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ON NONSTANDARD HULLS OF CONVEX SPACES

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A *nonstandard hull* of a TVS (locally convex topological vector space) (E, ξ) is a standard TVS $(\hat{E}, \hat{\xi})$ constructed from a nonstandard model for (E, ξ) [**3**]. If the nonstandard hulls of a TVS are independent of the non-standard model, we say that the TVS has *invariant* nonstandard hulls. This is (for complete spaces) the property that every finite element is infinitesimally close to a standard point. We build on the work of Henson and Moore [**4**], to show that invariance of nonstandard hulls is a self dual property equivalent to bounded sets being precompact, for F and DF spaces. (see Theorem 4.4).

In Section 3, we consider the weaker property of every finite element being weakly infinitesimally close to a standard point. Theorem 3.1 shows that this property is equivalent to the standard property of inductive semi-reflexivity [**2**]. (For standard results about inductive semi-reflexivity see [**1**; **2**; and **5**].) The question of invariance of nonstandard hulls being equivalent to inductive semi-reflexivity and bounded sets being precompact, is left open. However, we have a partial negative answer in Corollary 3.2 and the example in Section 5.

This example is of some standard interest. It shows that inductive semi-reflexivity is strictly stronger than semi-reflexivity and completeness (without the use of measurable cardinals.). Also of standard interest is the result that a DF space is a Schwartz space, if and only if, bounded sets are precompact (Corollary 4.3). This improves a result of Terzioglu [**12**]. The proofs of Corollary 4.3 and the preceeding Proposition 4.2 use no nonstandard analysis.

The first two sections are of a preliminary nature. Section 1 contains standard definitions, while Section 2 has the basics of the nonstandard analysis we need.

1. Preliminaries. By a TVS (E, ξ), we will always mean a vector space E, over the real or complex field, with a locally convex Hausdorff vector space topology ξ. The continuous (algebraic) dual of (E, ξ) will be denoted $E'(E^{\#})$. We will use $\sigma(E, E')(\beta(E, E'))$ for the weak (strong) topology on E given by E'.

The TVS (E, ξ) is *quasi-barrelled* (*σ-quasi-barrelled*) if every bounded subset (bounded sequence) of $(E', \beta(E', E))$ is ξ-equicontinuous. We note that (E, ξ) is σ-quasi-barrelled if every weakly separable bounded subset of $(E', \beta(E', E))$ is ξ-equicontinuous.

An *F space* is a Fréchet space (i.e. a complete metrizable TVS). A *DF space* is a TVS with a fundamental sequence of bounded sets and which satisfies a

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condition between quasi-barrelled and σ-quasi-barrelled. This condition is that every strongly bounded subset of E', which is a countable union of equicontinuous sets, is equicontinuous. The strong dual of a σ-quasi-barrelled space with a fundamental sequence of bounded sets is an F space [**8**, p. 11].

An M *space* is a Montel space (i.e. a quasi-barrelled TVS in which bounded sets are relatively compact). An S *space* is a Schwartz space (see [**6** or **12**]).

If $f \in E'^{\#}$ is in the canonical image of E, we will say f is *already in* E. Thus (E, ξ) is semi-reflexive if every $f \in (E', \beta(E', E))'$ is already in E. A TVS (E, ξ) is *inductively semi-reflexive* (Berezanskii [**2**]) if every $f \in E'^{\#}$, which is bounded on ξ-equicontinuous sets, is already in E. This property was called HC in [**1**].

A filter $\mathscr{F}$ on E, is an *almost Cauchy filter*, if for every neighborhood of the origin U, there is an integer n, such that $nU \in \mathscr{F}$. This condition appears as (*) in Theorem 4.1 of [**3**, p. 416].

2. Nonstandard hulls. We will use nonstandard analysis as developed in [**10**] or [**11**]. All our nonstandard models will be enlargements. The reference for this section is Henson and Moore [**3**].

