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ON CHARACTERIZING INJECTIVE SHEAVES

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1. Introduction and notation. Let T be a Grothendieck topology, Ab the category of abelian groups, and $\bar{S}$ the category of Ab-valued sheaves on T. It is known that $\bar{S}$ is an abelian $AB5$ category with a set of generators [**2**, Theorem 1.6(i), p. 30] and, hence, has injective envelopes [**10**, Theorem 3.2, p. 89]. Consider an object F of $\bar{S}$. Two necessary conditions that F be injective in $\bar{S}$ are that the values assumed by F are injective in Ab (i.e., divisible abelian groups) [**2**, Corollary 2.5, p. 17; Miscellany 1.8(ii), p. 33]; and that F is *cohomologically trivial* (dubbed "flask" in [**2**, p. 39]), in the sense that the Čech cohomology groups $H^n(\{U_i \to V\}, F)$ vanish for each $n \geqq 1$ and each cover $\{U_i \to V\}$ in T [**2**, Theorem 3.1, p. 19]. In this note, we seek instances in which these necessary conditions are jointly sufficient.

Our motivation is a result of Martinez [**9**] on cohomology of profinite groups which, as translated in Proposition 1 below, implies sufficiency of the above conditions in case T is the étale topology of Spec(k), where k is a field. If T corresponds to a topological space X (as, e.g., in [**14**, p. 193]), sufficiency is readily established in case X is either discrete or indiscrete (Remark 4), but fails in general (Example 3). Our main result (Theorem 6) establishes sufficiency for T arising from a Boolean space X. A corollary yields, for any profinite group G, a nontrivial injective-preserving left exact functor from the category of discrete G-modules to the category of Ab-valued sheaves on G.

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2. Results. Since any profinite group may be realized as the Galois group of a Galois field extension [**16**, Theorem 2], the motivating result of Martinez [**9**, Proposition 4] may be translated as follows.

PROPOSITION 1. *Let T be the sub-Grothendieck topology of the étale topology of* Spec(k), *constructed from a Galois field extension L/k as in* [**5**, p. 39]. (*If L is a separable closure of k, then T is the étale topology of* Spec(k).) *Then F* (*as above, an object of $\bar{S}$*) *is injective in $\bar{S}$ if* (*and only if*) *the following two conditions hold:*

(a) F(Spec K) *is injective in* Ab, *for each finite Galois subextension K/k of L/k; and*

(b) *the Čech cohomology group $\check{H}_T{}^i$*(Spec(K), F) $= 0$, *whenever $i = 1, 2$ and K/k is a finite subextension of L/k.*

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Proof (sketch). Let $G = \text{gal}(L/k)$. The equivalence between $\bar{S}$ and the category of discrete G-modules (cf. [**5**, Corollary 5.4, p. 54; **14**, Proposition 71, p. 207]) permits F to be identified with a discrete G-module M. In view of the construction of the equivalence (especially, of M^* in [**5**, p. 24]) and the definition of the Krull topology on G, we see that (a) is equivalent to that part of Martinez's criterion for injectivity which specifies that M^U be divisible for each open normal subgroup U of G. After subjecting the remaining part of Martinez's criterion to [**9**, Proposition 2] for translation, we obtain the condition that the profinite group cohomology $H^i(V, M)$ vanish whenever $i = 1, 2$ and V is an open subgroup of G. It remains only to identify the preceding condition with (b). To that end, let K be the subextension of L/k such that $V = \text{gal}(L/K)$ and observe from [**5**, p. 175] (with $F = \phi M$ and K "chosen") that $H^i(V, M) \cong \check{H}_T{}^i(\text{Spec}(K), F)$, to complete the proof.

Note that condition (b) in Proposition 1 is ostensibly weaker than the requirement that F be cohomologically trivial, as the Cech group $\check{H}_T{}^n(Y, F)$ is defined as the direct limit of the Čech groups $H^n(\{Y_i \to Y\}, F)$, the covers being ordered by refinement, as in [**2**, Definition 3.3, p. 21]. Moreover, (a) does not posit injectivity of *all* values of F. As Remark 4 and Theorem 6 also indicate, various contexts often admit affirmative results in which such weaker conditions characterize injectivity; however, Example 3 will develop a negative result in which "cohomologically trivial" is strengthened to "flabby" (in the sense of [**15**, p. 58]).

