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SHARP HEAT-KERNEL ESTIMATES FOR HIGHER-ORDER OPERATORS WITH SINGULAR COEFFICIENTS

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Abstract We obtain heat-kernel estimates for higher-order operators with measurable coefficients that can be singular or degenerate. Precise constants are given, which are sharp for small times.

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1. Introduction

Let

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} D^\alpha\{a_{\alpha\beta}(x) D^\beta f(x)\}, \quad x \in \Omega \subset \mathbb{R}^N, \tag{1.1}$$

be a self-adjoint uniformly elliptic operator of order $2m$ with measurable coefficients and subject to Dirichlet boundary conditions on $\partial\Omega$. It is known that if $2m > N$, then the associated heat semigroup e^{-Ht} has a kernel $K(t, x, y)$ which satisfies the estimate

$$|K(t,x,y)| < c_1 t^{-N/2m} \exp\biggl\{-c_2 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t\biggr\}$$

for some positive constants c_i. Under suitable conditions this was recently [**4**] sharpened to

$$|K(t,x,y)| < c_\epsilon t^{-N/2m} \exp\biggl\{-(\sigma_m - cD - \epsilon) \frac{d_M(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_{\epsilon,M} t\biggr\}, \tag{1.2}$$

where $\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m-2))$, $D \geqslant 0$, depends on the regularity of the coefficients and $d_M(x, y)$ is a Finsler-type metric that is induced by the principal symbol of H and depends on the arbitrarily large parameter M; as $M \to \infty$, $d_M(x, y)$ increases to a Finsler distance $d(x, y)$, but (1.2) is valid only for $M < \infty$. This estimate is sharp, as is seen by comparison with the small-time asymptotics for operators with smooth coefficients obtained in [**10**] (see (2.12) below). In the same direction, Dungey [**8**] used resolvent estimates to obtain a better estimate than (1.2) for powers of second-order operators. He showed in a general framework that if the self-adjoint operator H satisfies a standard Gaussian estimate with exponential constant $\frac{1}{4}-\epsilon$, then the heat kernel of H^m satisfies (1.2) with $D=0$ and $M=+\infty$. For an alternative approach valid also for higher-order systems see [**1**]. For a comprehensive review of recent results on the spectral theory of higher-order operators with measurable coefficients see [**7**].

Dungey's result also applies to operators with singular or degenerate coefficients, but it does not apply when the operator is not the power of a second-order operator. A sharp heat-kernel estimate for operators of the form (1.1) with singular and/or degenerate coefficients is the main result of this paper. At the same time, for the sake of greater generality, we do not assume that H is self-adjoint.

Concerning the singularity or degeneracy of H, we assume that there is a positive function $a(x)$ that controls in a suitable sense the behaviour of the coefficient matrix $\{a_{\alpha\beta}\}$ and we then impose two conditions (H1) and (H2) on $a(x)$. The first is a weighted Sobolev inequality and the second is a weighted interpolation inequality. These conditions were introduced in [**3**] and led to (non-sharp) off-diagonal estimates on the heat kernel of non-uniformly elliptic self-adjoint operators. Besides conditions (H1) and (H2) we shall assume that the symbol $A(x,\xi)$ is close—in a suitable sense—to a certain class of 'good' symbols denoted by $\mathcal{G}_a$. These symbols, besides satisfying (H1) and (H2), correspond to operators that are self-adjoint, their coefficients have some local regularity, and they are strongly convex in the sense of [**9**]. We make use of a certain stability property inherent in our approach and obtain bounds that are asymptotically sharp: they involve the exponential constant $\sigma_m - cD$, where c is an absolute constant and D is the distance of the symbol $A(x,\xi)$ from the class $\mathcal{G}_a$ in a certain weighted norm. In particular, the constant σ_m is obtained for symbols in $\mathcal{G}_a$. To the best of our knowledge such estimates are new even if the coefficients are assumed to be smooth and the symbol lies in $\mathcal{G}_a$.

2. Formulation of results

We first fix some notation. Given a multi-index $\alpha=(\alpha_1,\dots,\alpha_N)$ we write $\alpha! = \alpha_1!\cdots\alpha_N!$ and $|\alpha|=\alpha_1+\cdots+\alpha_N$. We write $\gamma\leqslant\alpha$ to indicate that $\gamma_i\leqslant\alpha_i$ for all i, in which case we also set $c_\gamma^\alpha=\alpha!/\gamma!(\alpha-\gamma)!$. We use the standard notation D^α for the differential expression $(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_N)^{\alpha_N}$, and for $k\geqslant 0$ we denote by $\nabla^k f$ the vector $(\mathrm{D}^\alpha f)_{|\alpha|=k}$. We denote by $\hat{f}$ the Fourier transform of a function f, $\hat{f}(\xi)=(2\pi)^{-N/2}\int \mathrm{e}^{\mathrm{i}\xi\cdot x}f(x)\,\mathrm{d}x$. We shall denote by $\|A\|_{p\to q}$ the norm of an operator A from $L^p(\Omega)$ to $L^q(\Omega)$. The letter c will stand for a positive constant whose value may change from line to line.

Let Ω be a domain in $\mathbb{R}^N$. We fix an integer $m\geqslant 1$ and consider the operator

$$Hf(x) = (-1)^m \sum_{\substack{|\alpha|=m\\|\beta|=m}} D^\alpha\{a_{\alpha\beta}(x)D^\beta f(x)\} \tag{2.1}$$

subject to Dirichlet boundary conditions on $\partial\Omega$; the precise definition shall be given below. The matrix-valued function $\{a_{\alpha\beta}\}$ is assumed to be measurable and to take its values in the set of all complex, $\nu \times \nu$ matrices, ν being the number of multi-indices α of length $|\alpha| = m$. We assume that each $a_{\alpha\beta}$ lies in $L^\infty_{\mathrm{loc}}(\Omega)$; we do not assume $\{a_{\alpha\beta}\}$ to be self-adjoint.