Let (E, ξ) be a TVS, let P be the set of ξ-continuous semi-norms on E and let $\mathscr{U}$ be a ξ-neighborhood basis of the origin in E. In the nonstandard TVS $({}^*E, {}^*\xi)$ we identify certain subsets as follows:

$$\text{fin}_\xi = \{x \in {}^*E : {}^*\rho(x) \text{ is finite for each } \rho \in P\}$$
$$\mu_\xi = \{x \in {}^*E : {}^*\rho(x) \text{ is infinitesimal for each } \rho \in P\}$$
$$\text{pns}_\xi = \{x \in {}^*E : x \in E + {}^*U \text{ for each } U \in \mathscr{U}\}.$$

For a filter $\mathscr{F}$ on E we define $\mu(\mathscr{F}) = \bigcap {}^*F (F \in \mathscr{F})$; note that $\mu(\mathscr{U}) = \mu_\xi$. A filter $\mathscr{F}$ converges to $x \in E$, if and only if, $\mu(\mathscr{F}) \subset x + \mu_\xi$. A filter $\mathscr{F}$ is an almost Cauchy filter, if and only if, $\mu(\mathscr{F}) \subset \text{fin}_\xi$. If $\mathscr{F}$ is a Cauchy filter, then $\mu(\mathscr{F}) \subset \text{pns}_\xi$.

$\hat{E}$ is defined to be the set of equivalence classes of fin_ξ modulo μ_ξ (i.e. $x \sim y$ if and only if $x - y \in \mu_\xi$). Hence there is a quotient map $\phi : \text{fin}_\xi \to \hat{E}$. For each $X \subset {}^*E$, we define $\hat{X} = \phi(X \cap \text{fin}_\xi)$. Let $\hat{\xi} = \{{}^*\hat{U} : U \in \xi\}$, which is just the quotient topology on $\hat{E}$, if fin_ξ is given the topology $\eta = \{{}^*U : U \in \xi\}$. The standard TVS $(\hat{E}, \hat{\xi})$ is a *nonstandard hull* of (E, ξ).

In general, $(\hat{E}, \hat{\xi})$ varies with the choice of the nonstandard model. If $(\hat{E}, \hat{\xi})$ is independent of the model, we say (E, ξ) has *invariant nonstandard hulls*. This will happen if and only if, (E, ξ) satisfies one of the following equivalent conditions [**3**, pp. 416–417]:

(1) $\text{fin}_\xi = \text{pns}_\xi$.

(2) $(\hat{E}, \hat{\xi})$ is the completion of (E, ξ).

(3) Every almost Cauchy ultrafilter is Cauchy.

For $f \in \text{fin}_\xi$, let ${}^0f$ be the linear functional on E' given by ${}^0f(e') =$ standard part of the *scalar $\langle f, e' \rangle$. We need the following theorem in the next section. A proof is given in [**4**, Theorem 6, p. 204].

THEOREM 2.1 (Henson and Moore). *Let* (E, ξ) *be a* TVS, *then* $\{{}^0f : f \in \text{fin}_\xi\} = \{f \in E'^{\#} : f$ *is bounded on* ξ*-equicontinuous sets of* E'.$\}$

Finally, let $\perp_{E'}$ be the subspace of $\hat{E}$ given by $\perp_{E'} = \phi(\mu_{\sigma(E,E')} \cap \text{fin}_\xi)$. Equivalently, $\perp_{E'} = \{f \in \text{fin}_\xi : {}^0f = 0\}$. We note that E and $\perp_{E'}$ are subspaces of $\hat{E}$ with $E \cap \perp_{E'} = \{0\}$.

3. Splitting nonstandard hulls. Those TVS's (E, ξ), for which $\hat{E} = E \oplus \perp_{E'}$, are characterized in this section. This is a strong semi-reflexive and completeness condition, equivalent to being inductive semi-reflexive. Also, for complete spaces (E, ξ), $\hat{E} = E \oplus \perp_{E'}$ is a necessary condition for (E, ξ) to have invariant nonstandard hulls.

THEOREM 3.1. *For a* TVS (E, ξ) *the following are equivalent*:

(1) $\hat{E} = E \oplus \perp_{E'}$, *algebraically.*

(2) $\hat{E} = E \oplus \perp_{E'}$, *topologically.*

(3) *Every almost Cauchy ultrafilter is* $\sigma(E, E')$ *convergent.*

(4) (E, ξ) *is inductive semi-reflexive* (*or any of the other equivalent conditions of Theorem* 4.1 *of* [**1**]).

Proof. We will show (1) $\Rightarrow$ (4) $\Rightarrow$ (2) and (1) $\Leftrightarrow$ (3). The implication (2) $\Rightarrow$ (1) is formal.