We pause to apply, and comment about, the conditions in Proposition 1.

Example 2.(i) In the context of Proposition 1, take L/k to be $\mathbf{C}/\mathbf{R}$; then, T is the étale topology of Spec $(\mathbf{R})$. As in [**5**], it is harmless to think of T as an $\mathbf{R}$-based topology, the objects of whose underlying category are the finite products of copies of $\mathbf{C}$ and $\mathbf{R}$. Note that the units functor U is an object of $\bar{S}$; similarly, the functor $U\mathbf{C}$ (to employ the notation of [**4**, p. 48]), given on an algebra A by $(U\mathbf{C})(A) = U(\mathbf{C} \otimes_{\mathbf{R}} A)$, is also an Ab-valued T-sheaf. We claim that $U\mathbf{C}$ is injective in $\bar{S}$.

Indeed, condition (a) is readily verified: $(U\mathbf{C})(\mathbf{R}) = U(\mathbf{C})$, $(U\mathbf{C})(\mathbf{C}) \cong U(\mathbf{C}) \times U(\mathbf{C})$, and both are divisible groups, since $\mathbf{C}$ is root-closed. As for (b), first note that $\check{H}_T{}^i(\text{Spec}(\mathbf{C}), U\mathbf{C}) = H^i(\{\text{Spec}(\mathbf{C}) \to \text{Spec}(\mathbf{C})\}, U\mathbf{C}) = 0$ for $i > 0$. Moreover, $\check{H}_T{}^i(\text{Spec}(\mathbf{R}), U\mathbf{C}) = H^i(\{\text{Spec}(\mathbf{C}) \to \text{Spec}(\mathbf{R})\}, U\mathbf{C})$ is the Amitsur cohomology group $H^i(\mathbf{C}/\mathbf{R}, U\mathbf{C}) \cong H^i(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}/\mathbf{C}, U)$; since there are $\mathbf{C}$-algebra homomorphisms $\mathbf{C} \to \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C}$, the fundamental homotopy property for Amitsur cohomology ([**1**, Lemma 2.7; **2**, Proposition 3.4, p. 22]) implies that the identity map on $H^i(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}/\mathbf{C}, U)$ factors through $H^i(\mathbf{C}/\mathbf{C}, U)$. Thus, $\check{H}_T{}^i(\text{Spec}(\mathbf{C}), U\mathbf{C}) = 0$ for all $i > 0$, to establish the above claim.

Consider the natural transformation α: $U \to U\mathbf{C}$ induced by the embeddings $A \to \mathbf{C} \otimes_{\mathbf{R}} A$. Although $U\mathbf{C}$ is injective and α is a monomorphism in $\bar{S}$, we next show that α is not an injective envelope, i.e., that α is not essential in $\bar{S}$.

To that end, let $G = \text{gal}(\mathbf{C}/\mathbf{R})$ and observe that the discussion in [**5**, pp. 48-49] identifies α with the homomorphism β: $U(\mathbf{C}) \to (U\mathbf{C})(\mathbf{C}) = U(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ of (necessarily discrete) G-modules given by $\beta(a) = 1 \otimes a$. (Note that β is a G-map since G acts on the second tensor factor of $U(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$, according to [**5**, p. 49].) Finally, if ω is a primitive cube root of unity, then $\{\omega \otimes 1, \omega^2 \otimes 1, 1 \otimes 1\}$ is a G-submodule of $U(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ which meets the image of β trivially (as ω and ω^2 are nonreal); thus β is not essential, and so neither is α. It would be convenient to have an explicit construction for the injective envelope of U.