We define a quadratic form $Q(\cdot)$ on $C_c^\infty(\Omega)$ by

$$Q(f) = \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x) D^\alpha f(x) D^\beta \bar{f}(x)\,\mathrm{d}x, \quad f \in C_c^\infty(\Omega).$$

We assume that there exists a positive weight $a(x)$ with $a^{\pm 1} \in L^\infty_{\mathrm{loc}}(\Omega)$ that controls the size of the matrix $\{a_{\alpha\beta}\}$ in the following sense: first,

$$|a_{\alpha\beta}(x)| \leqslant ca(x), \quad x \in \Omega, \tag{2.2}$$

for all multi-indices α, β; and second, the weighted Gårding inequality

$$\operatorname{Re} Q(f) \geqslant c \int_\Omega a(x) |\nabla^m f|^2\,\mathrm{d}x, \quad f \in C_c^\infty(\Omega), \tag{2.3}$$

is valid for some $c > 0$. We also assume the symbol version of (2.3), namely

$$\operatorname{Re} A(x,\xi) \geqslant ca(x)|\xi|^{2m}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N, \tag{2.4}$$

where $A(x,\xi) := \sum a_{\alpha\beta}(x)\xi^{\alpha+\beta}$. Relations (2.2) and (2.3) imply in particular that there exists $\beta > 0$ such that

$$|Q(f)| \leqslant \beta \operatorname{Re} Q(f), \quad f \in C_c^\infty(\Omega). \tag{2.5}$$

It is easily seen that Q is closable [**3**]. The domain of its closure is a weighted Sobolev space that we denote by $W^{m,2}_{a,0}(\Omega)$. We retain the same symbol, Q, for the closure of the above form and denote by H the associated accretive operator on $L^2(\Omega)$, so that $\langle Hf, f\rangle = Q(f)$, $f \in \mathrm{Dom}(H)$, and (2.1) is valid in a weak sense.

We make two hypotheses on the weight a: the first is a weighted Sobolev inequality and the second is a weighted interpolation inequality.

(H1) There exists $s \in [N/2m, 1]$ and $c > 0$ such that

$$\|f\|_\infty \leqslant c[\operatorname{Re} Q(f)]^{s/2} \|f\|_2^{1-s}, \quad f \in C_c^\infty(\Omega). \tag{2.6}$$

(H2) There exists a constant c such that

$$\int_\Omega a^{k/m} |\nabla^k f|^2\,\mathrm{d}x < \epsilon \int_\Omega a|\nabla^m f|^2\,\mathrm{d}x + c\epsilon^{-k/(m-k)} \int_\Omega |f|^2\,\mathrm{d}x, \tag{2.7}$$

for all $0 < \epsilon < 1$, $0 \leqslant k < m$ and all $f \in C_c^\infty(\Omega)$.

Both (H1) and (H2) are satisfied when H is uniformly elliptic, in which case the best value for the constant s is $s = N/2m$, showing that in the general case we cannot expect any value that is better (smaller) than $N/2m$; in particular, (H1) is valid trivially with

$s = N/2m$ if $a(x)$ is bounded away from zero. We refer to [**3**] for non-trivial examples for which (H1) and (H2) are satisfied; they involve suitable powers of either $1 + |x|$ or $\operatorname{dist}(x, K)$, where K is a smooth surface of lower dimension.

We note that condition (H2) implies that for any k, l with $0 \leqslant k, l \leqslant m$, $k + l < 2m$, there exists a constant c such that

$$(1+\lambda^{2m-k-l}) \int_{\Omega} a^{(k+l)/2m} |\nabla^k f|\, |\nabla^l f| \,\mathrm{d}x < \epsilon \operatorname{Re} Q(f) + c\epsilon^{-(k+l)/(2m-k-l)}(1+\lambda^{2m})\|f\|_2^2, \tag{2.8}$$

for all $\epsilon \in (0,1)$, $\lambda > 0$ and all $f \in C_c^\infty(\Omega)$. Indeed, for $\lambda = 1$, (2.8) is a consequence of (H2) and the Cauchy–Schwarz inequality; the case $\lambda < 1$ follows trivially from the case $\lambda = 1$; finally, writing (2.8) for $\lambda = 1$ and replacing ϵ by $\epsilon\lambda^{k+l-2m}$ we obtain the result for $\lambda > 1$.

Next we introduce the distance that shall be used in the heat-kernel estimates. Consider the set

$$\mathcal{E}_a = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \phi \text{ real valued and } a^{k/2m}\nabla^k \phi \in L^\infty(\Omega),\ 1 \leqslant k \leqslant m\}$$

and its subset (recall (2.4))

$$\begin{aligned}\mathcal{E}_{A,M} = \{\phi \in C^\infty(\Omega) \cap L^\infty(\Omega) : \operatorname{Re} A(x, \nabla\phi(x)) \leqslant 1,& \\ |\nabla^k \phi(x)| \leqslant M a(x)^{-k/2m},\ 2 \leqslant k \leqslant m,\ \text{a.e. } x \in \Omega\}.&\end{aligned} \tag{2.9}$$

Our estimates will be expressed in terms of the distance

$$d_M(x,y) = \sup\{\phi(y) - \phi(x) : \phi \in \mathcal{E}_{A,M}\} \tag{2.10}$$

for arbitrarily large but finite M. For $M = +\infty$ this reduces to the distance

$$d_\infty(x,y) = \sup\{\phi(y) - \phi(x) : \operatorname{Re} A(x, \nabla\phi(x)) \leqslant 1,\ x \in \Omega\}.$$

This is a Finsler distance, induced by the (singular/degenerate) Finsler metric with length element

$$\mathrm{d}s = \mathrm{d}s(x, \mathrm{d}x) = \sup_{\substack{\eta \in \mathbb{R}^N \\ \eta \neq 0}} \frac{\langle \mathrm{d}x, \eta\rangle}{(\operatorname{Re} A(x,\eta))^{m/2}}. \tag{2.11}$$

We refer the reader to the recent book [**2**] for a comprehensive introduction to Finsler geometry. The distance $d_\infty(x,y)$ relates to the short-time off-diagonal behaviour of the heat kernel: it was shown in [**10**] that if $\Omega = \mathbb{R}^N$ and H is self-adjoint, uniformly elliptic with strongly convex symbol (see (2.13)), then $d_\infty(\cdot,\cdot)$ controls the small-time behaviour of $K(t,x,y)$ in the sense that

$$\log t^{N/2m} K(t,x,y) = -\sigma_m \frac{d_\infty(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}}(1 + o(1)), \quad \text{as } t \to 0, \tag{2.12}$$

for x, y fixed and close enough; here and below we have

$$\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\pi/(4m-2)).$$

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Let us now proceed with the definition of the class $\mathcal{G}_a$. Let the functions $a_\gamma(\cdot)$, $|\gamma| = 2m$, be defined by requiring that

$$\sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)\xi^{\alpha+\beta} = \sum_{|\gamma|=2m} c_\gamma^{2m} a_\gamma(x)\xi^\gamma, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N$$

(recall that $c_\gamma^{2m} = (2m)!/\gamma!$). Following [**9**] we say that the principal symbol $A(x,\xi)$ of H is strongly convex if the quadratic form

$$\Gamma(x,p) = \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha+\beta}(x)p_\alpha\bar{p}_\beta, \quad p = (p_\alpha) \in \mathbb{C}^\nu, \tag{2.13}$$

is positive semidefinite for a.e. $x \in \Omega$.

