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Exceptional Θ-Correspondences I

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Abstract. Let G be either a split $\mathrm{SO}(2n+2)$, or a split adjoint group of type E_n, $(n=6,7,8)$, over a p-adic field. In this article we study correspondences arising by restricting the minimal representation of G to various dual pairs in G.

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Introduction

Let F be a p-adic field, and G be either a split $\mathrm{SO}(2n+2)$, or a split adjoint group of type of E_n, $(n=6,7,8)$ over F. In this article we study correspondences arising by restricting the minimal representation (Π, V) of G (introduced in [KS]) to various dual pairs in G.

Recall, from [S1], how one can measure the size of a smooth, admissible representation E of G: Fix K_n, $(n=1,2\ldots)$, a chain of principal congruence subgroups of G. Let E^{K_n} be the subspace of K_n-fixed vectors in E. Obviously,

$$E=\cup_{n=1}^{\infty}E^{K_n},\ E^{K_n}\subseteq E^{K_{n+1}}\quad\text{and}\quad\dim\,E^{K_n}<\infty.$$

Moreover, if the representation E has finite length, it follows from the character expansion of E that

$$\dim E^{K_n}=P(q^n)\quad\text{if}\quad n\gg 0,$$

where q is the order of the residual field of F, and P is a polynomial with the degree equal $\frac{1}{2}$ the dimension of a nilpotent orbit which appears as a leading term in the character expansion of E. The leading term for V is the unique minimal, non-trivial nilpotent orbit of G, so $\dim V^{K_n}$ grows at the slowest possible rate (amongst non-trivial representations). It is precisely in this sense that V is an analogue of the Weil representation of Sp_{2n}. So Rallis has asked if one can use V to obtain new dual pair correspondences.

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Recall that if $A \times B$ is a dual pair in $\mathrm{Sp}(2n)$ and π an irreducible representation of A, we say that an irreducible representation σ of B is a Θ-lift of π if $\pi \otimes \sigma$ is a quotient of the Weil representation (see [H1]). Let $\Theta(\pi)$ be the set of all such σ.

In this paper we study the restricition of V to the following dual pairs

$$\mathrm{SO}(2n-1) \times \mathrm{SO}(3) \subset \mathrm{SO}(2n+2)$$

and

$$G_2 \times H \tag{1}$$

with H adjoint,

$$H = \begin{cases} \mathrm{PGL}_3 & \text{if } G = E_6, \\ \mathrm{PGSp}_6 & \text{if } G = E_7 \\ F_4 & \text{if } G = E_8. \end{cases}$$

Although these exceptional dual pairs have been known, at least at the level of Lie algebras, since the work of Dynkin [D], the reader might not be very familiar with them. So, as an illustration, we describe the dual pair

$$G_2 \times \mathrm{PGL}_3.$$

First of all, let $\mathbb{O}$ be an 8-dimensional algebra of Octonions over F (see Section 3). Then G_2 is the automorphism group of $\mathbb{O}$ [J3]

$$G_2 = \mathrm{Aut}(\mathbb{O}).$$

Next, let J be the exceptional Jordan algebra, consisting of 3×3 Hermitian symmetric matrices with coefficients in the algebra Octonions $\mathbb{O}$ over F

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix},$$

where a, b, c are in F, and x, y, z are in $\mathbb{O}$ (the reader can find more details in Section 3). The algebra J plays an important role in this paper. Let

$$\det : J \to F$$
$$\det(A) = abc + \mathrm{Tr}(xyz) - a\mathbb{N}(x) - b\mathbb{N}(y) - c\mathbb{N}(z),$$

be a cubic F-valued form on J. Now, the group of isogenies of the form det is a reductive group of type E_6 (see [A1]). Obviously, this group contains G_2; the group G_2 acts on the entries of A in J. Also, GL_3 acts faithfully on J by the formula

$$A \mapsto \det(g)^{-1} g A g^t,$$

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where $\det(g)$ and g^t are the determinant and the transpose of the 3×3 matrix g, respectively. Clearly, these two actions commute, and the center of GL_3 coincides with the center of the reductive group. The dual pair $G_2 \times \mathrm{PGL}_3$ is obtained by passing to the adjoint quotients.

In this paper we first compute Θ-lifts of tempered spherical representations of $\mathrm{SO}(3) \cong \mathrm{PGL}_2$ to $\mathrm{SO}(2n-1)$, by restricting the minimal representation of $\mathrm{SO}(2n+2)$. This is the simplest case and as such it is a good introduction to exceptional dual pairs which form a more interesting part of this work.

