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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF PROBLEMS WITH CRITICAL GROWTH IN $\mathbb{R}^N$

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Abstract Using variational methods, we establish the existence and multiplicity of positive solutions for the following class of problems:

$$-\Delta u + (\lambda V(x) + Z(x))u = \beta u^q + u^{2^*-1}, \quad u > 0 \text{ in } \mathbb{R}^N,$$

where $\lambda, \beta \in (0, \infty)$, $q \in (1, 2^* - 1)$, $2^* = 2N/(N-2)$, $N \geqslant 3$, $V, Z : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some hypotheses.

Keywords: variational methods; critical growth; elliptic problems

2000 *Mathematics subject classification:* Primary 35J20; 35J60; 35B40

1. The problem

In this paper, we are concerned with the existence of positive solutions for the following class of problems:

$$\left.\begin{aligned} -\Delta u + (\lambda V(x) + Z(x))u = \beta u^q + u^{2^*-1} \text{ in } \mathbb{R}^N,\\ u > 0 \text{ in } \mathbb{R}^N,\\ u \text{ in } H^1(\mathbb{R}^N), \end{aligned}\right\} \tag{1.1}$$

where $\lambda, \beta \in (0, \infty)$, $q \in (1, 2^* - 1)$, $2^* = 2N/(N-2)$, $N \geqslant 3$ and $V, Z : \mathbb{R}^N \to \mathbb{R}$ are continuous functions with $V(x) \geqslant 0$ for all $x \in \mathbb{R}^N$. The function V has the property that $\operatorname{Int} V^{-1}(0) := \Omega$ is an open smooth domain composed of k open connected components denoted by Ω_j, $j \in \{1, \dots, k\}$, which satisfy $\mathrm{d}(\Omega_i, \Omega_j) > 0$ for $i \neq j$, that is,

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

with $V^{-1}(0) = \bar{\Omega}$.

For each $j \in \{1, 2, \dots, k\}$, we fix a bounded open subset Ω'_j with smooth boundary such that

(i) $\overline{\Omega_j} \subset \Omega'_j$,

(ii) $\overline{\Omega'_j} \cap \overline{\Omega'_l} = \emptyset$ for all $j \neq l$.

Moreover, we also fix a non-empty subset $\Gamma \subset \{1, 2, \dots, k\}$ and set

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j \quad \text{and} \quad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j.$$

Let us also assume that there exist two positive constants M_o and M_1 such that the functions V and Z verify

$$0 < M_o \leqslant V(x) + Z(x) \quad \forall x \in \mathbb{R}^N \tag{1.2}$$

and

$$|Z(x)| \leqslant M_1 \quad \forall x \in \mathbb{R}^N. \tag{1.3}$$

Many papers concerning existence and multiplicity of positive solutions for this kind of problem have been published in recent years. For example, in the case when the function $\lambda V(x) + Z(x)$ is coercive, Miyagaki [**21**] proved some existence results for a positive solution to (1.1). For the case when the function $\lambda V(x) + Z(x)$ is 1-periodic, Alves *et al*. [**3**] showed the existence of positive solutions to (1.1). If $\lambda V(x) + Z(x)$ is radial, Alves *et al*. [**4**] also established the existence of a positive solution to (1.1). The papers cited above proved only the existence of positive solutions; the multiplicity of solutions was established in [**6**–**8**, **10**, **11**, **22**].

In [**13**], Ding and Tanaka considered problem (1.1) without the critical term and assume that $\beta = 1$. Supposing that Ω has k connected components, the authors showed that, for this case, problem (1.1) has at least $2^k - 1$ solutions, for large λ, establishing the existence of solutions called multi-bumps.

In our work, due to the critical growth of the nonlinearity in $\mathbb{R}^N$, standard procedures adopted in the literature to treat the subcritical case do not hold. In view of this obstacle, a new approach has had to be applied. For instance, we have had to prove a bootstrap argument for the case we study (see Propositions 3.8 and 3.9). Motivated by [**13**] and by some arguments developed in [**1**], we have proved, even for the critical case, the existence of multiple solutions to (1.1), and that these solutions have the same characteristics of those found in [**13**]. We have employed variational arguments, and our main result completes the study made in [**13**], in the sense that we are working with a class of problems involving critical growth.

Our main result is the following.

Theorem 1.1. *Assume that (1.2) and (1.3) hold. Then, for any non-empty subset Γ of $\{1, 2, \dots, k\}$, there exist constants $\beta^* > 0$ and $\lambda^* = \lambda^*(\beta^*)$ such that, for all $\beta \geqslant \beta^*$ and $\lambda \geqslant \lambda^*$, problem (1.1) has a family $\{u_\lambda\}$ of positive solutions with the following property: for any sequence $\lambda_n \to \infty$, we can extract a subsequence (λ_{n_i}) such that $u_{\lambda_{n_i}}$*

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converges strongly in $H^1(\mathbb{R}^N)$ to a function u that satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$, and the restriction $u|_{\Omega_j}$ is a least-energy solution of the following problem:

$$
\begin{aligned}
-\Delta u + Z(x)u &= \beta u^q + u^{2^*-1} \in \Omega_j,\\
u &> 0 \quad \text{in } \Omega_j,\\
u &= 0 \quad \text{on } \partial\Omega_j
\end{aligned}
$$

for all $j \in \Gamma$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exist $\beta^* > 0$ and $\lambda^* = \lambda^*(\beta^*)$ such that, for $\lambda \geqslant \lambda^*$, problem (1.1) has at least $2^k - 1$ positive solutions.*

Notation

The integral $\int_{\mathbb{R}^N} \vartheta \, dx$ is denoted by $\int_{\mathbb{R}^N} \vartheta$. The usual norm of $H^1(\mathbb{R}^N)$ is denoted by $\|u\|$. The usual norm of $L^r(\mathbb{R}^N)$ is denoted by $|u|_r$, $r > 1$. The usual norm of $L^\infty(\mathbb{R}^N)$ is denoted by $|u|_\infty$. For an open set $\Theta \subset \mathbb{R}^N$, the symbols $\|u\|_\Theta$, $|u|_{r,\Theta}$, $r > 1$, and $|u|_{\infty,\Theta}$ denote the usual norms in the spaces $H^1(\Theta)$, $L^r(\Theta)$ and $L^\infty(\Theta)$, respectively.

2. Some preliminary results

In this section, we set some notation and the proper variational framework to be employed in this work.

Let us define the space of functions

$$\mathcal{H}_\lambda = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (\lambda V(x) + Z(x))u^2 < \infty \right\}$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + Z(x))u^2) \right)^{1/2}.$$

For $\lambda \geqslant 1$ it easy to see that $(\mathcal{H}_\lambda, \| \cdot \|_\lambda)$ is a Hilbert space and we have the following continuous imbedding: $\mathcal{H}_\lambda \hookrightarrow H^1(\mathbb{R}^N)$.

The non-negative weak solutions of (1.1) are the critical points of the functional $J : \mathcal{H}_\lambda \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + Z(x))u^2) - \frac{\beta}{q+1} \int_{\mathbb{R}^N} (u_+)^{q+1} - \frac{1}{2^*} \int_{\mathbb{R}^N} (u_+)^{2^*},$$

where $u_+(x) = \max\{u(x), 0\}$.

