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QUADRATIC FORMS AND LINKAGE OF QUATERNION ALGEBRAS

BY

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ABSTRACT. A field F satisfies n-linkage on a subset of $\dot{F}$ if whenever the quaternion algebras

$$\left(\frac{s_1, s_1'}{F}\right), \left(\frac{s_2, s_2'}{F}\right), \ldots, \left(\frac{s_n, s_n'}{F}\right)$$

are equal in $Br(F)$ there exist $z \in \dot{F}$ with

$$\left(\frac{s_i, s_i'}{F}\right) = \left(\frac{s_i, z}{F}\right)$$

for $i = 1, 2, \ldots, n$. This linkage of quaternion algebras is examined and its relationship to the torsion freeness of $I^2(F)$ and to the strong approximation property is investigated.

Throughout this note F will denote a field of characteristic different than 2.

There are two notions of linkage of quaternion algebras which have been examined quite extensively in the literature. On the one hand, Elman and Lam [1] introduced the following notion: Two quaternion algebras

$$\left[\frac{a, b}{F}\right] \text{ and } \left[\frac{c, d}{F}\right]$$

are linked if there are $x, y, z \in \dot{F}$ with

$$\left[\frac{a, b}{F}\right] = \left[\frac{x, z}{F}\right] \text{ and } \left[\frac{c, d}{F}\right] = \left[\frac{y, z}{F}\right].$$

In their papers [2] and [3], Elman and Lam showed that having all quaternion algebras linked in this manner is closely related to the torsion freeness of powers of the fundamental ideal $I(F)$ of the Witt ring $W(F)$ and to the strong approximation property SAP on the space of orderings of F. On the other hand, in [7] the power underlying the abstract theory of quadratic forms is the following linkage property of quaternion algebras: If

$$\left[\frac{a, b}{F}\right] = \left[\frac{c, d}{F}\right]$$

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then there exists $z \in \dot{F}$ with

$$\left[\frac{a, b}{F}\right] = \left[\frac{a, z}{F}\right] = \left[\frac{c, z}{F}\right] = \left[\frac{c, d}{F}\right].$$

In this note we study a stronger version of this linkage and show that it is also closely related to the torsion freeness of powers of $I(F)$ and to the SAP property.

We say that F is *linked on n-chains from a subset S of* $\dot{F}$ if whenever

$$\left[\frac{s_1, s_1'}{F}\right] = \left[\frac{s_2, s_2'}{F}\right] = \ldots = \left[\frac{s_n, s_n'}{F}\right]$$

with $s_i, s_i' \in S$, there exists $z \in \dot{F}$ with

$$\left[\frac{s_i, s_i'}{F}\right] = \left[\frac{s_i, z}{F}\right]$$

for $i = 1, 2, \ldots, n$. If F is linked on n-chains from $\dot{F}$ we will just say that F is *linked on n-chains*. Notice that the linkage axiom of [7] is just linkage on 2-chains. Every field is linked on 2-chains, see ([6], Ex. 12, p. 69).

In notation and terminology we primarily follow [6] and [7]. Thus $D(\varphi)$ is the value set of the quadratic form φ and $D(2) = D(\langle 1, 1\rangle)$. For $a \in \dot{F}$,

$$Q(a) = \left\{ \left[\frac{a, x}{F}\right] \middle| x \in \dot{F} \right\}$$

and $W(F)$ is non-degenerate if $Q(a) \neq \{1\}$ for every $a \in \dot{F} \backslash \dot{F}^2$.

PROPOSITION 1. *$I^2(F)$ is torsion free if and only if F is linked on 3-chains from $\{x, y, xy\}$ for every $x, y \in \dot{F}$.*

PROOF. Suppose $I^2(F)$ is torsion free and

$$A = \left[\frac{a_i, b_i}{F}\right]$$

$a_i, b_i \in \{x, y, xy\}$, $i = 1, 2, 3$. The result is trivial if $A = 1$ or $a_i = b_i$ for all i thus we may assume that

$$A = \left[\frac{x, y}{F}\right].$$

If

$$A = \left[\frac{xy, z}{F}\right]$$

then $2\langle\langle -xy, -z\rangle\rangle = \langle\langle -xy, -x, -y\rangle\rangle = 0$. By assumption,

$$A = \left[\frac{xy, z}{F}\right] = 1.$$

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The result now follows easily.

To show the converse, by ([4], Lemma 1, Corollary 1), it suffices to show that

$$\left[\frac{x, y}{F}\right] = 1$$

whenever $x \in D(2)$.