We then compute Θ-lifts of tempered spherical representations of PGL_3 to G_2. In particular, for such representations, this lift is functorial for the homomorphism

$$\mathrm{SL}_3(\mathbb{C}) \to G_2(\mathbb{C})$$

of the dual Langlands groups ([B]). Recall that spherical representations are parametrized by the Satake parameters, i.e. by semi-simple conjugacy classes in the dual group [Ca]. The main tool is the computation of the Jacquet functor of the minimal representation V with respect to a maximal parabolic subgroup of PGL_3. More precisely, let $\bar{P}$ be the maximal parabolic subgroup of G, whose preimage in the reductive cover is the group stabilizing the 10-dimensional subspace J_{10} of J, consisiting of elements

$$\begin{pmatrix} a & z & 0 \\ \bar{z} & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

([A1], 3.14). Then Theorem 1.1 gives a nice model, not for the minimal representation itself, but its restriction to $\bar{P}$. Since the stabilizer of J_{10} in GL_3 is the maximal parabolic subgroup consisitng of lower-triangular block matrices, and the Levi factor $\mathrm{GL}_2 \times \mathrm{GL}_1$, it follows that

$$(G_2 \times \mathrm{PGL}_3) \cap \bar{P} = G_2 \times \bar{Q},$$

where $\bar{Q}$ is a maximal parabolic subgroup of PGL_3, hence this model can be used, in a manner analogous to what is done in the classical case (Kudla [Ku] and Rallis [Ra]), to compute the Jacquet functor of V for $\bar{Q}$.

Next, we compute Θ-lifts of tempered spherical representations of $\mathrm{PGS}p_6$. Again, the main tool is a computation of the Jacquet functor, this time with respect to the Siegel maximal parabolic subgroup of PGSp_6. We finish the paper by computing Θ-lifts of tempered spherical representations of G_2 to H, in all three cases (assuming that $p \neq 2$). In particular, for such representations, the lift from G_2 to PGSp_6 obtained by restricting the minimal representation of E_7 is functorial for the homomorphism

$$G_2(\mathbb{C}) \to \mathrm{Spin}_7(\mathbb{C})$$

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of their dual groups.

Local computations are, in a way, a preparation for global correspondences (i.e. correspondences of automorphic forms). So it is worth mentioning that in a forthcoming work, D. Ginzburg, S. Rallis and D. Soudry are studying a global variant of exceptional correspondences. Another possible global application has recently been initiated by B. Gross in connection to a realization of a G_2-motive [G1]. Also, in [G2], B. Gross has given a conjectural answer for the exceptional correspondences. The evidence presented in this paper supports his conjectures.

1. Minimal representation

Let G be a simple, split, group of type A_{2n-1}, D_n or E_n. Let $\widehat{G}(\mathbb{C})$ be the dual Langlands group of G (see [B]). By a well known result of Kostant, the conjugacy classes of unipotent elements in $\widehat{G}(\mathbb{C})$ correspond to conjugacy classes of homomorphisms

$$\varphi\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C}).$$

Assume now that φ corresponds to the subregular unipotent orbit. Let

$$s = \varphi\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Then V is the spherical representation of G with the Satake parameter s.

We now describe the character expansion of V. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\langle\,,\rangle$ be the Killing form on $\mathfrak{g}$. Throughout this paper we also fix a non-trivial unitary character

$$\psi\colon F \to \mathbb{C}^\times.$$

Let $C_c^\infty(\mathfrak{g})$ denote the space of locally constant, compactly supported functions on $\mathfrak{g}$. Define the Fourier transform on $C_c^\infty(\mathfrak{g})$ by

$$\widehat{f}(y) = \int_{\mathfrak{g}} f(x)\psi(\langle x, y\rangle)\,\mathrm{d}x,$$

where $\mathrm{d}x$ is a self-dual measure on the vector space $\mathfrak{g}$. Let $\mathcal{O}_{\min}$ be the unique minimal nilpotent G-orbit in $\mathfrak{g}$ and $\mu_{\mathcal{O}_{\min}}$ a G-invariant measure on $\mathcal{O}_{\min}$ normalized as in [MW]. It is shown in [S1], that there exists a lattice L in $\mathfrak{g}$, containing 0, such that

$$\mathrm{Tr}\Pi(f) = \mathrm{Tr}\int_{\mathfrak{g}} f(x)\Pi(\exp x)\,\mathrm{d}x = \int \widehat{f}\mu_{\mathcal{O}_{\min}} + c\widehat{f}(0)$$

for any $f \in C_c^\infty(L)$.