(ii) We now proceed to establish independence of the conditions (a) and (b) in Proposition 1. First, we revisit a context mentioned by Rim [**12**, Remark, p. 708] (also cf. [**11**, p. 257]). Let T be constructed as above from a proper extension L/k of finite fields; take F in $\bar{S}$ to be UL (again using the notation of [**4**, p. 48]). Observe that F does not satisfy (a), since $F(k) = U(L)$ is not a divisible group. However, F does satisfy (b): it is a matter of evaluating $H^i(L/K, F)$ for $i = 1, 2$ and K any field between k and L, and these groups vanish, by using homotopy as in part (i). (For another example—essentially Rim's—with the same T, take $F = U$. Again (a) fails, but (b) holds: $H^1(L/K, U) = 0$ by Hilbert's Theorem 90, and $H^2(L/K, U)$ is the trivial split Brauer group $B(L/K)$. The H^2 computation may also be effected by group cohomology ([**3**, Theorem 5.4; **13**, p. 141]) since every element of K is a norm from L.)

For yet another example showing that a cohomologically trivial sheaf need not assume injective values, this time in the finite topology of any field (as defined in [**5**, p. 105]), take F to be A_U (constructed as in [**5**, pp. 93-94]), with A nondivisible. The required properties of F are given in [**5**, Chapter II, Propositions 1.3 and 4.7(a), Theorem 4.5].

Next, to show that (a) does not imply (b), let T arise from a finite cyclic n-dimensional field extension L/k with Galois group G, and let F correspond to a trivial G-module M which is divisible and has nontrivial n-torsion. As $F(K) = M$ for each field K between k and L (see construction of M^* in [**5**, p. 24]), F satisfies (a). (Indeed, *all* values of F are injective, since [**5**, Proposition 5.2, p. 51] shows that F is additive, in the sense of [**3**, p. 30]). However, (b) fails since $\check{H}_T{}^1(\text{Spec}(k), F) \cong H^1(L/k, F) \cong H^1(G, M) \cong \{x \text{ in } M: nx = 0\} \neq 0$.

(iii) For T as in Proposition 1, conditions (a) and (b) do not guarantee injectivity of an Ab-valued *T-presheaf* F (although (a) and (b) remain necessary conditions for injectivity in the category of T-presheaves). Indeed, [**6**, Theorem 3.2] provides an additive presheaf F on the étale topology of Spec$(\mathbf{Q})$ such that *all* values of F are injective, and *enough* Amitsur cohomology in F vanishes [**6**, Corollary 3.5] in order to assure that F satisfies (b); however, there is a cover whose $H^1(-, F)$ is nontrivial [**6**, Theorem 3.2], whence F is not injective. The moral is that the vanishing of the direct limit Čech groups

$\check{H}_T{}^*(-, F)$ does not guarantee vanishing of the Čech groups $H^*(-, F)$ taken with respect to individual covers. In view of such pathology, it would be interesting to obtain an analogue of Proposition 1 for presheaves.

Henceforth, T will arise from an ordinary topological space X, with $\bar{S}$ again denoting the sheaf category.

Example 3. Consider the two-point space $X = \{x, y\}$, whose open sets are ϕ, X and $\{x\}$. Define F in $\bar{S}$ by setting $F(X) = \mathbf{Q}$ and $F(\{x\}) = \mathbf{Q}/\mathbf{Z}$, with the restriction map $F(X) \to F(\{x\})$ being the canonical epimorphism, π. Now, F is flabby and, hence, cohomologically trivial; and, of course, all the values of F are injective. However, we claim that F is not injective.

Indeed, define P in $\bar{S}$ by setting $P(X) = \mathbf{Q} \oplus \mathbf{Q}/\mathbf{Z}$ and $P(\{x\}) = \mathbf{Q}/\mathbf{Z}$, with the restriction map $P(X) \to P(\{x\})$ being the second projection, π_2. Consider the monomorphism u: $F \to P$ in $\bar{S}$ given by $u_X = (1, \pi)$ and $u_{\{x\}} = 1$. Were F injective, there would be a natural transformation v: $P \to F$ such that $vu = 1$. Then v_X would be the unique retraction of u_X in Ab, namely $(1, 0)$; similarly, $v_{\{x\}} = 1$, and naturality of v would require $\pi(1, 0) = \pi_2$, the desired contradiction.

Sobered by the preceding example, we pause to record success for the trivial topologies.