For an open set $\Theta \subset \mathbb{R}^N$ we analogously define

$$\mathcal{H}_\lambda(\Theta) = \left\{ u \in H^1(\Theta); \int_\Theta (\lambda V(x) + Z(x))u^2 < \infty \right\}$$

and

$$\|u\|_{\lambda,\Theta} = \left(\int_\Theta |\nabla u|^2 + (\lambda V(x) + Z(x))u^2 \right)^{1/2}.$$

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In view of (1.2), we have

$$\|u\|^2_{\lambda,\Theta} \geqslant M_{\rm o}|u|^2_{2,\Theta} \quad \text{for all } u \in \mathcal{H}_\lambda(\Theta) \text{ and } \lambda \geqslant 1.$$

The next result is an immediate consequence of the last inequality.

Lemma 2.1. *There exist constants $\delta_0, \nu_0 > 0$ with $\delta_0 \approx 1$ and $\nu_0 \approx 0$ such that, for all open sets $\Theta \subset \mathbb{R}^N$,*

$$\delta_0\|u\|^2_{\lambda,\Theta} \leqslant \|u\|^2_{\lambda,\Theta} - \nu_0|u|^2_{2,\Theta} \quad \textit{for all } u \in \mathcal{H}_\lambda(\Theta) \textit{ and } \lambda \geqslant 1. \tag{2.1}$$

Once we consider the nonlinearity with critical growth, the next lemma will be useful and it is an immediate consequence of a result due to Lions [**18**–**20**].

Lemma 2.2. *Let $(v_n) \subset H^1(\mathbb{R}^N)$ be a bounded sequence such that $v_n \rightharpoonup v$ in $L^{2^*}(\mathbb{R}^N)$. If (v_n) is a subsequence such that $|v_n|^{2^*} \rightharpoonup \nu$ and $|\nabla v_n|^2 \rightharpoonup \mu$ for some measures ν and μ, then there are sequences of points $(x_n) \subset \mathbb{R}^N$ and $(\nu_n) \subset [0,\infty)$ satisfying*

$$|v_n|^{2^*} \rightharpoonup |v|^{2^*} + \sum_{i=1}^{\infty} \nu_i \delta_{x_i} \equiv \nu,$$

$$\sum_{n=1}^{\infty} \nu_n^{2/2^*} < \infty \quad \text{and} \quad \mu(x_n) \geqslant S\nu_n^{2/2^*} \quad \forall n \in \mathbb{N},$$

where δ_i is the Dirac measure and S is the best Sobolev constant of the immersion $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

3. An auxiliary problem

In this section, we adapt, for our case, some arguments developed by Ding and Tanaka [**13**] and del Pino and Felmer [**12**].

Henceforth, let us denote by $h : \mathbb{R} \to \mathbb{R}$ the function given by

$$h(s) = \begin{cases} \beta s^q + s^{2^*-1} & \text{if } s \geqslant 0, \\ 0 & \text{if } s \leqslant 0, \end{cases}$$

and fix a positive constant a verifying $h(a)/a = \nu_0$, where $\nu_0 > 0$ is the constant given in Lemma 2.1.

For technical reasons we define two functions $f, F : \mathbb{R} \to \mathbb{R}$, which play an important role in what follows:

$$f(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ h(s) & \text{if } s \in [0,a], \\ \nu_0 s & \text{if } s \geqslant a, \end{cases}$$

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and

$$F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau.$$

Using the set Ω'_Γ, let us also consider the functions

$$\chi_\Gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega'_\Gamma, \\ 0 & \text{for } x \notin \Omega'_\Gamma, \end{cases}$$

$$g(x,s) = \chi_\Gamma(x)h(s) + (1 - \chi_\Gamma(x))f(s) \tag{3.1}$$

and

$$G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t = \chi_\Gamma(x)H(s) + (1 - \chi_\Gamma(x))F(s), \tag{3.2}$$

where

$$H(s) = \int_0^s h(\tau)\,\mathrm{d}\tau.$$

Let us denote by $\Phi_\lambda : \mathcal{H}_\lambda \to \mathbb{R}$ the functional given by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + Z(x))u^2) - \int_{\mathbb{R}^N} G(x,u). \tag{3.3}$$

It is standard to prove that $\Phi_\lambda \in C^1(\mathcal{H}_\lambda, \mathbb{R})$ and that the critical points of Φ_λ are non-negative weak solutions of the equation

$$-\Delta u + (\lambda V(x) + Z(x))u = g(x,u) \quad \text{in } \mathbb{R}^N. \tag{3.4}$$

Note that positive solutions of the above equation are related with positive solutions of (1.1), once we see that if $u \in \mathcal{H}_\lambda$ is a positive solution of (3.4) verifying $u(x) \leqslant a$ in $\mathbb{R}^N \setminus \Omega'_\Gamma$, then it is a positive solution of (1.1).

3.1. The Palais–Smale condition and the study of some energy levels

A sequence $(u_n) \subset \mathcal{H}_\lambda$ is defined as a Palais–Smale sequence at the level $c \in \mathbb{R}$ (hereafter referred to as a $(\mathrm{PS})_c$ sequence) of the functional Φ_λ, when

$$\Phi_\lambda(u_n) \to c \in \mathbb{R} \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0 \in (\mathcal{H}_\lambda)'. \tag{3.5}$$

Remark 3.1. By the definition of the functions f and F, the Palais–Smale sequences may be assumed to be non-negative.

The next lemma establishes that all $(\mathrm{PS})_c$ sequences are bounded, and the proof follows using well-known arguments (see [**13**]).

Lemma 3.2. *Any $(PS)_c$ sequence $(u_n) \subset \mathcal{H}_\lambda$ of the functional Φ_λ is uniformly bounded with respect to $\lambda \geqslant 1$.*

Next, for each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the mountain-pass theorem associated with the functional $I_j : H^1_0(\Omega_j) \to \mathbb{R}$, given by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j} (|\nabla u|^2 + Z(x)u^2) - \frac{\beta}{q+1}\int_{\Omega_j} (u_+)^{q+1} - \frac{1}{2^*}\int_{\Omega_j} (u_+)^{2^*}. \tag{3.6}$$

It is well known that the critical points of I_j are weak solutions of the following problem:

$$\left.\begin{aligned} -\Delta u + Z(x)u = \beta u^q + u^{2^*-1} \quad & \text{in } \Omega_j, \\ u > 0 \quad & \text{in } \Omega_j, \\ u = 0 \quad & \text{on } \partial\Omega_j. \end{aligned}\right\} \tag{3.7}$$

The technique we shall apply in order to prove Theorem 1.1 includes the comparison between some energy levels of the functional associated with (1.1) with the energy levels associated with other auxiliary problems related to (1.1), as well as the study of the behaviour of some $(\mathrm{PS})_c$ sequences.

In this regard we prove the following results.