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Let Δ be the Dynkin diagram of G. Mark the diagram Δ as follows. Attach 0 to the unique branch vertex (or to the middle vertex of Δ is the type of G is A_{2n-1}) and 2 to all other vertices. This marking corresponds to the subregular nilpotent orbit [D]. Let $P = MN$ be a maximal parabolic subgroup of G. Let Δ_M be the Dynkin diagram of M. Assume that we are in the following favorable situation:

(1) N is a commutative group.
(2) The marking of Δ corresponding to the subregular nilpotent orbit of G restricts to the marking of Δ_M corresponding to the subregular nilpotent orbit of M.

The possible cases are given by the following table:

G	M	N
D_{n+1}	D_n	F^{2n}
E_6	D_5	F^{16}
E_7	E_6	F^{27}

Here F^{2n} is the standard representation of D_n, F^{16} is a spin-representation of D_5 and F^{27} is isomorphic to the exceptional Jordan algebra. We say that a point in N is singular if it is a highest weight vector for a Borel subgroup of M. Let ω and be the set of singular vectors in N. Note that ω is the smallest non-trivial M-orbit in N. If $G = D_{n+1}$ then ω is the null-cone in F^{2n} of the invariant quadratic form for D_n, with 0 excluded.

THEOREM 1.1. *Let $\bar{P} = M\bar{N}$ be the maximal parabolic subgroup, opposite to P. The minimal representation (Π, V) of G has a $\bar{P}$-invariant filtration*

$$0 \to \mathcal{C}_c^\infty(\omega) \to V \to V_{\bar{N}} \to 0.$$

Here $\mathcal{C}_c^\infty(\omega)$ denotes the space of locally constant, compactly supported functions on ω and $V_{\bar{N}}$ is the space of $\bar{N}$-coinvariants of V (Jacquet functor).

(1) *Let $f \in \mathcal{C}_c^\infty(\omega)$. The action of $\bar{P}$ is given by*

$$\Pi(n)f(x) = \psi(\langle x, \bar{n}\rangle)f(x), \quad \bar{n} \in \bar{N}$$

and

$$\Pi(m)f(x) = |\det(m)|^{s/d} f(m^{-1}xm), \quad m \in M.$$

(2) $$V_{\bar{N}} \cong V(M) \otimes |\det|^{t/d} + |\det|^{s/d},$$

where $V(M)$ is the minimal representation of M (center acting trivially).

Here $\langle\ ,\ \rangle$ is an F-valued pairing between N and $\bar{N}$ induced by the Killing form on $\mathfrak{g}$, and det *is determinant of the representation of M on $\bar{N}$. The values of s and t are given in the following table*

G	s	t
D_{n+1}	$n-1$	1
E_6	4	2
E_7	6	3

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and d is the dimension of N.

Proof. This is just Theorem 6.5 in [S1] if G is E_7. Note however that the other two cases also satisfy the conditions of Proposition 4.1 in [S1]. Hence the proof carries over with no changes. The proof given there, however, is valid only if $p \neq 2$, and this restricition enters through the work of Moeglin and Waldspurger [MW].

Let x be an element in N, and define a character ψ_x of $\bar{N}$ by

$$\psi_x(\bar{n}) = \psi(\langle x, \bar{n} \rangle).$$

Let $V_{\bar{N},\psi_x}$ be the quotient of V by the space spanned by the elements $\{\Pi(\bar{n})v - \psi_x(\bar{n})v \mid \bar{n} \in \bar{N}, v \in V\}$. The key point in the proof of Theorem 6.5 in [S1] is to show that

$$V_{\bar{N},\psi_x} = 0,$$

for $x \neq 0$ and not in ω, i.e. the $\bar{N}$-spectrum of V is supported on the closure of ω. This follows from the character expansion of V and and [MW], if $p \neq 2$.

To extend the theorem to $p = 2$, we use a global argument. Let k be a number field and $\mathbb{A}$ its ring of adelès. Ginzburg, Rallis and Soudry [GRS] have construced a square integrable automorphic form on $G_{\mathbb{A}}$, whose local components are the minimal representations. Arguing as Howe (Lemma 2.4 in [H2]), one shows that if the $\bar{N}$-spectrum is supported on ω at one place, then it is supported on ω at all places. This completes the proof of the theorem.

2. **Dual pair** $\mathrm{SO}(2n-1) \times \mathrm{SO}(3)$

Let $G = \mathrm{SO}(2n+2)$. We have an embedding

$$\mathrm{SO}(2n-1) \times \mathrm{SO}(3) \subseteq \mathrm{SO}(2n+2)$$

given by decomposing the standard representation F^{2n+2} of G as a direct sum of a $2n-1$-dimensional and a 3-dimensional orthogonal subspaces. We will assume that all three orthogonal groups are split.