Remark 4. If X has either the discrete or the indiscrete topology and F in $\bar{S}$ is such that $F(X)$ is divisible, then F is injective. Indeed, if X is indiscrete, the sheaf condition requires only that the tested presheaf send ϕ to 0, a natural transformation $F_1 \to F_2$ of sheaves amounts only to a group homomorphism $F_1(X) \to F_2(X)$, and the test diagrams for injectivity of F in $\bar{S}$ thus degenerate to test diagrams for injectivity of $F(X)$ in Ab. As for the case of discrete X, define P in $\bar{S}$ by $P(U) = \prod_{x \in U} F(\{x\})$ for each (open) subset U of X, with the restriction maps being the canonical projections. Observe that the sheaf property of F (applied to the cover of U by singleton sets) shows that F is naturally equivalent to P; but P is injective in $\bar{S}$ [**10**, Chapter X, Lemma 1.1 and Corollary 7.2], to complete the proof. (In fact, one proves similarly: the (flabby) sheaf Q, constructed on an arbitrary X in [**10**, p. 257], is injective if and only if its global sections, $Q(X)$, form a divisible group.)

Example 5. Let R be any commutative ring (with 1). In [**8**], Magid constructs a topological space X and a sheaf F in the corresponding $\bar{S}$ such that $F(X) \cong B(R)$, the Brauer group of R. Then F is injective if (and only if) $B(R)$ is divisible.

To begin the proof, recall that $X = \operatorname{Spec}(I(R))$, where $I(R)$ is the Boolean ring of idempotents of R (with suitably redefined addition). In this generality, the proof must await Theorem 6, which applies since X is a Boolean space.

A special case, which *can* be settled now, arises when R is the product of finitely many rings, $R_1, \ldots, R_n$, each having divisible Brauer group. (For instance, local classfield theory [**13**, Proposition 6, p. 200] permits the p-adic fields as suitable R_i.) Then $I(R)$ is isomorphic to the product of n copies of $\mathbf{Z}/2\mathbf{Z}$, and so X is a discrete n-point space. As $F(X) \cong B(R_1) \oplus \ldots \oplus B(R_n)$, Remark 4 implies that F is injective.

By way of generalizing the discrete case in Remark 4 and completing the proof begun in Example 5, we now present our main result. For us, a *Boolean space* will mean a Hausdorff space in which the compact open sets form a basis; for the purposes of Corollary 7, note that any compact Hausdorff totally disconnected space is Boolean (but not all Booleans are compact).

THEOREM 6. *Let F be an* Ab*-valued sheaf on a Boolean space X. Then F is injective in $\bar{S}$ if (and only if) $F(X)$ is divisible.*

Proof. We begin by establishing a fragment of cohomological triviality: if Y is a compact open subset of X, then $\check{H}_T{}^i(Y, F) = 0$. Indeed, since any compact subspace of a Hausdorff space is closed, the direct limit defining $\check{H}_T{}^n(Y, F)$ may be taken cofinally over finite covers by clopen (i.e., closed and open) sets which may be further assumed mutually disjoint. For any such cover $\{Y_i \to Y\}$, the coboundary in the corresponding Čech cochain complex alternates between 0 and isomorphisms, giving $H^n(\{Y_i \to Y\}, F) = 0$ for all $n > 0$. Thus, $\check{H}_T{}^n(Y, F) = 0$ for $n > 0$ (more than was claimed, the preceding argument being valid for *any* sheaf F).

Let u: $F \to I$ be an injective envelope in $\bar{S}$, with sheaf cokernel $I \to C$. If Y is open in X, there is an exact sequence

$$0 \to F(Y) \to I(Y) \to C(Y) \to H_T{}^1(Y, F) \to 0;$$

but $H_T{}^1(Y, F) \cong \check{H}_T{}^1(Y, F)$ [**2**, Corollary 3.6, p. 38]; so, if Y is also compact, the preceding observation yields $C(Y)$ as the cokernel in Ab of u_Y. Note $F(Y)$ is divisible for clopen Y, since the sheaf condition for F, applied to $\{Y \to X, X\backslash Y \to X\}$, gives

$$F(X) \xrightarrow{\cong} F(Y) \oplus F(X\backslash Y).$$

Hence, it suffices to prove that u_Y is essential in Ab for each compact open Y; for then u_Y is an isomorphism (being split essential), $C(Y) = 0$, C is the zero sheaf (having vanished on a basis), u is an isomorphism in $\bar{S}$, and F is injective, as required.