Lemma 3.3. *There exists $\beta^* > 0$ such that, for all $\beta \geqslant \beta^*$, we have*

$$c_j \in \left(0, \left(\frac{1}{2} - \frac{1}{q+1}\right)\frac{S^{N/2}}{k+1}\right) \quad \textit{for all } j \in \{1, \dots, k\}.$$

Proof. For each $j \in \{1, \dots, k\}$, we fix a non-negative function $\varphi_j \in H^1_0(\Omega_j) \setminus \{0\}$. Observe that there exists $t_{\beta,j} \in (0, +\infty)$ such that

$$c_j \leqslant I_j(t_{\beta,j}\varphi_j) = \max_{t \geqslant 0} I_j(t\varphi_j)$$

and thus, the following equality holds:

$$\int_{\Omega_j} [|\nabla\varphi_j|^2 + Z(x)|\varphi_j|^2] = \beta t_{\beta,j}^{q-1}\int_{\Omega_j} {\varphi_j}^{q+1} + t_{\beta,j}^{2^*-2}\int_{\Omega_j} {\varphi_j}^{2^*}.$$

This equality implies that

$$t_{\beta,j} \leqslant \left[\frac{\int_{\Omega_j} [|\nabla\varphi_j|^2 + Z(x)|\varphi_j|^2]}{\beta\int_{\Omega_j} {\varphi_j}^{q+1}}\right]^{1/(q-1)}$$

and hence

$$t_{\beta,j} \to 0 \quad \text{as } \beta \to +\infty.$$

Using the above limit, we have

$$I_j(t_{\beta,j}\varphi_j) \to 0 \quad \text{as } \beta \to +\infty,$$

whence it follows that there exists $\beta^* > 0$ such that

$$c_j < \left(\frac{1}{2} - \frac{1}{q+1}\right)\frac{S^{N/2}}{(k+1)} \quad \text{for all } j \in \{1, \dots, k\} \text{ and all } \beta \in [\beta^*, +\infty).$$

□

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Remark 3.4. In particular, the above lemma implies that

$$\sum_{j=1}^{k} c_j \in \left(0, \left(\frac{1}{2} - \frac{1}{q+1}\right) S^{N/2}\right). \tag{3.8}$$

The above result is very important, as we show in the following proposition.

Proposition 3.5. *For each $\lambda \geqslant 1$ and $c \in (0, (\frac{1}{2} - 1/(q+1))S^{N/2})$, any (PS)$_c$ sequence $(u_n) \subset \mathcal{H}_\lambda$ of the functional Φ_λ has a strongly convergent subsequence (in $\mathcal{H}_\lambda$).*

Proof. Let $(u_n) \subset \mathcal{H}_\lambda$ be a (PS)$_c$ sequence. By Lemma 3.3, the sequence (u_n) is bounded in H_λ and we may assume that

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{H}_\lambda \text{ and in } H^1(\mathbb{R}^N),$$
$$u_n \to u \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \quad \forall p \in [1, 2^*).$$

First, we observe that weak limit u is a critical point of Φ_λ.

By hypothesis, for any bounded sequence $(\varphi_n) \subset \mathcal{H}_\lambda$, we have $\Phi'_\lambda(u_n)\varphi_n = o_n(1)$. Let us choose a special φ_n for our purposes:

$$\varphi_n(x) = \eta(x) u_n(x),$$

where $\eta \in C^\infty(\mathbb{R}^N)$ is given by

$$\eta(x) = \begin{cases} 1 & \forall x \in B^c_R(0), \\ 0 & \forall x \in B_{R/2}(0), \end{cases}$$
$$\eta(x) \in [0, 1] \quad \text{with } \Omega'_\Gamma \subset B_{R/2}(0).$$

Here and below $B^c_R(0) = \{x \in \mathbb{R}^N;\ |x| \geqslant R\}$. Using the above data and adapting arguments used in [**12**, Lemma 1.1] one proves that, for each $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\{x\in\mathbb{R}^N : |x| \geqslant R\}} |\nabla u_n|^2 + (\lambda V(x) + Z(x)) u_n^2 \leqslant \varepsilon \quad \text{for large } n \in \mathbb{N}. \tag{3.9}$$

Claim 3.6. *The sequence (ν_n) obtained by applying Lemma 2.2 to the sequence (u_n) verifies $\nu_n = 0$ for all $n \in \mathbb{N}$.*

In fact, once it is proved that (u_n) is a (PS)$_c$ sequence, for each $\phi \in C^\infty_0(\Omega)$ we have that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \phi + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \phi + \int_{\mathbb{R}^N} (\lambda V + Z) u_n^2 \phi = \int_{\mathbb{R}^N} g(x, u_n) u_n \phi + o_n(1). \tag{3.10}$$

If (x_n) is the sequence given in Lemma 2.2, then let $\Phi_\varepsilon = \Phi(x - x_n)/\varepsilon$, $x \in \mathbb{R}^N$, $\varepsilon > 0$, where $\Phi \in C^\infty_0(\mathbb{R}^N, [0, 1])$ is such that $\Phi \equiv 1$ on $B_1(0)$, $\Phi \equiv 0$ on $B^c_2(0)$ and $|\nabla \Phi| \leqslant 2$. Considering $\phi = \Phi_\varepsilon$ in (3.10) and using the same type of arguments explored in [**16**], we obtain $\mu(x_n) \leqslant \nu_n$ for all $n \in \mathbb{N}$. If $\nu_n > 0$, the latter inequality combined with Lemma 2.2 implies that

$$\nu_n \geqslant S^{N/2} \quad \forall n \in \mathbb{N}, \tag{3.11}$$

whence it follows that (ν_n) is finite.

Next, we will prove that $\nu_n = 0$ for all $n \in \mathbb{N}$.

Again using the fact that (u_n) is a $(\mathrm{PS})_c$ sequence, we have

$$I(u_n) - \frac{1}{q+1} I'(u_n)u_n = c + o_n(1).$$

Consequently,

$$\left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + \left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} (\lambda V + Z) u_n^2 + \int_{\mathbb{R}^N} \left[\frac{1}{q+1} g(x, u_n)u_n - G(x, u_n)\right] = c + o_n(1).$$

Since

$$\int_{\mathbb{R}^N} (\lambda V + Z) u_n^2 + \int_{\mathbb{R}^N} \left[\frac{1}{q+1} g(x, u_n)u_n - G(x, u_n)\right] \geqslant 0,$$

it follows that

$$\left(\frac{1}{2} - \frac{1}{q+1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 \leqslant c + o_n(1),$$

and then

$$\left(\frac{1}{2} - \frac{1}{q+1}\right) \mu(x_n) \leqslant c \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Recalling that $\mu(x_n) \geqslant S\nu_n^{2/2^*}$, if there exists a $\nu_n > 0$ for some $n \in \mathbb{N}$, from (3.11) and (3.12) we obtain the inequality

$$c \geqslant \left(\frac{1}{2} - \frac{1}{q+1}\right) S^{N/2},$$

which is a contradiction; thus, $\nu_n = 0$ for all $n \in \mathbb{N}$.

From Claim 3.6 we have

$$u_n \to u \quad \text{in } L^{2^*}_{\mathrm{loc}}(\mathbb{R}^N) \tag{3.13}$$

and, by (3.9), it follows that

$$\int_{\mathbb{R}^N} g(x, u_n)u_n \to \int_{\mathbb{R}^N} g(x, u)u \quad \text{as } n \to \infty,$$

which implies that

$$u_n \to u \quad \text{in } \mathcal{H}_\lambda.$$

□

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A sequence $(u_n) \subset H^1(\mathbb{R}^N)$ is called a $(\mathrm{PS})_{\infty,c}$ sequence when the following hold:

$$\left.\begin{array}{c} u_n \in \mathcal{H}_{\lambda_n}, \\ \lambda_n \to \infty, \quad n \to \infty, \\ \Phi_{\lambda_n}(u_n) \to c \quad \text{as } \lambda_n \to \infty, \\ \|\Phi'_{\lambda_n}(u_n)\| \to 0 \quad \text{as } \lambda_n \to \infty. \end{array}\right\} \tag{$(\mathrm{PS})_{\infty,c}$}$$

Let us study the behaviour of a $(\mathrm{PS})_{\infty,c}$ sequence.