Since

$$\left[\frac{x, x}{F}\right] = 1$$

we have a chain

$$\left[\frac{x, y}{F}\right] = \left[\frac{y, x}{F}\right] = \left[\frac{xy, x}{F}\right].$$

By assumption there exists $z \in \dot{F}$ with

$$\left[\frac{x, y}{F}\right] = \left[\frac{x, z}{F}\right] = \left[\frac{y, z}{F}\right] = \left[\frac{xy, z}{F}\right].$$

The last equality implies

$$\left[\frac{x, z}{F}\right] = 1,$$

and the result follows. □

COROLLARY 2. *F is a Pythagorean field if and only if $|\dot{F}/\dot{F}^2| = 1$ or $W(F)$ is non-degenerate and F is linked on 3-chains from $\{x, y, xy\}$ for every $x, y \in \dot{F}$.*

PROOF. If F is Pythagorean and $x \in \dot{F}\backslash\dot{F}^2$ then

$$\left[\frac{x, x}{F}\right] \neq 1$$

hence $W(F)$ is non-degenerate. The result follows from Proposition 1. Conversely, by Proposition 1, $I^2(F)$ is torsion free. Hence if $x \in D(2)$, $\langle\langle -x, -y\rangle\rangle = 0$ for every $y \in \dot{F}$. Consequently,

$$\left[\frac{x, y}{F}\right] = 1$$

for every $y \in \dot{F}$. Since $W(F)$ is non-degenerate, $x = 1$ and F is Pythagorean. □

PROPOSITION 3. *F is a Pythagorean* SAP *field if and only if* $|\dot{F}/\dot{F}^2| = 1$ *or* $W(F)$ *is non-degenerate and F is linked on 3-chains from* $\{x, y, xy, -xy\}$ *for every* $x, y \in \dot{F}$.

PROOF. If F is a Pythagorean SAP field and A is any F-quaternion algebra, there exists $w \in F$ with

$$A = \left[\frac{-1, w}{F}\right]$$

by ([2], Theorem 5.3). If

$$A = \left[\frac{a, b}{F}\right]$$

then by ([6], Ex. 12, p. 69) there exists $z \in F$ with

$$\left[\frac{a, b}{F}\right] = \left[\frac{a, z}{F}\right] = \left[\frac{-1, z}{F}\right] = \left[\frac{-1, w}{F}\right].$$

It follows that $wz \in D(2) = \dot{F}^2$ so

$$\left[\frac{a, b}{F}\right] = \left[\frac{a, w}{F}\right].$$

Consequently, F is linked on n-chains for any n.

Conversely, by Corollary 2, we need only show that a Pythagorean field F satisfies SAP. By ([2], Theorem 5.3), it suffices to show that the form $\langle 1, x, -y, xy\rangle$ is isotropic. Notice that

$$\left[\frac{x, y}{F}\right] = \left[\frac{y, x}{F}\right] = \left[\frac{-xy, y}{F}\right].$$

By assumption there exists $z \in \dot{F}$ with

$$\left[\frac{x, y}{F}\right] = \left[\frac{x, z}{F}\right] = \left[\frac{y, z}{F}\right] = \left[\frac{-xy, z}{F}\right].$$

Hence

$$\left[\frac{-x, z}{F}\right] = 1$$

by the last equality and

$$\left[\frac{x, yz}{F}\right] = 1$$

by the first equality. Consequently, $z \in D\langle 1, x\rangle$ and $z \in D\langle y, -xy\rangle$. This shows that $\langle 1, x, -y, xy\rangle$ is isotropic. □

Proposition 3 and its proof immediately yield

THEOREM 4. *Suppose F is a field with $W(F)$ non-degenerate. The following statements are equivalent*:

1. *F is a Pythagorean* SAP *field.*
2. *F is linked on 3-chains from $\{x, y, xy, -xy\}$ for every $x, y \in \dot{F}$.*
3. *F is linked on 3-chains.*
4. *F is linked on n-chains for some $n > 2$.*
5. *F is linked on n-chains for every $n \geqq 2$.*

We remark that some results in quadratic form theory which depend mainly on linkage on 2-chains can be generalized for Pythagorean SAP fields using linkage on n-chains. We give Proposition 5 below as an example of such. Recall ([7], Lemma 5.2) which states that for $a, b \in \dot{F}$, $Q(a) \cap Q(b) \cong D\langle 1, -ab\rangle/(D\langle 1, -a\rangle \cap D\langle 1, -b\rangle)$.