Indeed, it is enough to show that u_X is essential in Ab. To prove this reduction, take Y compact open in X and H a nonzero subgroup of $I(Y)$. We must show, given u_X essential, that $u_Y(F(Y)) \cap H \neq 0$. The inclusions i: $Y \to X$ and j: $X\backslash Y \to X$ induce, as above, an isomorphism

$$f = (Ii, Ij): I(X) \to I(Y) \oplus I(X\backslash Y);$$

set $K = f^{-1}(H \oplus 0)$. By essentiality of u_X, there exists b in $F(X)$ such that $0 \neq u_X(b) \in K$. Since $f(u_X(b)) \neq 0$, project onto $I(Y)$ to obtain $0 \neq (Ii)(u_X(b)) = u_Y((Fi)(b)) \in H$, as required.

Suppose that u_X is *not* essential. Then, in particular, u_X is not surjective, and the injectivity of $F(X)$ provides a nonzero subgroup M of $I(X)$ such that $I(X) = u_X(F(X)) \oplus M$. Define an Ab-valued presheaf P on X by setting $P(X) = M$ and $P(Y) = 0$ for any open $Y \neq X$. Observe that P is a subpresheaf of I; let v: $I \to I/P = J$ be the canonical epimorphism. Note that vu: $F \to J$ is a monomorphism in the presheaf category; indeed, $\ker(v_X u_X) = 0$ since $u_X(F(X)) \cap M = 0$, while, for open $Y \neq X$, $\ker(v_Y u_Y) = \ker(u_Y) = 0$. If $J^\#$ is the associated sheaf to J and w: $J \to J^\#$ the canonical natural transformation [**2**, p. 24], then (left) exactness of $^\#$ implies that $(vu)^\# = wvu$ is a monomorphism in $\bar{S}$. Essentiality of u in $\bar{S}$ now implies that wv is a monomorphism in $\bar{S}$ and, hence, also a monomorphism in the presheaf category. Then $w_X v_X$ is a monomorphism in Ab, although $\ker(v_X) = M \neq 0$. This contradiction reveals that u_X *is* essential, to complete the proof.

Recall that the category of discrete modules over a profinite group G is equivalent to the category of Ab-valued sheaves on a Grothendieck topology associated to G([**2**, Example (0.6 bis); **5**, Corollary 5.4, p. 54; **14**, Proposition 71, p. 207]). Our final results treat the natural question of relating the former category to the category of Ab-valued sheaves on G itself.

COROLLARY 7. *Let G be a profinite group, $\bar{M}$ the category of discrete G-modules, $\bar{P}$ the category of* Ab*-valued presheaves on G, and $\bar{S}$ the category of* Ab*-valued sheaves on G. Then there exists an additive, left exact and fully faithful functor i: $\bar{M} \to \bar{P}$ such that the (additive, left exact) composite $j = {}_\# i$: $\bar{M} \to \bar{S}$ preserves injectives.*

Proof. For each object M of $\bar{M}$, let $i(M)$ be the object of $\bar{P}$ given by $i(M)(U) = M^U$ for each open set U of G. (The restriction map $i(M)(U) \to i(M)(V)$, arising from an inclusion $V \subset U$ of open subsets of G, is taken to be the inclusion $M^U \to M^V$.) If h: $M \to N$ is a morphism in $\bar{M}$,then $i(h)$ is taken to be the natural transformation $i(M) \to i(N)$ which, at level U, is the restriction of h to $M^U \to N^U$. Clearly, i is a functor $\bar{M} \to \bar{P}$.