Induced by the weight $a(x)$ is the weighted Sobolev space

$$W_a^{m-1,\infty}(\Omega) = \{f \in W_{\mathrm{loc}}^{m-1,\infty}(\Omega) : |\nabla^i f(x)| \leqslant ca(x)^{(2m-i)/2m}, \text{ a.e. } x \in \Omega, \ i \leqslant m-1\}. \tag{2.14}$$

Definition 2.1. We say that the symbol $A(x,\xi)$ lies in $\mathcal{G}_a$ if

(i) $A(x,\xi)$ is strongly convex;

(ii) $\{a_{\alpha\beta}\}$ is real and symmetric;

(iii) the coefficients $a_{\alpha\beta}$ lie in $W_a^{m-1,\infty}(\Omega)$.

We denote by D the distance of the coefficient matrix $\{a_{\alpha\beta}\}$ from $\mathcal{G}_a$ in the weighted uniform norm

$$\|f\|_{a,\infty} := \sup_{x\in\Omega} |f(x)/a(x)|;$$

that is

$$D = \inf_{\{\tilde{a}_{\alpha\beta}\}} \|\{a_{\alpha\beta}\} - \{\tilde{a}_{\alpha\beta}\}\|_{a,\infty}, \tag{2.15}$$

where the infimum is taken over all matrix-valued functions $\{\tilde{a}_{\alpha\beta}\}$ that induce a symbol in $\mathcal{G}_a$. Here we have used the notation

$$\|\{b_{\alpha\beta}\}\|_{a,\infty} := \sup_{x\in\Omega} \frac{|\{b_{\alpha\beta}(x)\}|}{a(x)},$$

where, for each $x \in \Omega$, $|\{b_{\alpha\beta}(x)\}|$ denotes the norm of $\{b_{\alpha\beta}(x)\}$ regarded as an operator on $\mathbb{C}^\nu$.

Our main result is as follows.

Theorem 2.2. *Assume that (H1) and (H2) are satisfied. Then for all* $\delta \in (0,1)$ *and all* M *large there exist positive constants* c_δ, $c_{\delta,M}$ *such that*

$$|K(t,x,y)| < c_\delta t^{-s} \exp\{-(\sigma_m - cD - \delta)d_M(x,y)^{2m/(2m-1)}t^{-1/(2m-1)} + c_{\delta,M}t\} \tag{2.16}$$

for all $x, y \in \Omega$ *and* $t > 0$*; the constant* c *is independent of* x, y, t, δ, D *and* M.

In the special case where H is uniformly elliptic and self-adjoint this estimate has already been obtained in [**4**].

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3. Proof of Theorem 2.2

Given $\phi \in \mathcal{E}_a$, the mapping $f \mapsto \mathrm{e}^{\phi} f$ maps $W^{m,2}_{a,0}(\Omega)$ into itself [**3**, Lemma 7]. Hence one can define a sesquilinear form $Q_\phi(\cdot,\cdot)$ with domain $W^{m,2}_{a,0}(\Omega)$ by

$$\begin{aligned} Q_\phi(f) &= Q(\mathrm{e}^{\phi} f, \mathrm{e}^{-\phi} f) \\ &= \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} D^\alpha(\mathrm{e}^{\phi} f) D^\beta(\mathrm{e}^{-\phi}\bar{f})\,\mathrm{d}x, \quad f \in W^{m,2}_{a,0}(\Omega). \end{aligned} \tag{3.1}$$

The associated operator is $H_\phi = \mathrm{e}^{-\phi} H \mathrm{e}^{\phi}$ and has domain $\mathrm{Dom}(H_\phi) = \mathrm{e}^{-\phi}\,\mathrm{Dom}(H)$. The form Q_ϕ is a lower-order perturbation of Q (cf. (3.8)) and it is a consequence of (H2) [**3**, Lemma 8] that for all $\epsilon > 0$ and $f \in W^{m,2}_{a,0}(\Omega)$, the following inequality holds:

$$|Q(f) - Q_\phi(f)| < \epsilon \operatorname{Re} Q(f) + c\epsilon^{-2m+1}(1+p(\phi))^{2m}\|f\|_2^2, \tag{3.2}$$

where we have used the seminorm

$$p(\phi) := \sup_{1\leqslant k\leqslant m} \operatorname*{ess\,sup}_{x\in\Omega} a(x)^{k/2m}|\nabla^k\phi(x)|. \tag{3.3}$$

Defining $s(\phi) = (1+p(\phi))^{2m}$, it follows in particular that

$$\operatorname{Re} Q_\phi(f) \geqslant -cs(\phi)\|f\|_2^2, \quad f \in C_c^\infty(\Omega), \tag{3.4}$$

where c is independent of ϕ, and this justifies the definition

$$-k_\phi = \inf\{\operatorname{Re} Q_\phi(f) : f \in C_c^\infty(\Omega),\ \|f\|_2 = 1\}. \tag{3.5}$$

The next lemma closely follows an argument used in [**5**].

Lemma 3.1. *Assume that (H2) is satisfied. Then for any $\phi \in \mathcal{E}_a$, the following inequalities hold:*

(i) $\|\mathrm{e}^{-H_\phi t}\|_{2\to 2} \leqslant \mathrm{e}^{k_\phi t}$,

(ii) $\|H_\phi \mathrm{e}^{-H_\phi t}\|_{2\to 2} \leqslant (c_\delta/t)\mathrm{e}^{k_\phi t}\mathrm{e}^{\delta s(\phi) t}$, *for all $\delta > 0$,*

where the constant c_δ is independent of $\phi \in \mathcal{E}_a$ and $t > 0$.