Proposition 3.7. *Let (u_n) be a $(PS)_{\infty,c}$ sequence with $c \in (0, (\frac{1}{2} - 1/(q+1))S^{N/2})$. Then, for some subsequence still denoted by (u_n), there exists $u \in H^1(\mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^1(\mathbb{R}^N).$$

Moreover,

(i) *$u \equiv 0$ in $\mathbb{R}^N \setminus \Omega_\Gamma$ and $u|_{\Omega_j}$ is a non-negative solution of*

$$\left.\begin{array}{c} -\Delta u + Z(x)u = \beta|u|^{q-1}u + |u|^{2^*-2}u \quad \text{in } \Omega_j, \\ u = 0 \quad \text{on } \partial\Omega_j \end{array}\right\} \tag{$(\mathrm{P})_j$}$$

for each $j \in \Gamma$.

(ii) *u_n converges to u in a stronger sense, namely*

$$\|u_n - u\|_{\lambda_n} \to 0.$$

Hence,

$$u_n \to u \quad \textit{strongly in } H^1(\mathbb{R}^N). \tag{3.14}$$

(iii) *As $\lambda_n \to \infty$, we have the following convergences:*

$$\lambda_n \int_{\mathbb{R}^N} V(x)u_n^2 \to 0, \tag{3.15}$$

$$\|u_n\|^2_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Gamma} \to 0, \tag{3.16}$$

$$\|u_n\|^2_{\lambda_n, \Omega'_j} \to \int_{\Omega_j} |\nabla u|^2 + Z(x)u^2 \quad \textit{for all } j \in \Gamma.$$

Proof. As in the proof of Lemma 3.2, there is a positive constant $K > 0$ such that

$$\|u_n\|_{\lambda_n} \leqslant K \quad \forall n \in \mathbb{N}.$$

Thus, (u_n) is a bounded sequence in $H^1(\mathbb{R}^N)$ and, for some subsequence still denoted by (u_n), we may assume that there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^N)$$

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and

$$u_n(x) \to u(x) \quad \text{a.e in } \mathbb{R}^N.$$

Using once more similar arguments explored in Proposition 3.5, we get

$$u_n \to u \quad \text{in } H^1(\mathbb{R}^N). \tag{3.17}$$

To show (i), we fix the set $C_m = \{x \in \mathbb{R}^N : V(x) \geqslant 1/m\}$. Then

$$\int_{C_m} u_n^2 \leqslant \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x) u_n^2,$$

that is,

$$\int_{C_m} u_n^2 \leqslant \frac{m}{\lambda_n} \|u_n\|_{\lambda_n}^2.$$

The above inequality, together with Fatou's lemma, implies that

$$\int_{C_m} u^2 = 0 \quad \forall m \in \mathbb{N}.$$

Thus, $u(x) = 0$ on $\bigcup_{m=1}^{+\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega}$ and we can assert that $u|_{\Omega_j} \in H_{\mathrm{o}}^1(\Omega_j)$ for all $j \in \{1, \dots, k\}$.

Once we have proved that $\Phi'_{\lambda_n}(u_n)\varphi \to 0$ as $n \to \infty$ for each $\varphi \in C_{\mathrm{o}}^{\infty}(\Omega_j)$ (and hence for each $\varphi \in H_{\mathrm{o}}^1(\Omega_j)$), it follows from (3.17) that

$$\int_{\Omega_j} \nabla u \nabla \varphi + Z(x) u \varphi - \int_{\Omega_j} g(x, u)\varphi = 0, \tag{3.18}$$

showing that $u|_{\Omega_j}$ is a solution of $(\mathrm{P})_j$ for each $j \in \{1, 2, \dots, k\}$.

For each $j \in \{1, 2, \dots, k\} \setminus \Gamma$, setting $\varphi = u|_{\Omega_j}$ in (3.18), we have

$$\int_{\Omega_j} |\nabla u|^2 + Z(x) u^2 - \int_{\Omega_j} f(u) u = 0,$$

that is,

$$\|u\|_{\lambda, \Omega_j}^2 - \int_{\Omega_j} f(u) u = 0.$$

By (2.1) and the fact that $f(s)s \leqslant \nu_0 s^2$ for all $s \in \mathbb{R}$, we have

$$\delta_0 \|u\|_{2, \Omega_j}^2 \leqslant \|u\|_{\lambda, \Omega_j}^2 - \nu_0 |u|_{2, \Omega_j}^2 \leqslant \|u\|_{\lambda, \Omega_j}^2 - \int_{\Omega_j} f(u) u = 0.$$

Thus, $u = 0$ in Ω_j, for $j \in \{1, 2, \dots, k\} \setminus \Gamma$, proving (i).

For (ii), we have

$$\|u_n - u\|^2_{\lambda_n} - \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} (f(u_n) - f(u))(u_n - u) - \int_{\Omega'_\Gamma} (h(u_n) - h(u))(u_n - u) = \Phi'_{\lambda_n}(u_n)(u_n - u) - \Phi'_{\lambda_n}(u)(u_n - u).$$

Using the equalities

$$\int_{\Omega'_\Gamma} (h(u_n) - h(u))(u_n - u) = o_n(1),$$

$$\Phi'_{\lambda_n}(u)(u_n - u) = \int_{\Omega_\Gamma} \nabla u \nabla (u_n - u) + Z(x) u (u_n - u) - \int_{\Omega_\Gamma} f(u)(u_n - u) = o_n(1)$$

and employing the inequality

$$|\Phi'_{\lambda_n}(u_n)(u_n - u)| \leqslant \|\Phi'_{\lambda_n}(u_n)\| (\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) = o_n(1),$$

it follows that

$$\|u_n - u\|^2_{\lambda_n} - \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} (f(u_n) - f(u))(u_n - u) = o_n(1).$$

Now, using (2.1), the fact that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega'_\Gamma$ and the above estimate, we obtain

$$\|u_n - u\|^2_{\lambda_n} \to 0 \quad \text{as } n \to \infty.$$

To prove (iii), note that, from (1.3),

$$\int_{\mathbb{R}^N} \lambda_n V(x) u_n^2 \leqslant C \|u_n - u\|^2_{\lambda_n},$$

so

$$\int_{\mathbb{R}^N} \lambda_n V(x) u_n^2 \to 0 \quad \text{as } n \to \infty.$$

□

In order to establish a uniform L^∞ bound result for (u_λ), we need the next two propositions. The first is a version of [**9**, Theorem 2.3] due to Brezis and Kato (see also [**23**]) and we omit its proof.