Verification that i is additive and left exact may safely be omitted. (Note that $j = {}_\# i$ is then also additive and left exact, as $_\#$ surely is.) To show that i is faithful, suppose that h: $M \to N$ in $\bar{M}$ satisfies $i(h) = 0$. Then h restricts to the zero map $M^U \to N^U$ for *all* U and, since discreteness of M gives $M = \bigcup M^U$, we conclude $h = 0$, as needed. Fullness of i also follows readily from discreteness of M.

It remains to show that, if M is injective in $\bar{M}$, then $j(M)$ is injective in $\bar{S}$. As G is, in particular, a Boolean space, injectivity of $j(M)$ in $\bar{S}$ amounts, by Theorem 6, to divisibility of the abelian group $j(M)(G)$. Before calculating $j(M)(G)$, we next establish divisibility of some related groups.

We claim that, if W is an open normal subgroup of G and if $g \in G$, then M^{Wg} is divisible. Indeed, since $\langle W, g\rangle$, the subgroup of G generated by W and g, is again open and $M^{Wg} = M^{\langle W, g\rangle}$, it suffices to prove that M^U is divisible for

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each open subgroup U of G. This, in turn, follows since M corresponds to an injective sheaf H on a sub-Grothendieck topology of the étale topology of Spec(k), where k is a field, and corresponding to U there is a finite separable field extension K/k satisfying $H(\text{Spec}(K)) = M^U$.

In order to apply the preceding observation to a computation of $j(M)(G)$, first recall that $\#$ is built by the "double $+$" construction [**2**, pp. 24-30]. Note, for any nonempty open subset W of G, that $i(M)^+(W) = \check{H}_T{}^0(W, i(M)) = \lim_\to H^0(\{W_k \to W\}, i(M)) = \lim_\to (\cap M^{W_k}) = \lim_\to M^W = M^W$; similarly, $i(M)^+(\phi) = 0$. (The last assertion reveals why replacement of i by j was necessary; namely, $i(M)$ is typically not a sheaf, as $i(M)(\phi) = M$.) Now, consider any element β in $j(M)(G) = \check{H}_T{}^0(G, i(M)^+) = \lim_\to H^0(\{W_k \to G\}, i(M)^+)$. Since G is compact, we may assume that β is the class of an element b in $H^0(\{W_k \to G\}, i(M)^+)$, where $\{W_k \to G\}$ is a finite irredundant cover of G by n basic open sets. It is convenient to use the basis of clopens of the form Wg where W is an open normal subgroup of G and $g \in G$, for then b is in $\prod_{k=1}^n i(M)^+(W_k) = \prod M^{W_k}$ which, by the remarks of the preceding paragraph, is a divisible group.

Now, if m is a positive integer and one seeks γ in $j(M)(G)$ such that $m\gamma = \beta$, then the preceding comment supplies c in $\prod M^{W_k}$ such that $mc = b$. We would like to take γ to be the class of c, but cannot (yet), since torsion in M may prevent c from being in the difference kernel $H^0(\{W_k \to G\}, i(M)^+)$. To circumvent this possibility, refine $\{W_k \to G\}$ to the cover $\{V_k \to G\}$, where $V_n = W_n$ and $V_k = W_k \backslash (W_{k+1} \cup \ldots \cup W_n)$ for all $1 \leqq k \leqq n - 1$. As the V_k are mutually disjoint and $i(M)^+(\phi) = 0$, we have $H^0(\{V_k \to G\}, i(M)^+) = \prod M^{V_k}$, which *does* contain c (along with b and the rest of $\prod M^{W_k}$). Now it makes sense to refer to the class of c in $j(M)(G)$, and such a choice for γ assures $m\gamma = \beta$, thus establishing divisibility.

We close with three observations about the functors introduced in Corollary 7.

Remark 8. (a) Since i is fully faithful and left exact, i reflects injectives. The next example illustrates that j does *not*, in general, reflect injectives. Accordingly, j is not fully faithful (and we obtain an amusing proof that neither is $\#$).