Proof. Part (i) is the standard energy estimate that follows by integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{e}^{-H_\phi t} f\|_2^2 = -2\operatorname{Re}\langle H_\phi \mathrm{e}^{-H_\phi t} f, \mathrm{e}^{-H_\phi t} f\rangle \leqslant 2k_\phi \|\mathrm{e}^{-H_\phi t} f\|_2^2.$$

Now by (3.2) the following inequality holds:

$$|Q_\phi(f) - Q(f)| \leqslant \tfrac{1}{2}\operatorname{Re} Q(f) + c' s(\phi)\|f\|_2^2, \quad f \in C_c^\infty(\Omega), \tag{3.6}$$

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where $c' > 0$ depends only on m. Hence, for any $\epsilon \in (0,1)$,

$$\begin{aligned}\operatorname{Re} Q_\phi(f) &= \epsilon \operatorname{Re} Q_\phi(f) + (1-\epsilon) \operatorname{Re} Q_\phi(f) \\ &\geqslant \tfrac{1}{2}\epsilon \operatorname{Re} Q(f) - [c'\epsilon s(\phi) + (1-\epsilon)k_\phi]\|f\|_2^2,\end{aligned}$$

and hence

$$\operatorname{Re}[Q(f) - Q_\phi(f)] \leqslant (1-\tfrac{1}{2}\epsilon)\operatorname{Re} Q(f) + [c'\epsilon s(\phi) + (1-\epsilon)k_\phi]\|f\|_2^2.$$

Fix $f \in L^2(\Omega)$ and $\theta \in (-\pi/2, \pi/2)$, and for $\rho > 0$ set $f_\rho = \exp(-H_\phi \rho \mathrm{e}^{\mathrm{i}\theta})f$. We then have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\|f_\rho\|_2^2 &= -2\operatorname{Re}[\mathrm{e}^{\mathrm{i}\theta} Q_\phi(f_\rho)] \\ &= -2\cos\theta \operatorname{Re} Q(f_\rho) + 2\sin\theta \operatorname{Im} Q_\phi(f_\rho) + 2\cos\theta[\operatorname{Re} Q(f_\rho) - \operatorname{Re} Q_\phi(f_\rho)] \\ &\leqslant -2\cos\theta \operatorname{Re} Q(f_\rho) + 2\sin|\theta|[(\tfrac{1}{2}+\beta)\operatorname{Re} Q(f_\rho) + c's(\phi)\|f_\rho\|_2^2] \\ &\qquad + 2\cos\theta[(1-\tfrac{1}{2}\epsilon)\operatorname{Re} Q(f_\rho) + [c'\epsilon s(\phi) + (1-\epsilon)k_\phi]\|f_\rho\|_2^2] \\ &= [-\epsilon\cos\theta + (2\beta+1)\sin|\theta|]\operatorname{Re} Q(f_\rho) \\ &\qquad + [2\cos\theta\{c'\epsilon s(\phi) + (1-\epsilon)k_\phi\} + 2c'\sin|\theta| s(\phi)]\|f_\rho\|_2^2.\end{aligned}$$

Let $\alpha \in (0, \pi/2)$ be such that $\tan\alpha = \epsilon/(2\beta+1)$. For $|\theta| \leqslant \alpha$ we then have $-\epsilon\cos\theta + (2\beta+1)\sin|\theta| \leqslant 0$ and hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\|f_\rho\|_2^2 &\leqslant 2\cos\theta\biggl[c'\epsilon s(\phi) + (1-\epsilon)k_\phi + s(\phi)\frac{c'\epsilon}{2\beta+1}\biggr]\|f_\rho\|_2^2 \\ &\leqslant 2(k_\phi + 2c'\epsilon s(\phi))\|f_\rho\|_2^2 \\ &=: 2A_\epsilon\|f_\rho\|_2^2.\end{aligned}$$

It follows that $\|\mathrm{e}^{-H_\phi z}\|_{2\to 2} \leqslant \mathrm{e}^{A_\epsilon|z|}$ in the sector $|\arg z| \leqslant \alpha$. We conclude that letting

$$\tau_\epsilon = \frac{A_\epsilon}{\cos\alpha}$$

we have

$$\|\exp\{-(H_\phi + \tau_\epsilon)z\}\|_{2\to 2} \leqslant 1,$$

and hence [**6**, Lemma 2.38]

$$\|(H_\phi + \tau_\epsilon)\mathrm{e}^{-(H_\phi+\tau_\epsilon)t}\| \leqslant \frac{c}{\alpha t},$$

for all $t > 0$. Multiplying both sides by $\mathrm{e}^{\tau_\epsilon t}$ and using the triangle inequality we obtain

$$\|H_\phi \mathrm{e}^{-H_\phi t}\|_{2\to 2} \leqslant \frac{c}{\alpha t}\exp\biggl\{\frac{k_\phi + 2c'\epsilon s(\phi)}{\cos\alpha}t\biggr\} + \tau_\epsilon \mathrm{e}^{k_\phi t}.$$

This last expression can be made smaller than the right-hand side of Lemma 3.1 (ii) provided ϵ is chosen small enough; this completes the proof. □

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Proposition 3.2. *Assume that (H1) and (H2) are satisfied. Then for any $\delta > 0$ there exists $c_\delta > 0$ independent of $\phi \in \mathcal{E}_a$ such that*

$$\|\mathrm{e}^{-H_\phi t}\|_{1\to\infty} \leqslant c_\delta t^{-s} \mathrm{e}^{k_\phi t} \mathrm{e}^{\delta s(\phi) t}. \tag{3.7}$$

Proof. Let $f \in L^2(\Omega)$ and set $f_t = \mathrm{e}^{-H_\phi t} f$, $t > 0$. Using (H1) we have

$$\begin{aligned}
\|f_t\|_\infty &\leqslant c[\operatorname{Re} Q(f_t)]^{s/2} \|f_t\|_2^{1-s} \\
&\leqslant c[\operatorname{Re} Q_\phi(f_t) + s(\phi)\|f_t\|_2^2]^{s/2} \|f_t\|_2^{1-s} \quad \text{(by (3.6))} \\
&\leqslant c[\|H_\phi f_t\|_2 \|f_t\|_2 + s(\phi)\|f_t\|_2^2]^{s/2} \|f_t\|_2^{1-s} \\
&\leqslant c[(c_\epsilon/t)\mathrm{e}^{\epsilon s(\phi)t} + s(\phi)]^{s/2} \mathrm{e}^{k_\phi t} \|f\|_2 \quad \text{(by Lemma 3.1 (ii) and Lemma 3.1 (i))} \\
&= c t^{-s/2} [c_\epsilon \mathrm{e}^{\epsilon s(\phi) t} + s(\phi) t]^{s/2} \mathrm{e}^{k_\phi t} \|f\|_2.
\end{aligned}$$

Taking ϵ to be small enough we conclude that given $\delta > 0$ there exists c_δ such that

$$\|\mathrm{e}^{-H_\phi t}\|_{2\to\infty} \leqslant c_\delta t^{-s/2} \mathrm{e}^{k_\phi t} \mathrm{e}^{\delta s(\phi) t}.$$