Proposition 3.8. *Let b be a non-negative measurable function and let the function $g : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following. For each non-negative function $v \in H^1(\mathbb{R}^N)$ there exists a function $h \in L^{N/2}(\mathbb{R}^N)$ such that*

$$g(x, v(x)) \leqslant (h(x) + C_g) v(x) \quad \forall x \in \mathbb{R}^N.$$

If $v \in H^1(\mathbb{R}^N)$ is a weak solution of $-\Delta v + b(x) v = g(x, v)$, we have $v \in L^p(\mathbb{R}^N)$ for all $2 \leqslant p < \infty$. Moreover, there exists a positive constant $C_p = C(p, C_g, h)$ such that

$$\|v\|_p \leqslant C_p \|v\|_{H^1}. \tag{3.19}$$

Moreover, if (v_k), (b_k) and (h_k) satisfy the above hypothesis and $h_k \to h$ in $L^{N/2}(\mathbb{R}^N)$, the sequence $C_{p,k} = C(p, C_g, h_k)$ is bounded.

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Proposition 3.9. *Suppose that b is as set in Proposition 3.8, $q > N/2$ and that, for each non-negative function $v \in H^1(\mathbb{R}^N)$, there exists $h \in L^q(\mathbb{R}^N)$ with*

$$g(x, v(x)) \leqslant h(x)v(x) \quad \forall x \in \mathbb{R}^N.$$

Then, if v is a non-negative weak solution of $-\Delta v + b(x)v = g(x, v)$, there exists $C = C(q, \|h\|_q) > 0$ such that

$$\|v\|_\infty \leqslant C\|v\|_{H^1}.$$

Moreover, if (v_k), (b_k) and (h_k) satisfy the above hypothesis and $(\|h_k\|_q)$ is bounded, it is possible to show that $(C_k = C(q, \|h_k\|_q))$ is a bounded sequence.

Proof. We use the Moser iteration technique, adapting the arguments found in [**15**,**17**] (see also [**5**]). The basic idea is as follows.

For each $n \in \mathbb{N}$ and $\alpha > 1$ such that $v \in L^{2\alpha q_1}(\mathbb{R}^N)$, consider $A_n = \{x \in \mathbb{R}^N : |v|^{\alpha-1} \leqslant n\}$, $B_n = \mathbb{R}^N \setminus A_n$ and the function v_n given by

$$v_n = v|v|^{2(\alpha-1)} \text{ on } A_n \quad \text{and} \quad v_n = n^2 v \text{ on } B_n.$$

Once we have proved that $v_n \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (\nabla v \nabla v_n + b(x) v v_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} g(x, v) v_n\,\mathrm{d}x.$$

Considering $q_1 = q/(q-1)$, $r > 2q_1$,

$$\omega_n = v|v|^{(\alpha-1)} \in A_n \quad \text{and} \quad \omega_n = nv \in B_n$$

and repeating the arguments explored in [**5**, **15**], we obtain

$$|v|_{r\alpha} \leqslant \alpha^{1/\alpha}(S_r|h|_q)^{\alpha/2}|v|_{2\alpha q_1}. \tag{3.20}$$

Now, we will prove the estimate involving the L^∞ norm.

Step 1. Fixing $\chi = r/(2q_1) > 1$ and $\alpha = \chi$, we have $2q_1\alpha = r$ and we can rewrite (3.20) in the following way:

$$|v|_{r\chi} \leqslant \chi^{1/\chi}(S_r|h|_q)^{1/2\chi}|v|_r. \tag{3.21}$$

Step 2. Considering $\alpha = \chi^2$, we have $2q_1\alpha = r\chi$. Thus, by Step 1 and (3.20), we get

$$|v|_{r\chi^2} \leqslant \chi^{2/\chi^2}(S_r|h|_q)^{1/(2\chi^2)}|v|_{r\chi}. \tag{3.22}$$

From (3.21) and (3.21), it follows that

$$|v|_{r\chi^2} \leqslant \chi^{1/\chi+2/\chi^2}(S_r|h|_q)^{(1/\chi+1/\chi^2)/2}|v|_r. \tag{3.23}$$

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Step 3. Choosing $\alpha = \chi^3$, we have $2q_1\alpha = r\chi^2$. Thus, by Step 2 and (3.20),

$$|v|_{r\chi^3} \leqslant \chi^{3/\chi^3}(S_r|h|_q)^{1/(2\chi^3)}|v|_{r\chi^2}. \tag{3.24}$$

From (3.23) and (3.24),

$$|v|_{r\chi^3} \leqslant \chi^{1/\chi+2/\chi^2+3/\chi^3}(S_r|h|_q)^{(1/\chi+1/\chi^2+1/\chi^3)/2}|v|_r. \tag{3.25}$$

Repeating the above arguments, for each $m \in \mathbb{N}$ we have the following inequality:

$$|v|_{r\chi^m} \leqslant \chi^{1/\chi+2/\chi^2+3/\chi^3+\cdots+m/\chi^m}(S_r|h|_q)^{(1/\chi+1/\chi^2+1/\chi^3+\cdots+1/\chi^m)/2}|v|_r. \tag{3.26}$$

Since

$$\sum_{m=1}^{\infty}\frac{m}{\chi^m} = \frac{1}{(\chi-1)} \quad \text{and} \quad \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{\chi^m} = \frac{1}{2(\chi-1)},$$

from (3.26) we have

$$|v|_{r\chi^m} \leqslant C|v|_r,$$

where $C = \chi^{1/(\chi-1)}(S_r|h|_q)^{1/(2(\chi-1))}$. Consequently,

$$|v|_\infty \leqslant C|v|_r.$$

□

Now we are ready to prove the key result in order to conclude the proof of Theorem 1.1.

Proposition 3.10. *Let* $\{u_\lambda\}$ *be a family of positive solutions of (3.4) satisfying*

$$\sup_{\lambda\leqslant 1}\{\Phi_\lambda(u_\lambda)\} < \left(\frac{1}{2} - \frac{1}{q+1}\right)S^{N/2}.$$

Then, there exists $\lambda^* > 0$ *such that*

$$|u_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Gamma} \leqslant a \quad \forall\lambda \geqslant \lambda^*.$$

Hence, u_λ *is a positive solution of (1.1) for* $\lambda \geqslant \lambda^*$.

Proof. Let (λ_n) be a sequence with $\lambda_n \to \infty$ and define $u_n(x) = u_{\lambda_n}(x)$. Then u_{λ_n} is a bounded sequence of positive solution of (3.4). Using Proposition 3.7, it follows that $u_n \to u$ in $H^1(\mathbb{R}^N)$, where u is the weak limit of (u_n) in $H^1(\mathbb{R}^N)$. Moreover, recall that there exists $C > 0$ such that

$$g(x, u_n) \leqslant u_n + Cu_n^{2^*-1} \leqslant (1 + a_n(x))u_n,$$

where $a_n(x) = C|u_n|^{2^*-2}$, which converges in $L^{N/2}(\mathbb{R}^N)$ to u^{2^*-2}. Using Proposition 3.8, it follows that, for each $r > 1$, the sequence $(|u_n|_r)$ is uniformly bounded. In the following we set $r > 2^*$. Let us rewrite (3.4) in the following way:

$$-\Delta u_n + (\lambda_n V(x) + Z(x) - \nu_o)u_n = \tilde{g}(x, s) := g(x, u_n) - \nu_o u_n \in \mathbb{R}^N.$$

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Note that

$$\tilde{g}(x, u_n) \leqslant C u_n^{2^*-1} = a_n(x) u_n,$$

and we can check that $a_n(x) = C|u_n|^{2^*-2} \in L^q(\mathbb{R}^N)$ with $q = r/(2^* - 2)$ and $q > N/2$. Proposition 3.9 ensures that

$$|u_n|_\infty \leqslant K_{\rm o} \quad \forall n \in \mathbb{N},$$

for some $K_{\rm o} > 0$.