For the example, take G and M as in the last part of Example 2(ii). Applying the sheaf condition for $j(M)$ to the cover of G by singletons leads to $j(M)(G) \cong \prod_{g \in G} j(M)(\{g\})$; now, $j(M)(\{g\}) = H^0(\{\{g\} \to \{g\}\}, i(M)^+) = i(M)^+(\{g\}) = M^g = M$. Hence, $j(M)(G)$ is divisible and, by either Remark 4 or Theorem 6, $j(M)$ is injective in $\bar{S}$. However, M is not injective in $\bar{M}$, since it was seen earlier that $H^1(G, M) \neq 0$.

(b) It is worthwhile to note that j is never the zero functor. Indeed, if M is a nonzero but trivial G-module, then $j(M)(W) \neq 0$, for any nonempty clopen subset W of G. For a proof, observe that covers by finitely many, mutually disjoint nonempty clopens are cofinal in the family of covers of W;

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for such a cover $H^0(\{W_k \to W\}, i(M)^+) = \prod M^{W_k} = \prod M$, a nonzero group which, by [**2**, (1.4) and (1.5), pp. 26-27], survives in the direct limit defining $j(M)(W)$.

(c) With regard to the possible (right) exactness of j, we shall first obtain a formula for $R^n j$, the n-th right derived functor of j. Fix an open subset W of G; let $\Gamma = \Gamma_W$: $\bar{S} \to \mathrm{Ab}$ be the section functor given by $\Gamma(F) = F(W)$ for each sheaf F. By Corollary 7, j and Γ satisfy the conditions of Grothendieck's composite functor theorem [**7**, Théorème 2.4.1]; thus, for each M in $\bar{M}$, one obtains a first-quadrant spectral sequence

$$E_2^{pq} = (R^p\Gamma)((R^q j)(M)) \Rightarrow E^n = (R^n(\Gamma j))(M).$$

The initial term E_2^{pq} is the Grothendieck cohomology group $H_T^p(W, (R^q j)(M))$; since the clopens form a basis for the topology of G, the parenthetical remark in the first paragraph of the proof of Theorem 6 combines with [**7**, Corollaire 4, p. 176] to permit E_2^{pq} to be identified with $\check{H}_T^p(W, (R^q j)(M))$. (As G is paracompact, cf. also [**15**, Chapter VIII].)

Assume, moreover, that W is clopen. The aforementioned parenthetical remark now implies that the spectral sequence collapses: $E_2^{pq} = 0$ whenever $p > 0$. Hence, $E_2^{0,n} \cong E^n$ for all n; i.e., $(R^n j)(M)(W) \cong R^n(\Gamma_W j)(M)$, the promised formula for $(R^n j)(M)$ on the basis of clopens. In particular, j is exact if and only if $\Gamma_W j$ is exact for each clopen subset W of G.

Finally, we claim that, in case G is finite, j is exact if and only if G is the trivial group. Indeed, suppose that j is exact. Consider a short exact sequence $0 \to M \to I \to C \to 0$ in $\bar{M}$, with I injective. Let $W = \{g\}$ be a singleton subset of G. Exactness of $\Gamma_W j$ implies exactness of the sequence $0 \to j(M)(W) \to j(I)(W) \to j(C)(W) \to 0$, that is (arguing as in (a)), exactness of $0 \to M^g \to I^g \to C^g \to 0$. (One may also see this without appeal to spectral sequences, by arguing that the sequence of stalks, $0 \to j(M)_g \to j(I)_g \to j(C)_g \to 0$, is exact.) Hence, $H^1(E, -) = 0$ for each cyclic subgroup E of G. By dimension-shifting, each E has strict cohomological dimension 0 and hence, by [**14**, Proposition 16], is trivial. Then G is trivial, proving the "only if" part of the claim.

Conversely, if G is trivial, then exactness of j amounts to exactness of $\Gamma_{\{1\}} j$, since $\Gamma_\phi j = 0$ is exact. Now, $\Gamma_{\{1\}} j$ converts an exact sequence $M_1 \to M_2 \to M_3$ from $\bar{M}$ into the sequence $(M_1)^1 \to (M_2)^1 \to (M_3)^1$ in Ab. Triviality of G permits $\bar{M}$ to be identified with Ab, so that the two sequences become identified, making inheritance of exactness obvious, and completing the proof.

We conjecture that j fails to be (right) exact for any infinite profinite group G.

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