(1) $\Rightarrow$ (4). Let $f \in E'^{\#}$ which is bounded on ξ-equicontinuous sets. By Theorem 2.1, there is a $g \in \text{fin}_\xi$ that agrees with f, up to an infinitesimal, on standard points of E'. By hypothesis, there exists $x \in E$ that agrees with g, up to an infinitesimal, on standard points of E'. Hence $x = f$, and (E, ξ) is inductive semi-reflexive.

(4) $\Rightarrow$ (2). If $f \in \text{fin}_\xi$, then by Theorem 2.1, ${}^0f$ is bounded on ξ-equicontinuous sets and by hypothesis ${}^0f \in E$. Define a projection $P : \hat{E} \to E$ by $P(f) = {}^0f$. It is easy to see that P is well defined since $\mu_\xi \subset \mu_{\sigma(E,E')}$. If U is a $\sigma(E, E')$ closed absolute convex ξ-neighborhood of the origin, and if $f \in {}^*U$, then $f \in {}^*U^{00} = {}^*U$. So ${}^0f \in U$, and hence P is continuous and $\hat{E} = E \oplus \perp_{E'}$, topologically [**9**, p. 95].

(1) $\Rightarrow$ (3). Let $\mathscr{F}$ be an almost Cauchy ultrafilter, hence $\mu(\mathscr{F}) \subset \text{fin}_\xi$. By hypothesis there exists an $x \in E$ such that $\mu(\mathscr{F})$ meets $x + \perp_{E'}$ which is in turn contained in $x + \mu_{\sigma(E,E')}$. Since $\mathscr{F}$ is an ultrafilter, this implies that $\mu(\mathscr{F}) \subset x + \mu_{\sigma(E,E')}$ and therefore $\mathscr{F}$ converges $\sigma(E, E')$ to x.

(3) $\Rightarrow$ (1). It suffices to show that for all $x \in \text{fin}_\xi$ there is a $y \in E$ such that $x \in y + \mu_{\sigma(E,E')}$. For then $x \in (y + \mu_{\sigma(E,E')}) \cap \text{fin}_\xi = y + (\mu_{\sigma(E,E')} \cap \text{fin}_\xi) = y + \perp_{E'}$. Let $x \in \text{fin}_\xi$, then there is an almost Cauchy ultrafilter $\mathscr{F}$ such that $x \in \mu(\mathscr{F})$. By hypothesis, $\mathscr{F}$ converges weakly to some $y \in E$. That is, $x \in \mu(\mathscr{F}) \subset y + \mu_{\sigma(E,E')}$. The proof is complete.

COROLLARY 3.2. *A complete* TVS (E, ξ) *has invariant nonstandard hulls, if and only if*, (E, ξ) *is inductive semi-reflexive and*

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(*) *every almost Cauchy ultrafilter which is* $\sigma(E, E')$ *convergent is also* ξ *convergent.*

The corollary gives another characterization of spaces with invariant nonstandard hulls. (The restriction to complete spaces is minor, since a TVS has invariant nonstandard hulls, if and only if, its completion does [**3**, p. 419].) However, the condition (*) is hard to get a hold on. It is easy to show that (*) implies that bounded sets are precompact for semi-reflexive spaces (use [**9**, Proposition 6, p. 50]). The converse is false as the example in Section 5 shows. The corresponding condition (3) of Theorem 3.1 is a new standard characterization of inductive semi-reflexivity.

4. *F* and *DF* spaces.

Theorem 4.4 exposes the self-duality of possessing invariant nonstandard hulls for F and DF spaces. In particular, we show a DF space has invariant nonstandard hulls, if and only if, bounded sets are precompact. Along the way, we give a standard proof of Proposition 4.2, which is of standard interest in itself (see Corollary 4.3). We need the following result [**12**, § 4, (2), p. 240].

THEOREM 4.1. (Terzioglu). *The strong dual of an FM space is an S space.*

PROPOSITION 4.2. *If the* TVS (E, ξ) *is* σ*-quasi-barrelled, has a fundamental sequence of bounded sets and bounded sets are precompact, then* (E, ξ) *is a quasi-barrelled DFS space and* $(E', \beta(E', E))$ *is an FM space.*

Proof. First we show that in $(E', \beta(E', E))$ bounded sets are precompact. Suppose not and let B be a bounded set which is not precompact in $(E', \beta(E', E))$. There exists a $\beta(E', E)$-neighborhood of the origin U and a sequence $(x_n) \subset B$ such that, $n \neq m$ implies $x_n - x_m \notin U$. The sequence (x_n) is not strongly precompact, but is strongly bounded and hence is ξ-equicontinuous. This is impossible, since on the ξ-equicontinuous sets (which are relatively weakly compact) the weak and strong topologies agree [**7**, § 21, 6.(3), p. 264]. We have that $(E', \beta(E', E))$ is an FM space.