Now let $v_n(x) = u_{\lambda_n}(\varepsilon_n x + \bar{x}_n)$, $\varepsilon_n^2 = 1/\lambda_n$ and $(\bar{x}_n) \subset \partial\Omega'_\Gamma$. Without loss of generality we will assume that $\bar{x}_n \to \bar{x} \in \partial\Omega'_\Gamma$. We have $|v_n|_\infty \leqslant K_{\rm o}$,

$$-\Delta v_n + (V(\varepsilon_n x + \bar{x}_n) + \varepsilon_n^2 Z(\varepsilon_n x + \bar{x}_n)) v_n = \varepsilon_n^2 g(\varepsilon_n x + \bar{x}_n, v_n)$$

and

$$|g(\varepsilon_n x + \bar{x}_n, v_n)| \leqslant |v_n| + C|v_n|^{2^*-1}.$$

These facts, together with bootstrap arguments, imply that there exists $K_1 > 0$ such that

$$\|v_n\|_{C^2(B_1(0))} \leqslant K_1 \quad \forall n \in \mathbb{N}.$$

The above estimate implies that the weak limit v of the sequence $(v_n) \subset H^1(\mathbb{R}^N)$ belongs to $C^1(B_1(0))$ with

$$v_n \to v \in C^1(B_1(0)) \quad \text{as } n \to \infty.$$

Assuming by contradiction that there exists $\eta > 0$ verifying

$$u_{\lambda_n}(\bar{x}_n) \geqslant \eta \quad \forall n \in \mathbb{N},$$

it follows that

$$v_n(0) \geqslant \eta \quad \forall n \in \mathbb{N}.$$

Thus, $v \neq 0$ in $B_1(0)$.

On the other hand, the function v satisfies the equation

$$-\Delta v + V(\bar{x}) v = 0 \in \mathbb{R}^N.$$

This implies that $v \equiv 0$, and contradicts the fact that $v \neq 0$ in $B_1(0)$. Thus, there exists $\lambda^* > 0$ such that

$$|u_\lambda|_{\infty, \partial\Omega'_\Gamma} \leqslant a \quad \forall \lambda \geqslant \lambda^*.$$

Repeating the arguments explored in [**12**], we have

$$|u_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\Gamma} \leqslant a \quad \forall \lambda \geqslant \lambda^*,$$

from which the proposition follows. □

4. Positive solutions for the original problem

In this section, for each $\lambda \geqslant 1$ and $j \in \Gamma$, let us denote by $\Phi_{\lambda,j} : H^1(\Omega'_j) \to \mathbb{R}$ the functional

$$\Phi_{\lambda,j}(u) = \frac{1}{2}\int_{\Omega'_j} |\nabla u|^2 + (\lambda V(x) + Z(x))u^2 - \frac{\beta}{q+1}\int_{\Omega'_j} (u_+)^{q+1} - \frac{1}{2^*}\int_{\Omega'_j} (u_+)^{2^*}. \quad (4.1)$$

We know that the critical points of $\Phi_{\lambda,j}$ are the weak solutions of the elliptic equation with Neumann boundary condition

$$\left.\begin{aligned} -\Delta u + (\lambda V(x) + Z(x))u = \beta u^q + u^{2^*-1} \in \Omega'_j, \\ u > 0 \in \Omega'_j, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega'_j. \end{aligned}\right\} \quad (4.2)$$

It is easy to check that the $\Phi_{\lambda,j}$ satisfy the mountain-pass geometry. In what follows, we denote by $c_{\lambda,j}$ the minimax level related to the functional $\Phi_{\lambda,j}$ and defined by

$$c_{\lambda,j} = \inf_{\gamma \in \Upsilon_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)),$$

where

$$\Upsilon_{\lambda,j} = \{\gamma \in C([0,1], H^1(\Omega'_j));\ \gamma(0) = 0,\ \Phi_{\lambda,j}(\gamma(1)) < 0\}.$$

Since β is small, using well-known arguments found in [**2, 14**], it is possible to prove that there exist two non-negative functions $w_j \in H^1_0(\Omega_j)$ and $w_{\lambda,j} \in H^1(\Omega'_j)$ verifying

$$I_j(w_j) = c_j \quad \text{and} \quad I'_j(w_j) = 0 \quad (I_j \text{ was defined in (3.6)})$$

and

$$\Phi_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \Phi'_{\lambda,j}(w_{\lambda,j}) = 0.$$

4.1. A special critical value for the functional Φ_λ

In what follows, let us fix $R > 1$ such that

$$\left| I_j\left(\frac{1}{R}w_j\right)\right| < \tfrac{1}{2}c_j \quad \forall j \in \Gamma$$

and

$$|I_j(Rw_j) - c_j| \geqslant 1 \quad \forall j \in \Gamma.$$

From the definition of c_j, it is standard to prove the equality

$$\max_{s \in [1/R^2, 1]} I_j(sRw_j) = I_j(w_j) = c_j \quad \forall j \in \Gamma, \quad (4.3)$$

where the interval $[1/R^2, 1]$ is chosen conveniently for our purposes.

Reordering the set Γ, we may consider $\Gamma = \{1, \dots, l\}$, $l \leqslant k$.

Let us define

$$[1/R^2,1]^l = \underbrace{[1/R^2,1]\times\cdots\times[1/R^2,1]}_{l \text{ times}},$$

the l-dimensional closed cube in $\mathbb{R}^l$ and $(1/R^2,1)^l$, the l-dimensional open cube in $\mathbb{R}^l$.

We also need to define the application

$$\gamma_{\mathrm{o}} : [1/R^2,1]^l \to \bigcup_{j\in\Gamma} H_0^1(\Omega_j) \subset H^1(\Omega'_\Gamma)$$

as

$$\gamma_{\mathrm{o}}(s_1,s_2,\dots,s_l)(x) = \sum_{j=1}^{l} s_j R w_j(x) \tag{4.4}$$

and the number

$$b_{\lambda,\Gamma} = \inf_{\gamma\in\Upsilon_*} \max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma(s_1,\dots,s_l)),$$

where

$$\Upsilon_* = \{\gamma\in C([1/R^2,1]^l, H^1(\Omega'_\Gamma)\setminus\{0\})\gamma=\gamma_{\mathrm{o}} \text{ on } \partial([1/R^2,1]^l)\}.$$

We remark that $\gamma_{\mathrm{o}}\in\Upsilon_*$, so that $\Upsilon_*\neq\emptyset$ and $b_{\lambda,\Gamma}$ is well defined.