The same arguments are valid for $(H_\phi)^* = (H^*)_{-\phi}$, the constant k_ϕ clearly staying the same. Hence by duality and the semigroup property, (3.7) follows. □

In order for Proposition 3.2 to be useful we need a precise upper estimate on k_ϕ, which amounts to a precise lower estimate on $\operatorname{Re} Q_\phi(\cdot)$ (cf. (3.5)). This will be established in Lemma 3.11 following a series of intermediate lemmas. Recalling that $c_\gamma^\alpha = \alpha!/\gamma!(\alpha-\gamma)!$ it follows immediately from (3.1) that for $\lambda > 0$, $\phi \in \mathcal{E}_a$ we have

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma,\lambda\phi} P_{\delta,-\lambda\phi} D^{\alpha-\gamma} f D^{\beta-\delta} \bar{f} \,\mathrm{d}x, \tag{3.8}$$

where

$$P_{\gamma,\lambda\phi}(x) := \mathrm{e}^{-\lambda\phi(x)} D^\gamma [\mathrm{e}^{\lambda\phi(x)}]$$

is a polynomial in various derivatives of $\lambda\phi$. Now, the induction relation $P_{\gamma+e_j,\lambda\phi} = (\lambda\partial_j\phi + \partial_j) P_{\gamma,\lambda\phi}$ implies that $P_{\gamma,\lambda\phi}$ has the form

$$P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \sum c_{\gamma;\gamma_1,\dots,\gamma_k} (D^{\gamma_1}\phi) \cdots (D^{\gamma_k}\phi), \tag{3.9}$$

where the second sum is taken over all non-zero multi-indices $\gamma_1, \dots, \gamma_k$ such that $\gamma_1 + \cdots + \gamma_k = \gamma$ and $c_{\gamma;\gamma_1,\dots,\gamma_k}$ are constants. Hence, recalling that $|\nabla^k \phi| \leqslant c a^{-k/2m}$, we can write

$$P_{\gamma,\lambda\phi}(x) = \sum_{k=1}^{|\gamma|} \lambda^k \tilde{P}_{k,\phi}(x),$$

where $|\tilde{P}_{k,\phi}(x)| \leqslant ca^{-|\gamma|/2m}$. It follows from (3.8) that

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha\\\delta\leqslant\beta}} \sum_{\substack{k\leqslant|\gamma|\\j\leqslant|\delta|}} \lambda^{k+j} w_{\alpha\beta\gamma\delta kj}(x) D^{\alpha-\gamma} f D^{\beta-\delta}\bar{f}\,\mathrm{d}x, \tag{3.10}$$

where $w_{\alpha\beta\gamma\delta kj} := a_{\alpha\beta}c^\alpha_\gamma c^\beta_\delta \tilde{P}_{k,\phi}\tilde{P}_{j,-\phi}$ satisfies $|w_{\alpha\beta\gamma\delta kj}| \leqslant ca^{(2m-|\gamma+\delta|)/2m}$. Replacing γ and δ by $\alpha-\gamma$ and $\beta-\delta$, respectively, we conclude from (3.10) the following lemma.

Lemma 3.3. *$Q_{\lambda\phi}(f)$ is a linear combination of terms of the form*

$$T(f) = \lambda^s \int_\Omega w(x) D^\gamma f D^\delta \bar{f}\,\mathrm{d}x, \tag{3.11}$$

where $|w| \leqslant ca^{(|\gamma+\delta|)/2m}$ on Ω and

(i) *s is an integer with $0 \leqslant s \leqslant 2m$;*

(ii) *γ and δ are multi-indices with $|\gamma|, |\delta| \leqslant m$;*

(iii) *$s + |\gamma+\delta| \leqslant 2m$.*

Definition 3.4. We call the number $s+|\gamma+\delta|$ the *essential order* of T.

Hence the essential order is an integer between 0 and $2m$. We denote by $\mathcal{L}_{a,m}$ the linear space consisting of (finite) linear combinations of forms whose essential order is smaller than $2m$. In Lemma 3.9 we will see that terms in $\mathcal{L}_{a,m}$ are in a sense negligible. We also point out for later use that (2.8) implies the interpolation inequality

$$|T(f)| < c\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\}, \quad f \in W^{m,2}_{a,0}(\Omega), \tag{3.12}$$

valid for all terms $T(\cdot)$ of essential order $2m$.

We have the following lemma.

Lemma 3.5. *Given $\phi \in \mathcal{E}_a$ and $\lambda > 0$ define*

$$Q_{1,\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\substack{\gamma\leqslant\alpha\\\delta\leqslant\beta}} a_{\alpha\beta}c^\alpha_\gamma c^\beta_\delta (\lambda\nabla\phi)^\gamma(-\lambda\nabla\phi)^\delta D^{\alpha-\gamma} f D^{\beta-\delta}\bar{f}\,\mathrm{d}x.$$

Then the difference $Q_{\lambda\phi}(f) - Q_{1,\lambda\phi}(f)$ lies in $\mathcal{L}_{a,m}$.

Proof. One simply has to recall (3.8) and observe from (3.9) that $P_{\gamma,\lambda\phi}$, considered as a polynomial in λ, has $\lambda^{|\gamma|}(\nabla\phi)^\gamma$ as its highest-degree term. □

3.1. Symbols in $\mathcal{G}_a$

At this point and for the whole of this subsection we restrict our attention to operators H whose symbol belongs to $\mathcal{G}_a$. For $x \in \Omega$, $\xi, \eta \in \mathbb{C}^N$ and $\zeta \in \mathbb{R}^N$ let us define

$$\begin{aligned} k_m &= [\sin(\pi/(4m-2))]^{-2m+1}, \\ A(x,\xi,\eta) &= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha \bar{\eta}^\beta, \\ S(x,\zeta;\xi,\eta) &= \operatorname{Re} A(x,\xi - \mathrm{i}\zeta, \eta + \mathrm{i}\zeta) + k_m \operatorname{Re} A(x,\zeta). \end{aligned}$$

Lemma 3.6. *Assume that the symbol $A(x,\xi)$ lies in $\mathcal{G}_a$. Then*

$$\begin{aligned} &\operatorname{Re} Q_{1,\lambda\phi}(f) + k_m \lambda^{2m} \int_\Omega \operatorname{Re} A(x, \nabla\phi(x)) |f|^2 \, \mathrm{d}x \\ &\qquad = (2\pi)^{-N} \iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N} S(x,\lambda\nabla\phi;\xi,\eta) \mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} \, \mathrm{d}x \, \mathrm{d}\xi \, \mathrm{d}\eta \qquad (3.13) \end{aligned}$$

for all $\phi \in \mathcal{E}_a$, $\lambda > 0$ and $f \in C_c^\infty(\Omega)$.