Next we show that (E, ξ) is quasi-barrelled, hence a DF-space. Since an FM space is separable [**7**, § 27, 2.(5), p. 370], every strongly bounded subset of E' is separable. Thus, by the σ-quasi-barrelledness of (E, ξ), every strongly bounded set is ξ-equicontinious.

Now, since (E, ξ) is quasi-barrelled, the canonical injection of (E, ξ) into $(E'', \beta(E'', E')$ is a homeomorphism. By Theorem 4.1, $(E'', \beta(E'', E'))$ is an S space. As a subspace of an S space, (E, ξ) is an S space [**6**, pp. 278–279].

COROLLARY 4.3. *A DF space is an S space, if and only if, bounded sets are precompact.*

The corollary improves a result of Terzioglu [**12**, § 4, (8), p. 241].

THEOREM 4.4. *If* (E, ξ) *is an F space or a DF space, then the following are equivalent:*

(1) *(E, ξ) has invariant nonstandard hulls.*
(2) *Bounded sets are precompact in (E, ξ).*
(3) *$(E', \beta(E', E))$ has invariant nonstandard hulls.*
(4) *The completion of (E, ξ) is an M space.*

Before proving Theorem 4.4 some remarks on the work of Henson and Moore are in order. In [**4**], they show the equivalence of (1), (2) and (4) for F spaces and that these imply (3). In [**3**], they show that for any TVS, (1) implies (2). Finally, we shall need their Theorem 4 of [**4**] which we state as:
(*) *An S space has invariant nonstandard hulls.*
For a quick proof of (4) $\Rightarrow$ (3) for F spaces, combine Theorem 4.1 and (*).

Proof. First we complete the proof of Theorem 4.4 for F spaces (i.e. (3) $\Rightarrow$ (4)). By [**7**, § 28, 5.(1), p. 385], $(E', \beta(E', E))$ is a complete DF space. From this and the hypothesis, it follows that bounded sets are relatively compact in $(E', \beta(E', E))$. $(E', \beta(E', E))$ is quasi-barrelled by Proposition 4.2, hence it is an M space. Thus the strong bidual of (E, ξ) is an FM space [**7**, § 27, 2.(2), p. 269]. And finally, by [**7**, § 29, 2.(5), p. 396], (E, ξ) is reflexive and hence an FM space itself.

Now let (E, ξ) be a DF space.

(2) $\Rightarrow$ (3): From Proposition 4.2, we have $(E', \beta(E', E))$ is an FM space. Now (3) follows from the theorem for F spaces.

(3) $\Rightarrow$ (4): Bounded sets are precompact in $(E, \beta(E', E))$ by the theorem for F spaces. Thus $(E', \beta(E', E))$ is an FM space. As in the proof of Proposition 4.2, we have (E, ξ) is quasi-barrelled. Therefore, (E, ξ) is a subspace of the complete DFS space $(E'', \beta(E'', E'))$ by Theorem 4.1. So bounded subsets of the completion of (E, ξ) are relatively compact. Since quasi-barrelledness is preserved by completion [**7**, § 27, 1.(2), p. 368], the completion of (E, ξ) is an M space.

(4) $\Rightarrow$ (1): (E, ξ) is a subspace of its completion, which is a DFS space by Proposition 4.2 and the reflexivity of M spaces. Thus (E, ξ) is an S space [**6**, pp. 278–279]. And so by (*) we have (1). This completes the proof of Theorem 4.4.

5. Example. The example is borrowed from [**4**], which is an example of a complete semi-reflexive space which is not inductive semi-reflexive. This shows that $\hat{E} = E \oplus \perp_{E'}$ is strictly stronger than semi-reflexivity and completeness.

The example. A complete TVS whose bounded sets are relatively compact but is not inductive semi-reflexive.

Construction. Let N be the set of natural numbers and let X be the set or real valued functions on N with finite support. Let $\mathscr{U}$ be a free ultra-filter on N. A function $\theta : N \rightarrow R$ is *admissible*, if there exists a $M \in \mathscr{U}$, such that, θ is bounded on M. For each admissible θ, let ρ_θ be the semi-norm on X defined

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by $\rho_\theta(x) = \sum|\theta(n)|\,|x(n)|$. Let ξ be the topology on X generated by the set of semi-norms $\{\rho_\theta : \theta \text{ admissible}\}$. Finally, for $n \in N$, let $e_n \in X$ be the function that is one at n and zero otherwise.