We identify $\mathrm{SO}(3)$ with PGL_2, and let e, h, f be the standard basis for $\mathrm{sl}(2)$, the Lie algebra of PGL_2. Let $\mathfrak{g}$ be the Lie algebra of G, and define

$$\begin{cases} \bar{\mathfrak{n}} = \{x \in \mathfrak{g} \mid [h, x] = -2x\}, \\ \mathfrak{m} = \{x \in \mathfrak{g} \mid [h, x] = 0\}, \\ \mathfrak{n} = \{x \in \mathfrak{g} \mid [h, x] = 2x\}. \end{cases}$$

Then

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$$

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and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ is the Lie algebra of the maximal parabolic subgroup $P = MN$ defined in Section 1.

Note that $e \in \mathfrak{n}$, $f \in \bar{\mathfrak{n}}$, and their centralizer in M is

$$\mathrm{SO}(2n-1) \times \langle \pm 1 \rangle = C_M(e) = C_M(f).$$

Let $Q = LU = P \cap \mathrm{PGL}_2$. It is a Borel subgroup, and if we represent elements in PGL_2 by 2×2 matrices, we will assume that Q is represented by upper-triangular matrices. In particular, an element in L will be represented by a diagonal matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

PROPOSITION 2.1. *Let V be the minimal representation of G. Let*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in L \subset \mathrm{PGL}_2 \cong \mathrm{SO}(3).$$

Then

(1) $$\Pi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) f(x) = \left|\frac{b}{a}\right|^{n-1} f\left(\frac{b}{a}x\right), \qquad f \in \mathcal{C}_c^\infty(\omega).$$

(2) *The eigenvalues of* $\Pi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)$ *on* $V_{\bar{N}}$ *are* $\left|\frac{b}{a}\right|$ *and* $\left|\frac{b}{a}\right|^{n-1}$.

Proof. This is a special case of Theorem 1.1.

Let χ be a multiplicative character of F. Let $\rho\chi$ denote the character of L defined by

$$\rho\chi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \left|\frac{b}{a}\right|^{1/2} \chi\left(\frac{b}{a}\right).$$

Let $\tau_\chi = \mathrm{Ind}_{\bar{Q}}^{\mathrm{SO}(3)} \rho\chi$. If χ is unitary then τ_χ is an irreducible tempered spherical representation of $\mathrm{SO}(3)$.

Let σ be an irreducible representation of $\mathrm{SO}(2n-1)$. Then, by the Frobenius reciprocity

$$\mathrm{Hom}_{\mathrm{SO}(2n-1)\times \mathrm{SO}(3)}(V, \sigma \otimes \tau_\chi) = \mathrm{Hom}_{\mathrm{SO}(2n-1)\times L}(V_U, \sigma \otimes \rho\chi).$$

Hence, $\sigma \in \Theta(\tau_\chi)$ if and only if $\sigma \otimes \rho\chi$ is a quotient of $V_{\bar{U}}$. Since

$$0 \to \mathcal{C}_c^\infty(\omega)_{\bar{U}} \to V_{\bar{U}} \to V_{\bar{N}} \to 0$$

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we need to understand $\mathcal{C}_c^\infty(\omega)_{\bar{U}}$. Let NN be the complement of $\bar{U}$ in N with respect to the form $\langle\ ,\ \rangle$. Put

$$\omega\omega = \omega \cap NN.$$

LEMMA 2.2.

$$\mathcal{C}_c^\infty(\omega)_{\bar{U}} = \mathcal{C}_c^\infty(\omega\omega).$$

Proof. Let us recall few known facts about Jacquet functors. Let (π, E) be a $\bar{U}$-module. Then $E_{\bar{U}} = E/E(\bar{U})$ where $E(\bar{U})$ can be defined either as the space spanned by the elements $\{\pi(\bar{u})v - v \mid \bar{u} \in \bar{U}, v \in E\}$ or the space of all v such that

$$\int_{\bar{U}_K} \pi(\bar{u})v \,\mathrm{d}\bar{u} = 0$$

for some open compact subgroup $\bar{U}_K \subset \bar{U}$ depending on v (2.33 [BZ]).

Obviously, $\mathcal{C}_c^\infty(\omega\omega)$ is a quotient of $\mathcal{C}_c^\infty(\omega)$ and by Theorem 1.1 (1), $\bar{U}$ acts trivially on $\mathcal{C}_c^\infty(\omega\omega)$. Let $f \in \mathcal{C}_c^\infty(\omega)$ such that $f|_{\omega\omega} = 0$. To prove the lemma, we need to find an open compact subgroup $\bar{U}_K$ such that

$$\int_{\bar{U}_K} \psi(\langle x, \bar{u}\rangle) f(x) d\bar{u} = 0$$

for all $x \in \omega$.

