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THE DISCRETENESS OF SPECTRUM FOR HIGHER-ORDER DIFFERENTIAL OPERATORS IN WEIGHTED FUNCTION SPACES

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Abstract

In this paper we consider the discreteness of spectrum for higher-order differential operators in weighted function spaces. Using the method of embedding theorems of weighted Sobolev spaces H_p^n in weighted spaces $L_{s,r}$, we obtain a new sufficient and necessary condition to ensure that the spectrum is discrete, which can be easily used to judge the discreteness of some differential operators.

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1. Introduction

The spectrum of differential operators is a fundamental and important problem in the theory of differential operators and over the years much attention has been paid to the situation in which the spectrum is discrete. Since Molcanov in 1953 established the celebrated criterion on the discreteness of the spectrum, this result has been developed by many authors [1–7]. Particularly in recent years much work has been done on higher-order differential operators in weighted function spaces, that is, the self-adjoint operators generated by the differential expression

$$Ly = \frac{1}{r^2}\sum_{k=0}^{n}(-1)^k(\rho_k^2 y^{(k)})^{(k)}, \qquad x \in \mathbb{R}. \tag{1.1}$$

In 1991 and 1994 Edmunds and Sun [2, 7], obtained some conditions to ensure that the spectrum is discrete. In particular, under some assumptions, if r^2 is bounded below and above by positive constants, then the spectrum of all the self-adjoint extensions is discrete if and only if

$$\lim_{|x|\to\infty}\int_x^{x+w}\rho_0^2(t)\,dt = \infty$$

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for all $w > 0$. Using these results, we can judge the discreteness of some differential operators. However, for some differential operators satisfying similar conditions, these results are hard to deal with. Such examples can be found in Section 3. Motivated by this, we modify the earlier papers [2, 7] and obtain that the spectrum of the operator generated by (1.1) is discrete under the additional assumption that the potential function ρ_0^2 is bounded above and below by positive constants if and only if

$$\lim_{|x|\to\infty} \int_x^{x+w} r^2(t)\,dt = 0$$

for all $w > 0$. This can be easily used to judge the discreteness of the spectrum or the existence of the essential spectrum of a certain class of higher-order differential operators.

As is well known, the theory of the spectrum is related to the embedding of Sobolev space in a Hilbert space. Let H_p^n and $L_{s,r}$ represent the completion of $C_0^\infty(R)$ with respect to the norm $\|\cdot\|_{H_p^n}$ and $\|\cdot\|_{s,r}$ respectively, where

$$\begin{aligned} \|f\|_{H_p^n} &= \Big(\sum_{k=0}^{n} \|f^{(k)}\rho_k\|_p^p\Big)^{1/p}, \\ \|f\|_{s,r} &= \|fr\|_s. \end{aligned}$$

Our results are based on the following lemma; see [1, 2, 6, 7].

LEMMA 1.1. *The spectrum of the self-adjoint operator T generated by L is discrete if and only if H_2^n is compactly embedded in $L_{2,r}$.*

This paper is organised as follows. Section 2 is devoted to the main results, including the embedding of H_p^n in $L_{s,r}$ and the discreteness of the spectrum. In Section 3, some examples are described.

2. Main results

Throughout the paper, n will denote a positive integer and $1 < p \le s < \infty$, $p' = p/(p-1)$; I will stand for an arbitrary interval in $\mathbb{R}$ of length $|I|$. The coefficients and the weight function satisfy the following fundamental assumptions: $\rho_0, \rho_1, \dots, \rho_n, r$ are functions on $\mathbb{R}$ which are measurable and positive almost everywhere; it is assumed that $r \in L_s^{\mathrm{loc}}(\mathbb{R})$, $\rho_0, \rho_1, \dots, \rho_n \in L_p^{\mathrm{loc}}(\mathbb{R})$, $1/\rho_1, \dots, 1/\rho_n \in L_{p'}^{\mathrm{loc}}(\mathbb{R})$ and $\rho_0 \notin L_p(-\infty, 0) \cup L_p(0, \infty)$.

Assume that there exist positive constants c, c_1, c_2, c_3 such that:

(A) $(\int_I \rho_k^{-p'})^{1/p'}(\int_I \rho_{k+1}^p)^{1/p} \le c|I|$, for all I, $k = 1, 2, \dots, n-1$;
(B) $(\int_I \rho_0^{-p'})^{1/p'}(\int_I \rho_1^p)^{1/p} \le c|I|$, for all I;
(C) $c_1 \le \rho_0$, $c_3 \le \rho_n$ almost everywhere on $\mathbb{R}$;
(D) $\rho_0 \le c_2$ almost everywhere on $\mathbb{R}$.