Lemma 4.1. *For any $\gamma\in\Upsilon_*$, there exists $(t_1,\dots,t_l)\in[1/R^2,1]^l$ such that*

$$\Phi'_{\lambda,j}(\gamma(t_1,\dots,t_l))(\gamma(t_1,\dots,t_l)) = 0 \quad \text{for } j\in\{1,\dots,l\}.$$

Proof. For a given $\gamma\in\Upsilon_*$, let us consider the map $\tilde{\gamma}:[1/R^2,1]^l\to\mathbb{R}^l$ defined by

$$\tilde{\gamma}(s_1,\dots,s_l) = (\Phi'_{\lambda,1}(\gamma)(\gamma),\dots,\Phi'_{\lambda,l}(\gamma)(\gamma)),$$

where

$$\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}(\gamma(s_1,\dots,s_l))(\gamma(s_1,\dots,s_l)) \quad \forall j\in\Gamma.$$

For $(s_1,\dots,s_l)\in\partial([1/R^2,1]^l)$, it follows that

$$\gamma(s_1,\dots,s_l) = \gamma_0(s_1,\dots,s_l)$$

and, by (4.3), that

$$\Phi'_{\lambda,j}(\gamma_0(s_1,\dots,s_l))(\gamma_0(s_1,\dots,s_l)) = 0 \implies s_j = \frac{1}{R} \quad \forall j\in\Gamma.$$

Thus, $(0,\dots,0)\notin\tilde{\gamma}(\partial([1/R^2,1]^l))$. After some algebraic manipulation, we get the following equality involving the topological degree:

$$\deg(\tilde{\gamma},[1/R^2,1]^l,(0,\dots,0)) = (-1)^l.$$

Therefore, using topological degree properties, there exist $(t_1,\dots,t_l)\in(1/R^2,1)^l$ such that

$$\Phi'_{\lambda,j}(\gamma(t_1,\dots,t_l))(\gamma(t_1,\dots,t_l)) = 0 \quad \text{for } j\in\{1,\dots,l\}. \tag{4.5}$$

□

In the following, the number $c_\Gamma := \sum_{j=1}^{l} c_j$ is very important in the proof of Theorem 1.1. Let us analyse the interaction between $\sum_{j=1}^{l} c_{\lambda,j}$, $b_{\lambda,\Gamma}$ and c_Γ, using the fact that

$$c_\Gamma \in \left(0, \left(\frac{1}{2} - \frac{1}{q+1}\right) S^{N/2}\right)$$

(see Remark 3.4).

Proposition 4.2.

(a) $\sum_{j=1}^{l} c_{\lambda,j} \leqslant b_{\lambda,\Gamma} \leqslant c_\Gamma$ *for all* $\lambda \geqslant 1$.

(b) $\Phi_\lambda(\gamma(s_1, \dots, s_l)) < c_\Gamma$ *for all* $\lambda \geqslant 1$, $\gamma \in \Upsilon_*$ *and* $(s_1, \dots, s_l) \in \partial([1/R^2, 1]^l)$.

Proof. The proof of this proposition follows by adapting some arguments found in [**1**, Proposition 4.2]. □

Corollary 4.3.

(a) $b_{\lambda,\Gamma} \to c_\Gamma$, *as* $\lambda \to \infty$.

(b) $b_{\lambda,\Gamma}$ *is a critical value of* Φ_λ *for large* λ.

Proof. (a) For all $\lambda \geqslant 1$ and for each j we have $0 < c_{\lambda,j} \leqslant c_j$. Using the same type of idea explored in the proof of Proposition 3.7, we can prove that $c_{\lambda,j} \to c_j$ as $\lambda \to \infty$ and thus, from Proposition 4.2, $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.

(b) By Corollary 4.3 (a) and (3.8), we may choose a large λ such that

$$b_{\lambda,\Gamma} \simeq c_\Gamma \in \left(0, \left(\frac{1}{2} - \frac{1}{q+1}\right) S^{N/2}\right).$$

Proposition 3.5 implies that any $(\mathrm{PS})_{b_{\lambda,\Gamma}}$ sequence of the functional Φ_λ has a strongly convergent subsequence (in $\mathcal{H}_\lambda$). Employing this fact, we can use the well-known arguments involving the deformation lemma to conclude that $b_{\lambda,\Gamma}$ is a critical level of Φ_λ for $\lambda \geqslant 1$. □

4.2. Proof of the main theorem

To prove Theorem 1.1, we need to find a positive solution u_λ for a large λ, which approaches a least-energy solution in each $\Omega_j (j \in \Gamma)$ and vanishes elsewhere as $\lambda \to \infty$. To this end, we will prove two propositions that, together with the estimates made in the above section, imply that Theorem 1.1 holds.

Henceforth,

$$M = 1 + \sum_{j=1}^{k} \sqrt{\left(\frac{1}{2} - \frac{1}{q+1}\right)^{-1} c_j},$$
$$\bar{B}_{M+1}(0) = \{u \in \mathcal{H}_\lambda;\ \|u\|_\lambda \leqslant M+1\}$$

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and, for small $\mu > 0$, we define

$$A_\mu^\lambda = \{u \in \bar{B}_{M+1}(0);\ \|u\|_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma} \leqslant \mu \text{ and } |\Phi_{\lambda,j}(u) - c_j| \leqslant \mu \text{ for all } j \in \Gamma\}.$$

We also use the notation

$$\Phi_\lambda^{c_\Gamma} = \{u \in \mathcal{H}_\lambda;\ \Phi_\lambda(u) \leqslant c_\Gamma\}$$

and remark that $w = \sum_{j=1}^{l} w_j \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, showing that $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma} \neq \emptyset$. Fixing

$$0 < \mu < \tfrac{1}{3}\min\{c_j; j \in \Gamma\}, \tag{4.6}$$

we have the following uniform estimate of $\|\Phi'_\lambda(u)\|_\lambda$ on the annulus $(A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 4.4. *Let $\mu > 0$ satisfy (4.6). Then there exist $\sigma_o > 0$ and $\Lambda_* \geqslant 1$ independent of λ such that*

$$\|\Phi'_\lambda(u)\|_\lambda \geqslant \sigma_o \quad \text{for } \lambda \geqslant \Lambda_* \text{ and all } u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}. \tag{4.7}$$

Proof. The proof of this proposition follows the arguments found in [**1**, Proposition 4.4]. □

Proposition 4.5. *Let μ satisfy (4.6) and let $\Lambda_* \geqslant 1$ be a constant given in Proposition 4.4. Then, for $\lambda \geqslant \Lambda_*$, there exists a positive solution u_λ of $(P)_\lambda$ in the set $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$.*

Proof. Assuming by contradiction that there are no critical points in $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, since the Palais–Smale condition holds for Φ_λ in

$$\left(0, \left(\frac{1}{2} - \frac{1}{q+1}\right) S^{N/2}\right)$$

(see Proposition 3.9), there exists a constant $d_\lambda > 0$ such that

$$\|\Phi'_\lambda(u)\| \geqslant d_\lambda \quad \text{for all } u \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}.$$