Proof. Writing

$$D^\gamma f(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} (\mathrm{i}\xi)^\gamma \mathrm{e}^{\mathrm{i}\xi\cdot x} \hat{f}(\xi) \, \mathrm{d}\xi$$

we have

$$\begin{aligned} Q_{1,\lambda\phi}(f) &= (2\pi)^{-N} \iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leqslant\alpha \\ \delta\leqslant\beta}} c^\alpha_\gamma c^\beta_\delta (-\mathrm{i}\lambda\nabla\phi)^\gamma (-\mathrm{i}\lambda\nabla\phi)^\delta \\ &\qquad\qquad \times \xi^{\alpha-\gamma}\eta^{\beta-\delta} \mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}x \\ &= (2\pi)^{-N} \iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} (\xi - \mathrm{i}\lambda\nabla\phi)^\alpha (\eta - \mathrm{i}\lambda\nabla\phi)^\beta \\ &\qquad\qquad \times \mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}x \\ &= (2\pi)^{-N} \iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N} A(x, \xi - \mathrm{i}\lambda\nabla\phi(x), \eta + \mathrm{i}\lambda\nabla\phi(x)) \\ &\qquad\qquad \times \mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x} \hat{f}(\xi)\overline{\hat{f}(\eta)} \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}x. \end{aligned}$$

This last integral has the form $\int_\Omega q[g] \, \mathrm{d}x$, where, for fixed $x \in \Omega$,

$$\begin{aligned} g(\xi) &= \mathrm{e}^{\mathrm{i}\xi\cdot x} \hat{f}(\xi), \\ q[g] &= \int_{\mathbb{R}^N\times\mathbb{R}^N} p(\xi,\eta) g(\xi)\overline{g(\eta)} \, \mathrm{d}\xi \, \mathrm{d}\eta, \\ p(\xi,\eta) &= A(x, \xi - \mathrm{i}\lambda\nabla\phi(x), \eta + \mathrm{i}\lambda\nabla\phi(x)). \end{aligned}$$

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Since the matrix $\{a_{\alpha\beta}\}$ is symmetric we have $p(\xi,\eta)=p(\eta,\xi)$ and therefore

$$\overline{q[g]}=\int_{\mathbb{R}^N\times\mathbb{R}^N}\overline{p(\xi,\eta)}g(\xi)\overline{g(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Hence

$$\operatorname{Re}q[g]=\int_{\mathbb{R}^N\times\mathbb{R}^N}\operatorname{Re}p(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and integration over $x\in\Omega$ yields

$$\begin{aligned}\operatorname{Re}Q_{1,\lambda\phi}(f)&+k_m\int_\Omega\operatorname{Re}A(x,\lambda\nabla\phi(x))|f|^2\,\mathrm{d}x\\&=(2\pi)^{-N}\iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N}\operatorname{Re}[A(x,\xi-\mathrm{i}\lambda\nabla\phi(x),\eta+\mathrm{i}\lambda\nabla\phi(x))+k_mA(x,\lambda\nabla\phi)]\\&\qquad\qquad\times\mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\\&=(2\pi)^{-N}\iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N}S(x,\lambda\nabla\phi;\xi,\eta)\mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x.\end{aligned}$$

□

We now proceed to estimate the triple integral on the right-hand side of (3.13). It is shown in [**9**, Theorem 2.1] that there exist positive numbers $w_0,\dots,w_{m-2}$ such that

$$S(x,\zeta;\xi,\xi)=\sum_{s=0}^{m-2}w_s\Gamma(x,p^{(s)}_{\xi,\zeta}),\quad x\in\Omega\zeta,\quad\xi\in\mathbb{R}^N,\tag{3.14}$$

where $\Gamma(x,\cdot)$ is the quadratic form associated with the principal symbol of H (cf. (2.13)) and $p^{(s)}_{\xi,\zeta}$ is the vector in $\mathbb{R}^\nu$ defined for fixed $\xi,\zeta\in\mathbb{R}^N$ by requiring that

$$\sum_{|\alpha|=m}p^{(s)}_{\xi,\zeta,\alpha}a^\alpha=(\sin\theta_m)^{-s-2}(\xi\cdot a)^{m-s-2}(\zeta\cdot a)^s\{(\sin\theta_m)^2(\xi\cdot a)^2-(\cos\theta_m)^2(\zeta\cdot a)^2\}\tag{3.15}$$

for all $a\in\mathbb{R}^N$; here $\theta_m=\pi/(4m-2)$. To simplify the notation let us define the sesquilinear forms $\boldsymbol{\Gamma}(x,\cdot,\cdot)$ on $\mathbb{C}^{m-1}\otimes\mathbb{C}^\nu\simeq\mathbb{C}^{\nu(m-1)}$ by

$$\boldsymbol{\Gamma}(x,u,v)=\sum_{s=0}^{m-2}w_s\Gamma(x,u^{(s)},v^{(s)})=\sum_{s=0}^{m-2}\sum_{\substack{|\alpha|=m\\|\beta|=m}}w_sa_{\alpha+\beta}(x)u^{(s)}_\alpha\overline{v^{(s)}_\beta}$$

for all $u=(u^{(s)}_\alpha)$, $v=(v^{(s)}_\beta)\in\mathbb{C}^{\nu(m-1)}$. Then $\boldsymbol{\Gamma}$ is positive semi-definite by the strong convexity of $A(x,\xi)$. To handle the above expressions we introduce two auxiliary elliptic differential forms $S_{\lambda\phi}$ and $\Gamma_{\lambda\phi}$ on $L^2(\Omega)$. They have common domain $W^{m,2}_{a,0}(\Omega)$ and are given by

$$S_{\lambda\phi}(f)=(2\pi)^{-N}\iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N}S(x,\lambda\nabla\phi;\xi,\eta)\mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x,\tag{3.16}$$

$$\Gamma_{\lambda\phi}(f)=(2\pi)^{-N}\iiint_{\Omega\times\mathbb{R}^N\times\mathbb{R}^N}\boldsymbol{\Gamma}(x,p_{\xi,\lambda\nabla\phi},p_{\eta,\lambda\nabla\phi})\mathrm{e}^{\mathrm{i}(\xi-\eta)\cdot x}\hat{f}(\xi)\overline{\hat{f}(\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x,\tag{3.17}$$

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where

$$p_{\xi,\lambda\nabla\phi} = (p^{(s)}_{\xi,\lambda\nabla\phi,\alpha})^{|\alpha|=m}_{0\leqslant s\leqslant m-2} \in \mathbb{C}^{\nu(m-1)}$$

is defined by (3.15).