PROPOSITION 5. *Let F be a Pythagorean* SAP *field. For $a_1, a_2, \ldots, a_n \in \dot{F}$,*

$$\bigcap_{j=1}^{n} Q(a_j) \cong \bigcap_{j=1}^{n} D\langle 1, -a_1a_j\rangle / \bigcap_{j=1}^{n} D\langle 1, -a_j\rangle.$$

PROOF. If $x \in \cap_{j=1}^{n} D\langle 1, -a_1a_j\rangle$ then

$$\left[\frac{a_1a_j, x}{F}\right] = 1$$

thus

$$\left[\frac{a_1, x}{F}\right] = \left[\frac{a_j, x}{F}\right] \text{ and } \left[\frac{a_1, x}{F}\right] \in \bigcap_{j=1}^{n} Q(a_j).$$

Consequently,

$$x \to \left[\frac{a_1, x}{F}\right]$$

is a group homomorphism from $\cap_{j=1}^{n} D\langle 1, -a_1a_j\rangle$ to $\cap_{j=1}^{n} Q(a_j)$. If $A \in \cap_{j=1}^{n} Q(a_j)$, then there exist $y_i \in \dot{F}$ with

$$A = \left[\frac{a_i, y_i}{F}\right]$$

hence by assumption there exists $z \in \dot{F}$ with

$$A = \left[\frac{a_i, z}{F}\right].$$

Thus

$$\left[\frac{a_1a_j, z}{F}\right] = 1, z \in \bigcap_{j=1}^{n} D\langle 1, -a_1a_j\rangle$$

and the mapping is surjective. One verifies easily that the kernel is $\cap_{j=1}^{n} D\langle 1, -a_j\rangle$. □

We close with a relationship between linkage and Witt rings of local type in the finitely generated case.

PROPOSITION 6. *Let F be a field with $|\dot{F}/\dot{F}^2| = 2^n$. If for every basis $\{x_1, x_2, \ldots, x_n\}$ of $\dot{F}/\dot{F}^2$ there is a chain*

$$\left[\frac{x_1, a_1}{F}\right] = \left[\frac{x_2, a_2}{F}\right] = \ldots = \left[\frac{x_n, a_n}{F}\right] \neq 1$$

which is linked then each proper subgroup of $\dot{F}/\dot{F}^2$ is contained in some value set $D\langle 1, -z\rangle$, $z \neq 1$.

PROOF. Let H be a subgroup of $\dot{F}/\dot{F}^2$ with say $|H| = 2^m$, $m < n$. Pick a basis $\{y_1, \ldots, y_m\}$ for H and extend it to a basis $\{y_1, \ldots, y_m, x_{m+1}, \ldots, x_n\}$ for $\dot{F}/\dot{F}^2$. Notice that $\{y_1x_{m+1}, \ldots, y_mx_{m+1}, x_{m+1}, \ldots, x_n\}$ is also a basis for $\dot{F}/\dot{F}^2$. By our assumption, there is a chain

$$\left[\frac{y_1x_{m+1}, a_1}{F}\right] = \ldots = \left[\frac{y_mx_{m+1}, a_m}{F}\right] = \left[\frac{x_{m+1}, a_{m+1}}{F}\right] = \ldots = \left[\frac{x_n, a_n}{F}\right]$$

which is linked. Thus there is a $z \in \dot{F}$ with

$$\left[\frac{y_1x_{m+1}, z}{F}\right] = \ldots = \left[\frac{y_mx_{m+1}, z}{F}\right] = \left[\frac{x_{m+1}, z}{F}\right].$$

This implies

$$\left[\frac{y_1, z}{F}\right] = \left[\frac{y_2, z}{F}\right] = \ldots = \left[\frac{y_m, z}{F}\right] = 1$$

and hence $y_1, y_2, \ldots, y_m \in D\langle 1, -z\rangle$. □

COROLLARY 7. *If $W(F)$ is non-degenerate and if F satisfies the hypothesis of Proposition 6 then F has exactly one non-trivial quaternion algebra.*

PROOF. By Proposition 6, each hyperplane is contained in a value set $D\langle 1, -z\rangle$, $z \neq 1$. Since $W(F)$ is non-degenerate, every hyperplane must be a value set $D\langle 1, -z\rangle$. Consequently, every value set $D\langle 1, -z\rangle z \neq 1$, has index 2 in $\dot{F}/\dot{F}^2$. The result now follows from a result of Kaplansky [5], see ([7], Proposition 5.15). □

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