By hypothesis we also have

$$\|\Phi'_\lambda(u)\| \geqslant \sigma_o \quad \text{for all } u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma},$$

where $\sigma_o > 0$ is independent of λ. In what follows, $\Psi : \mathcal{H}_\lambda \to \mathbb{R}$ and $W : \Phi_\lambda^{c_\Gamma} \to \mathbb{R}$ are continuous functions that verify

$$\Psi(u) = \begin{cases} 1 & \text{for } u \in A_{3\mu/2}^\lambda, \\ 0 & \text{for } u \notin A_{2\mu}^\lambda, \end{cases}$$

$$0 \leqslant \Psi(u) \leqslant 1 \quad \text{for } u \in \mathcal{H}_\lambda$$

and

$$W(u) = \begin{cases} -\Psi(u)\|Y(u)\|^{-1}\|Y(u)\|, & u \in A_{2\mu}^\lambda, \\ 0, & u \notin A_{2\mu}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ'_λ on $\mathcal{M} = \{u \in \mathcal{H}_\lambda : \Phi'_\lambda(u) \neq 0\}$. Hence, using the properties involving Y and Φ_λ, we have the following inequality:

$$\|W(u)\| \leqslant 1 \quad \forall \lambda \geqslant \Lambda_* \text{ and } u \in \Phi_\lambda^{c_\Gamma}.$$

Considering the deformation flow $\eta : [0,\infty) \times \Phi_\lambda^{c_\Gamma} \to \Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = W(\eta), \quad \eta(0,u) = u \in \Phi_\lambda^{c_\Gamma},$$

and observing that there exists $K_* > 0$ such that

$$|\Phi_{\lambda,j}(u) - \Phi_{\lambda,j}(v)| \leqslant K_* \|u - v\|_{\lambda,\Omega'_j} \quad \text{for all } u, v \in \bar{B}_{M+1}(0) \text{ and all } j \in \Gamma,$$

using similar arguments explored by Ding and Tanaka [**13**], we obtain two numbers $T = T(\lambda) > 0$ and $\varepsilon_* > 0$ independent of $\lambda \geqslant \Lambda_*$ satisfying

$$\gamma^*(s_1, \dots, s_l) = \eta(T, \gamma_{\mathrm{o}}(s_1, \dots, s_l)) \in \Gamma_*$$

and

$$\max_{(s_1,\dots,s_l) \in [1/R^2,1]^l} \Phi_\lambda(\gamma^*(s_1, \dots, s_l)) \leqslant c_\Gamma - \varepsilon_*.$$

Combining the definition of $b_{\lambda,\Gamma}$ and the above results, we obtain the inequality

$$b_{\lambda,\Gamma} \leqslant c_\Gamma - \varepsilon_* \quad \forall \lambda \geqslant \Lambda_*,$$

which contradicts Corollary 4.3. □

We now conclude the proof of Theorem 1.1.

From Proposition 4.5 there exists a family $\{u_\lambda\}$ of positive solutions to (A_λ) verifying the following properties.

(i) For a fixed $\mu > 0$ there exists λ^* such that

$$\|u_\lambda\|_{\lambda,\mathbb{R}^N \setminus \Omega'_\Gamma} \leqslant \mu \quad \forall \lambda \geqslant \lambda^*.$$

Thus, from the proof of Proposition 3.10, fixed μ sufficiently small we can conclude that

$$|u_\lambda|_{\infty,\mathbb{R}^N \setminus \Omega'_\Gamma} \leqslant a \quad \forall \lambda \geqslant \lambda^*,$$

showing that u_λ is a positive solution of $(\mathrm{P})_\lambda$.

(ii) Fixing $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence $\{u_{\lambda_n}\}$ verifies that

$$\Phi_{\lambda_n}(u_{\lambda_n}) = 0 \quad \forall n \in \mathbb{N},$$
$$\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^N \setminus \Omega_\Gamma} \to 0,$$
$$\Phi_{\lambda_n,j}(u_{\lambda_n}) \to c_j \quad \forall j \in \Gamma$$

and

$$u_{\lambda_n} \to u \in H^1(\mathbb{R}^N) \quad \text{with } u \in H^1_{\mathrm{o}}(\Omega_\Gamma),$$

from which the proof of Theorem 1.1 follows.

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References

1. C. O. Alves, Existence of multi-bump solutions for a class of quasilinear problems, *Adv. Nonlin. Studies* **6** (2006), 491–509.
2. C. O. Alves and A. El Hamidi, Nehari manifold and existence of positive solutions to a class of quasilinear problems, *Nonlin. Analysis* **60** (2005), 611–624.
3. C. O. Alves, P. C. Carrião and O. H. Miyagaki, Nonlinear pertubations of a periodic elliptic problem with critical growth, *J. Math. Analysis Applic.* **260** (2001), 133–146.
4. C. O. Alves, D. C. de Morais Filho and M. A. S. Souto, Radially symmetric solutions for a class of critical exponent elliptic problems in $\mathbb{R}^N$, *Electron. J. Diff. Eqns* **1996** (1996), 1–12.
5. C. O. Alves and G. M. Figueiredo, Multiplicity of positive solutions for a quasilinear problem in $\mathbb{R}^N$ via the penalization method, *Adv. Nonlin. Studies* **4** (2005), 551–572.
6. T. Bartsch, A. Pankov and Z.-Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.* **3** (2001), 549–569.
7. T. Bartsch and Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems in $\mathbb{R}^N$, *Commun. PDEs* **20** (1995), 1725–1741.
8. T. Bartsch and Z. Q. Wang, Multiple positive solutions for a nonlinear Schrödinger equation, *Z. Angew. Math. Phys.* **51** (2000), 366–384.
9. H. Brezis and T. Kato, Remarks on the Schrödinger operator with regular complex potentials, *J. Math. Pures Appl.* **58** (1979), 137–151.
10. M. Clapp and Y. H. Ding, Positive solutions of a Schrödinger equation with critical nonlinearity, *Z. Angew. Math. Phys.* **55** (2004), 592–605.
11. D. G. de Figueiredo and Y. H. Ding, Solutions of a nonlinear Schrödinger equation, *Discrete Contin. Dynam. Syst.* **8** (2002), 563–584.
12. M. del Pino and P. L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. PDEs* **4** (1996), 121–137.
13. Y. H. Ding and K. Tanaka, Multiplicity of positive solutions of a nonlinear Schrödinger equation, *Manuscr. Math.* **112** (2003), 109–135.
14. J. Garcia Azorero and I. Peral Alonzo, Multiplicity of solutions for elliptic problems with critical exponent with a nonsymmetric term, *Trans. Am. Math. Soc.* **2** (1991), 877–895.
15. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order* (Springer, 1998).
16. J. V. Gonçalves and C. O. Alves, Existence of positive solutions for m-Laplacian equation in $\mathbb{R}^N$ involving critical Sobolev exponents, *Nonlin. Analysis TMA* **32** (1998), 53–70.
17. L. Gongbao, Some properties of weak solutions of nonlinear scalar field equations, *Annales Acad. Scient. Fenn.* A **14** (1989), 27–36.
18. P. L. Lions, The concentration–compactness principle in the calculus of variations, the limit case, I, *Rev. Mat. Ibero.* **1** (1985), 46–20.
19. P. L. Lions, The concentration–compactness principle in the calculus of variations, the limit case, II, *Rev. Mat. Ibero.* **1** (1985), 145–201.
20. P. L. Lions, *La méthode de concentration–compacité en calculs des variations* (ICTP, Trieste, 1988).

21. O. H. Miyagaki, On a class of semilinear elliptic problems in $\mathbb{R}^N$ with critical growth, *Nonlin. Analysis TMA* **29** (1997), 773–781.
22. E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.* **209** (1992), 27–42.
23. M. Struwe, *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems* (Springer, 1990).

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