The space (X, ξ) is an example of Henson and Moore [**4**, pp. 196–197]. They have shown that bounded sets of (X, ξ) are finite dimensional, hence relatively compact. Thus (X, ξ) is semi-reflexive. Furthermore, it was shown that, for $n \in \mu(\mathscr{U})$, $e_n \in \text{fin}_\xi \backslash \text{pns}_\xi$. So (X, ξ) does not have invariant nonstandard hulls. They also have shown that $X' = \{f : N \to R : f \text{ is admissible}\}$.

Let's show that (X, ξ) is not inductive semi-reflexive. For the sake of the argument, suppose that there is an even integer $n \in \mu(\mathscr{U})$. If (X, ξ) were inductive semi-reflexive, then by Theorem 3.1, the linear functional F on X', given by $F(X') = {}^0\langle x', e_n\rangle$, is already in X. Let $x \in X$ and let m be an integer, such that, $k \geqq m$ implies $x(k) = 0$. Let $f \in X'$ be the function which is one on even integers past m and zero otherwise. Now $F(f) = {}^0\langle f, e_n\rangle = 1$ and $f(x) = 0$. Therefore F cannot be in X, and so (X, ξ) is not inductive semi-reflexive.

To show that (X, ξ) is complete, let $\mathscr{F}$ be a ξ-Cauchy filter on X. Clearly, $\mathscr{F}$ converges pointwise to some function y on N. Suppose $y \notin X$, then the set $A = \{n : y(n) \neq 0\}$ is infinite. We can write A as the disjoint union of two infinite sets, and one of them, say B, does not belong to $\mathscr{U}$. Define $\theta : N \to R$ by $\theta(n) = 2|y^{-1}(n)|$, if $n \in B$, and zero otherwise. θ is admissible since $N \backslash B \in \mathscr{U}$. Let $U = \{x \in X : \rho_\theta(x) \leqq 1\}$. There exists a sequence of sets $(F_n) \subset \mathscr{F}$ such that:

(1) $F_n - F_n \subset U$.

(2) $x \in F_n$ implies $|x(i) - y(i)| < n^{-1}$, $i = 1, 2, \ldots, n$.

(3) $F_n \subset F_{n+1}$, for $n = 1, 2, \ldots$.

Let $m \in B$ and $x \in F_1$. Now for each $k \in N$, we have $F_1 - F_{m+k} \subset F_1 - F_1 \subset U$. So for $z \in F_{m+k}$, $|x(m) - z(m)| \leqq 2^{-1}|y(m)|$ and $|z(m) - y(m)| < (m + k)^{-1}$. By choosing k large enough, we have $|x(m) - y(m)| < |y(m)|$ or that $x(m) \neq 0$ for m in the infinite set B. This contradiction shows that $y \in X$.

Let θ be any admissible function and let $U = \{x \in X : \rho_\theta(x) \leqq 1\}$. Let $F \in \mathscr{F}$, such that $F - F \subset 2^{-1}U$. Let $z \in F$ and let n be the largest integer such that $z(n) \neq 0$. For any $x \in F$, $\sum_{n+1}^{\infty}|\theta(i)|\,|x(i)| \leqq \rho_\theta(z - x) \leqq 2^{-1}$. Let m be the largest integer such that $y(m) \neq 0$, and let $q = \max(n, m)$. Since $\mathscr{F}$ converges pointwise to y, for each $k \in N$, there is a $G_k \in \mathscr{F}$, such that $x \in G_k$ implies $|x(i) - y(i)| < k^{-1}$, for $i = 1, 2, \ldots, q$. For large enough k, we have $x \in G_k$ implies $\sum_1^q|x(i) - y(i)|\,|\theta(i)| < 2^{-1}$. Thus for $x \in G_k \cap F$, $\rho_\theta(x - y) = \sum_1^q|x(i) - y(i)|\,|\theta(i)| + \sum_{q+1}^{\infty}|x(i)|\,|\theta(i)| \leqq 2^{-1} + 2^{-1}$. Therefore $y + U \in \mathscr{F}$ and $\mathscr{F}$ ξ-converges to y.

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