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For any $x \in \mathbb{R}$ and a fixed $j \in (0, \infty)$, define

$$d_j(x) = \sup\Big\{d > 0 : \Big(\int_{x-\frac{1}{2}d}^{x+\frac{1}{2}d} \rho_0^p \, dt\Big)^{1/p} d^{1/p'} \le j\Big\}.$$

For $k \in (j, \infty)$, define

$$d_k(x) = \sup\Big\{d > 0 : \Big(\int_{x-\frac{1}{2}d}^{x+\frac{1}{2}d} \rho_0^p \, dt\Big)^{1/p} d^{1/p'} \le k\Big\}.$$

Let

$$x_j^+ = x + \tfrac{1}{2}d_j(x), \quad x_j^- = x - \tfrac{1}{2}d_j(x), \quad \triangle_j(x) = (x_j^-, x_j^+),$$
$$d(x) = d_k(x) - d_j(x).$$

The case of $j = 1, k = 2$ is a special case here, which can be found in [2, 7]. Here we introduce three new functions $d_j(x)$, $d_k(x)$ and $d(x)$, by which we can obtain the following theorems.

Theorem 2.1. *Suppose that conditions (A) and (C) hold.*

(1) *If*

$$M = \sup_{x\in\mathbb{R}} M(x) = \sup_{x\in\mathbb{R}} d_j(x)^{1/p'} \Big(\int_{\triangle_j(x)} r^s\Big)^{1/s} < \infty,$$

then H_p^n is embedded in $L_{s,r}$.

(2) *If*

$$\lim_{|x|\to\infty} M(x) = 0,$$

then H_p^n is compactly embedded in $L_{s,r}$.

Proof. This is similar to [2, proof of Theorem 1]. □

Theorem 2.1 gives conditions sufficient to ensure that the embedding exists or is compact. The following theorem provides a necessary condition for this to happen.

Theorem 2.2. *Suppose that conditions (A), (B) and (C) hold.*

(1) *If H_p^n is embedded in $L_{s,r}$, then*

$$\sup_{x\in\mathbb{R}} (d(x))^n M(x) < \infty.$$

(2) *If H_p^n is compactly embedded in $L_{s,r}$, then*

$$\lim_{|x|\to\infty} (d(x))^n M(x) = 0.$$

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Proof. Set

$$f_x(t) = (d(x))^n (d_j(x))^{1/p'} g_x(t), \qquad \forall x \in \mathbb{R}, \forall t \in \mathbb{R},$$

where

$$g_x(t) = \begin{cases} f(2(t - x_k^-)/d(x)), & x_k^- \le t < x_j^-; \\ 1, & x_j^- \le t < x_j^+; \\ f(2(x_k^+ - t)/d(x)), & x_j^+ \le t < x_k^+; \\ 0, & \mathbb{R} \setminus (x_k^-, x_k^+), \end{cases}$$

where $f \in C^\infty(0, 1)$ such that $0 \le f \le 1$, $f(0) = 0$, $f(1) = 1$, $f' \in C_0^\infty(0, 1)$, and

$$\max_{1 \le i \le n} \max_{0 \le t \le 1} |f^{(i)}(t)| \le K.$$

Using methods similar to those used to prove [7, Theorem 2], we obtain the results. □

So far we have obtained some sufficient conditions or necessary conditions to ensure that the embedding exists or is compact. Below we will use the above conclusions to give a necessary and sufficient condition to ensure that the embedding is compact under the restrictions that the potential function is bounded above and below by two positive constants. Here we extend the functions $d_1(x)$, $d_2(x)$ and $d(x)$ in [2, 7] to $d_j(x)$, $d_k(x)$ and $d(x)$ in order to obtain the following necessary and sufficient condition to ensure that the spectrum is discrete. We state it in the following theorem.

Theorem 2.3. *Suppose that ρ_i, r satisfy the fundamental assumption and conditions (A)–(D). Then H_p^n is compactly embedded in $L_{s,r}$ if and only if*

$$\lim_{|x| \to \infty} \int_x^{x+w} r^s \, dt = 0$$

for every fixed $w > 0$.