Lemma 3.7. *Assume that the symbol $A(x,\xi)$ lies in $\mathcal{G}_a$. Then the form $S_{\lambda\phi}(\cdot) - \Gamma_{\lambda\phi}(\cdot)$ lies in $\mathcal{L}_{a,m}$.*

Proof. It follows from (3.14) that $S_{\lambda\phi}$ and $\Gamma_{\lambda\phi}$ have integral kernels which are polynomials of ξ and η and whose values coincide for $\xi = \eta$. Using the inverse Fourier transform this implies that the difference $S_{\lambda\phi}(f) - \Gamma_{\lambda\phi}(f)$ is a linear combination of terms of the form

$$T(f) = \lambda^s \int_\Omega w(x)[D^{\gamma+\kappa} f D^\delta \bar{f} - (-1)^\kappa D^\gamma f D^{\delta+\kappa}\bar{f}]\,\mathrm{d}x, \tag{3.18}$$

where w is some function and κ is a multi-index of length $|\kappa| \leqslant m-1$. In fact, recalling (3.13) and the definition of $Q_{1,\lambda\phi}$ we see that $w = a_{\alpha\beta}(\nabla\phi)^\mu$, where $|\mu| = s$ and $\gamma+\delta+\kappa+\mu = \alpha+\beta$. Since $a_{\alpha\beta} \in W^{m-1,\infty}_a(\Omega) \subset W^{m-1,\infty}_{\mathrm{loc}}(\Omega)$, we can integrate by parts $|\kappa|$ times and use Leibnitz's rule to obtain

$$T(f) = (-1)^{|\kappa|}\lambda^s \sum_{0<\kappa_1\leqslant\kappa} c^\kappa_{\kappa_1} \int_\Omega D^{\kappa_1} w D^\gamma f D^{\delta+\kappa-\kappa_1}\bar{f}\,\mathrm{d}x. \tag{3.19}$$

We estimate $D^{\kappa_1}w$: clearly,

$$|D^{\kappa_1}(a_{\alpha\beta}(\nabla\phi)^\mu)| \leqslant c\sum_{i=0}^{|\kappa_1|} |\nabla^{|\kappa_1|-i}a_{\alpha\beta}||\nabla^i(\nabla\phi)^\mu| \quad \text{in } \Omega.$$

Recalling the definition of $\mathcal{E}_{A,M}$ it is easily seen that $|\nabla^i(\nabla\phi)^\mu| \leqslant ca^{-(|\mu|+i)/2m}$; recalling also from (2.14) the definition of the space $W^{m-1,\infty}_a(\Omega)$ where the $a_{\alpha\beta}$ lie we conclude that

$$|D^{\kappa_1}(a_{\alpha\beta}(\nabla\phi)^\mu)| \leqslant c_M a(x)^{(2m-|\kappa_1+\mu|)/2m} = c_M a^{(|\gamma+\delta+\kappa-\kappa_1|)/2m}.$$

Hence (3.19) implies that T has essential order $s + |\gamma+\delta+\kappa-\kappa_1| < 2m$, as required. □

Proposition 3.8. *Let $A(x,\xi) \in \mathcal{G}_a$. Then for any $\phi \in \mathcal{E}_a$, $\lambda > 0$ and all $f \in C^\infty_c(\Omega)$, the following inequality holds:*

$$\mathrm{Re}\, Q_{\lambda\phi}(f) \geqslant -k_m\lambda^{2m}\,\mathrm{Re}\int_\Omega A(x,\nabla\phi(x))|f|^2\,\mathrm{d}x + T(f), \tag{3.20}$$

where $T(\cdot) \in \mathcal{L}_{a,m}$.

Proof. Combining Lemmas 3.5, 3.6 and 3.7 we have

$$\mathrm{Re}\, Q_{\lambda\phi}(f) + k_m\int_\Omega \mathrm{Re}\, A(x,\lambda\nabla\phi(x))|f|^2\,\mathrm{d}x = \Gamma_{\lambda\phi}(f) + T(f), \tag{3.21}$$

for a form $T(\cdot) \in \mathcal{L}_{a,m}$. Now let

$$u(x) = \int_{\mathbb{R}^N} p_{\xi,\lambda\nabla\phi} \mathrm{e}^{\mathrm{i}\xi\cdot x} \hat{f}(\xi)\,\mathrm{d}\xi$$

(a $\mathbb{C}^{\nu(m-1)}$-valued integral defined component wise); it follows immediately from definition (3.17) that

$$\Gamma_{\lambda\phi}(f) = \int_\Omega \boldsymbol{\Gamma}(x, u(x), u(x))\,\mathrm{d}x, \tag{3.22}$$

and hence $\Gamma_{\lambda\phi}(\cdot)$ is non-negative by the strong convexity of $A(x,\xi)$. □

3.2. The general case

We now remove the assumption $A \in \mathcal{G}_a$ and return to the general setting described in § 2. We recall that the quantity D measures the distance of A from $\mathcal{G}_a$ and has been defined in (2.15).

Lemma 3.9. *Let $T \in \mathcal{L}_{a,m}$. Then for any $\epsilon \in (0,1)$ the following inequality holds for all $\lambda > 0$ and $f \in C_c^\infty(\Omega)$:*

$$|T(f)| < \epsilon\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + c_\epsilon\|f\|_2^2. \tag{3.23}$$

Proof. By definition, $T(f)$ is a finite linear combination of expressions of the form

$$I(f) = \lambda^s \int_\Omega w(x) D^\gamma f(x) D^\delta \bar{f}(x)\,\mathrm{d}x,$$

where $|w(x)| \leqslant ca(x)^{|\gamma+\delta|/2m}$ and $s + |\gamma+\delta| \leqslant 2m-1$. Setting $\mu^{2m-|\gamma+\delta|} = \lambda^s$ and recalling (2.8) we have

$$\begin{aligned}
|I(f)| &\leqslant c\mu^{2m-|\gamma+\delta|} \int_\Omega a(x)^{|\gamma+\delta|/2m} |D^\gamma f|\,|D^\delta f|\,\mathrm{d}x \\
&\leqslant \epsilon \operatorname{Re} Q(f) + c\epsilon^{-2m+1}(1+\mu^{2m})\|f\|_2^2 \\
&\leqslant \epsilon \operatorname{Re} Q(f) + c\epsilon^{-2m+1}(1+\lambda^{2m-1})\|f\|_2^2 \\
&\leqslant \epsilon\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + c\epsilon^{-4m^2+1}\|f\|_2^2.
\end{aligned}$$

□

Remark 3.10. It is seen from the proof that the size of the constant c_ϵ in (3.23) depends only on $\epsilon > 0$ and the (finite) quantity $\max_I \sup\{|w(x)|a(x)^{-|\gamma+\delta|/2m}\}$, where the maximum is taken over all forms $I(\cdot)$ that make up $T(\cdot)$. In particular, when we restrict our attention to functions $\phi \in \mathcal{E}_{A,M}$ we obtain a constant $c_\epsilon = c_{\epsilon,M}$ which is otherwise independent of ϕ.

