[X, Y]$ in two letters over a field k , $M = (X, Y)$. Let $Q = M^3 + (X^2+XY)R + (Y^2+XY)R$. Then Q has index 3 . Q is not irreducible, in fact Q is not even index-unmixed since $Q = [M^2+(X-Y)R] \cap [M^3+(X+Y)R]$, which is a representation of Q as an intersection of irreducible ideals of index 2 and 3 respectively. It should be noted however that for any distinct elements a and b of k , $Q = [M^3+(X+Y+aX^2)R] \cap [M^3+(X+Y+bX^2)R]$ so that Q admits several decompositions into irreducible ideals where the indices are all equal.

References

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