Proof. From the condition $\rho_0 \ge c_1$ and Hölder's inequality, together with the definition of $d_j(x)$, we can see that

$$j = d_j(x)^{1/p'} \Big(\int_{\triangle_j(x)} \rho_0^p \, dt \Big)^{1/p} \ge c_1 d_j(x),$$

that is, $d_j(x) \le c_1^{-1} j$. By the definition of $d_j(x)$ and condition (D), we see that

$$j = \Big(\int_{\triangle_j(x)} \rho_0^p \, dt \Big)^{1/p} d_j(x)^{1/p'} \le c_2 d_j(x),$$

that is, $d_j(x) \ge c_2^{-1} j$. Again by Theorem 2.1(2), the sufficiency of the theorem can be easily obtained.

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If H_p^n is compactly embedded in $L_{s,r}$, then by Theorem 2.2(2),

$$\lim_{|x|\to\infty}(d(x))^n d_j(x)^{1/p'}\Big(\int_{\triangle_j(x)} r^s\,dt\Big)^{1/s}=0$$

for any $k\in(j,\infty)$. Since there exists a positive constant $k>j$ such that $c_2^{-1}k-c_1^{-1}j>0$, and from $d_j(x)\le c_1^{-1}j$, we can get

$$c_1^{-1}k-c_2^{-1}j\ge d(x)=d_k(x)-d_j(x)\ge c_2^{-1}k-c_1^{-1}j>0.$$

Thus we can obtain that

$$\lim_{|x|\to\infty}\int_{\triangle_j(x)} r^s\,dt=0.$$

The required conclusion is attained. □

We now consider the spectrum of problem (1.1). The discreteness of spectrum of a self-adjoint operator T generated by differential expressions L is equivalent to the compactness of the embedding of H_2^n in $L_{2,r}$. Therefore Theorem 2.3 reduces to the following result.

Theorem 2.4. *Suppose that ρ_i, r satisfy the fundamental assumption and conditions (A)–(D) with $p=s=2$. Then the spectrum of problem (1.1) is discrete if and only if*

$$\lim_{|x|\to\infty}\int_x^{x+w} r^2\,dt=0$$

for any fixed $w>0$.

Corollary 2.5. *Suppose that there are positive constants c_1, c_2 such that $c_1\le\rho_i(x)\le c_2$ $(i=0,1,2,\dots,n)$ for all $x\in\mathbb{R}$. Then the spectrum of problem (1.1) is discrete if and only if*

$$\lim_{|x|\to\infty}\int_x^{x+w} r^2\,dt=0$$

for all $w>0$.

Proof. It is easy to prove that the ρ_i satisfy all the conditions (A)–(D) with $p=s=2$. Then the result follows from Theorem 2.4. □

Remark 2.6. If some of the ρ_i $(i=2,3,\dots,n-1)$ are zero, these theorems also hold. This is the same as the remark of [2].

3. Example

Consider the differential expression

$$Ly=\frac{1}{r^2}\Big(y^{(4)}-y''+\Big(1+\frac{3}{4}\sin x\Big)y\Big),\qquad x\in\mathbb{R}. \tag{3.1}$$

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Here

$$\rho_2^2 = \rho_1^2 = 1, \quad \rho_0^2 = 1 + \tfrac{3}{4}\sin x.$$

If we use [2, Theorem 6] and [7, Theorem 2], we need to compute $d_1(x)$, $d_2(x)$ and $d(x)$, where

$$d_1(x) = \sup\Big\{d > 0 : \Big(\int_{x-\frac{1}{2}d}^{x+\frac{1}{2}d} \rho_0^p\,dt\Big)^{1/p} d^{1/p'} \le 1\Big\},$$

$$d_2(x) = \sup\Big\{d > 0 : \Big(\int_{x-\frac{1}{2}d}^{x+\frac{1}{2}d} \rho_0^p\,dt\Big)^{1/p} d^{1/p'} \le 2\Big\},$$

and

$$d(x) = d_2(x) - d_1(x).$$

Indeed, we find that these functions are very hard to compute. Thus using the previous results is difficult. However, using the results obtained here we can easily judge the discreteness of the spectrum of the above differential operator. It is easy to see that all the functions ρ_0, ρ_1, ρ_2 are bounded below and above. Thus by Corollary 2.5, we have that the spectrum of (3.1) is discrete if and only if

$$\lim_{|x|\to\infty} \int_x^{x+w} r^2\,dt = 0$$

for every $w > 0$.

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