Lemma 3.11. *For any $\phi \in \mathcal{E}_{A,M}$, $\lambda > 0$ and $\epsilon > 0$ the following inequality holds:*

$$\operatorname{Re} Q_{\lambda\phi}(f) \geqslant -\{(k_m + cD + \epsilon)\lambda^{2m} + c_{\epsilon,M}\}\|f\|_2^2, \quad f \in C_c^\infty(\Omega), \tag{3.24}$$

where the constant c is independent of D, M, ϵ, λ and ϕ and the constant $c_{\epsilon,M}$ is independent of D, λ and ϕ.

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Proof. Let $\tilde{A} \in \mathcal{G}_a$ be such that $\|A - \tilde{A}\|_{a,\infty} \leqslant 2D$. It follows from (3.12) that

$$|\operatorname{Re} \tilde{Q}_{\lambda\phi}(f) - \operatorname{Re} Q_{\lambda\phi}(f)| < cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\},$$
$$\left|\lambda^{2m}\int_\Omega [A(x,\nabla\phi(x)) - \tilde{A}(x,\nabla\phi(x))]\,\mathrm{d}x\right| < cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\}.$$

Combining these relations with (3.20)—as applied to the operator $\tilde{H}$—we obtain

$$\operatorname{Re} Q_{\lambda\phi}(f) \geqslant -k_m\lambda^{2m}\int_\Omega \operatorname{Re} A(x,\nabla\phi(x))|f|^2\,\mathrm{d}x - cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f).$$

We have $\operatorname{Re} A(x,\nabla\phi(x)) \leqslant 1$ and therefore (allowing c to change from line to line and ϵ to rescale)

$$\begin{aligned}
\operatorname{Re} Q_{\lambda\phi}(f) &\geqslant -k_m\lambda^{2m}\|f\|_2^2 - cD\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + T(f)\\
&\geqslant -k_m\lambda^{2m}\|f\|_2^2 - (cD+\epsilon)\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M}\|f\|_2^2 \qquad \text{(by (3.23))}\\
&\geqslant -k_m\lambda^{2m}\|f\|_2^2 - (cD+\epsilon)\{\operatorname{Re} Q_{\lambda\phi}(f) + \lambda^{2m}\|f\|_2^2\} - c_{\epsilon,M}\|f\|_2^2 \quad \text{(by (3.6))}.
\end{aligned}$$

Now, either $\operatorname{Re} Q_{\lambda\phi}(f)$ is positive, in which case (3.24) is true, or it is not, in which case it can be discarded from the right-hand side of the last inequality. This completes the proof. □

Proof of Theorem 2.2. The rest of the proof is standard. Combining Proposition 3.2 with (3.24) and using the relation $K_{\lambda\phi}(t,x,y) = \mathrm{e}^{-\lambda\phi(x)}K(t,x,y)\mathrm{e}^{-\lambda\phi(y)}$ we obtain

$$|K(t,x,y)| < c_\delta t^{-s}\exp\{\lambda[\phi(y)-\phi(x)] + [(k_m + cD + \delta)\lambda^{2m} + c_{\delta,M}]t\}.$$

Optimizing over $\phi \in \mathcal{E}_{A,M}$ introduces $d_M(x,y)$ and choosing

$$\lambda = \left(\frac{d_M(x,y)}{2mk_m t}\right)^{1/(2m-1)}$$

we obtain

$$-\lambda d_M(x,y) + k_m\lambda^{2m}t = -\sigma_m\frac{d_M(x,y)^{2m/(2m-1)}}{t^{1/(2m-1)}},$$

which completes the proof. □

Remark 3.12. It is shown in [**4**] that the term cD cannot be eliminated from (3.24). Thus for it to be removed from Theorem 2.2 an essentially different approach is needed—if indeed the term is removable at all.

Remark 3.13. We point out that the above method can also work for operators of the form $H + W$, where W is a lower-order perturbation of H. It is clear that the estimate of Theorem 2.2 is valid for $H + W$ provided $W_{\lambda\phi}$ can be estimated by

$$|W_{\lambda\phi}(f)| < \epsilon\{\operatorname{Re} Q(f) + \lambda^{2m}\|f\|_2^2\} + c_\epsilon\|f\|_2^2$$

for all $\phi \in \mathcal{E}_a$ and $\lambda > 0$ and any $\epsilon > 0$. Such estimates can be obtained by means of weighted Hardy- and Sobolev-type inequalities. We do not elaborate on this and prove a theorem for zero-order real perturbations.

Proposition 3.14. *Let $V = V_+ - V_-$, where $V_+ \in L^1_{\mathrm{loc}}(\Omega)$ and $V_- \in L^1(\Omega)$ are, respectively, the positive and negative parts of the real-valued potential V. Then the heat kernel of $H + V$ satisfies the estimate of Theorem 2.2.*

Proof. We have

$$\begin{aligned}\int_\Omega V_-|f|^2 &\leqslant \|V_-\|_1\|f\|_\infty^2 \\ &\leqslant c\|V_-\|_1[\operatorname{Re} Q(f)]^s\|f\|_2^{2-2s} \quad \text{(by (H1))} \\ &\leqslant \epsilon \operatorname{Re} Q(f) + c_{\epsilon,V}\|f\|_2^2\end{aligned}$$

(hence $H + V$ is defined with the same form domain as for $H + V_+$). Moreover,

$$\operatorname{Re}(H+V)_{\lambda\phi} = \operatorname{Re} H_{\lambda\phi} + V \geqslant \operatorname{Re} H_{\lambda\phi} - V_-.$$

Hence the estimate of Lemma 3.11 is also valid for $H + V$ and the rest of the argument goes through. □

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