Fix a chain $\{\bar{U}_i\}$, $i \in \mathbb{Z}$, of open compact subgroups of $\bar{U}$ such that

$$\bar{U}_i \subseteq \bar{U}_{i+1} \quad \text{and} \quad \bigcup_i \bar{U}_i = \bar{U}.$$

Let x be such that $f(x) \neq 0$. Since x is not in NN, there exists an open compact subgroup $\bar{U}_x$ in the family, such that $\psi(\langle x, \bar{u}\rangle)$ is a non-trivial character of $\bar{U}_x$. Also, there exists an open compact neighbourhood $\mathcal{O}_x$ of x such that $\psi(\langle y, \bar{u}\rangle)$ is a non-trivial character of $\bar{U}_x$ for any $y \in \mathcal{O}_x$. Since the support of f is compact, a finite collection of $\mathcal{O}_x$ covers the support of f. The union of the corresponding $\bar{U}_x$ is the desired $\bar{U}_K$. The lemma follows.

We can, therefore, summarize the situation with the following proposition.

PROPOSITION 2.3. *$V_{\bar{U}}$ has a filtration with two succesive quotients*

$$\mathcal{C}_c^\infty(\omega\omega), \quad \textit{and} \quad V_{\bar{N}},$$

where $\mathcal{C}_c^\infty(\omega\omega)$ is a submodule, and $V_{\bar{N}}$ a quotient. As $\mathrm{SO}(2n-1) \times L$*-modules:*

$$(1) \quad \Pi_{\bar{U}}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times g\right) f(x) = \left|\frac{b}{a}\right|^{n-1} f\left(\frac{b}{a} g^{-1} x g\right), \quad f \in \mathcal{C}_c^\infty(\omega\omega).$$

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$$(2) \quad V_{\bar{N}} \cong V(M) \otimes \left|\frac{b}{a}\right| + 1 \otimes \left|\frac{b}{a}\right|^{n-1}$$

where $V(M)$ is the minimal representation of M (center acting trivially).

Note that $NN = F^{2n-1}$ and $\omega\omega$ is the null-cone of the $\mathrm{SO}(2n-1)$-invariant quadratic form (with 0 excluded). Let $\mathcal{C}^\infty(\omega\omega)$ be the space of locally constant functions on $\omega\omega$. We can define degenerate principal series representations σ_χ of $\mathrm{SO}(2n-1)$ by

$$\sigma_\chi = \{f \in \mathcal{C}^\infty(\omega\omega) \mid f(cx) = \chi(c)|c|^{(3/2)-n} f(x)\}.$$

Analogously, σ_χ can be defined as a quotient of $\mathcal{C}_c^\infty(\omega\omega)$ consisting of $\bar{f}$ such that

$$\bar{f}(cx) = \chi(c)|c|^{(3/2-n)} \bar{f}(x).$$

If χ is unramified and unitary then σ_χ is an irreducible unitarizable spherical representation by a result of Tadić [T2], Theorem 9.2. We are now ready to state and prove the main result of this section.

PROPOSITION 2.4. *Let χ be an unramified, unitary multiplicative character. Then*

$$\Theta(\tau_\chi) = \{\sigma_\chi\}.$$

Proof. By the Frobenius reciprocity, $\sigma \otimes \tau_\chi$ is a quotient of V if and only if $\sigma \otimes \rho\chi$ is a quotient of $V_{\bar{U}}$. We need the following.

LEMMA 2.5. *Let Γ be a p-adic reductive group and*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

a sequence of smooth Γ-modules. Assume that there exists an element T in the Bernstein center (see [BD]*) of Γ such that V_3 decomposes as a sum of finitely many eigenspaces for T*

$$V_3 = V_3(\lambda_1) \oplus \cdots \oplus V_3(\lambda_n).$$

Let W be a smooth Γ-module on which T acts as a scalar μ. If μ is different form all λ_i, then W is a quotient of V_2 if and only if it is a quotient of V_1.

Proof. Obvious.

We apply the lemma to

$$0 \to \mathcal{C}_c^\infty(\omega\omega) \to V_{\bar{U}} \to V_{\bar{N}} \to 0$$

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and $\Gamma = L$. Let ϖ be the uniformizing element of F, $|\varpi| = q^{-1}$. Put

$$T = \Pi \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $q^{1/2}\chi(\varpi)$ is different from q^{n-1} and q if χ is unitary, it follows from the lemma that $\sigma \otimes \tau_\chi$ is a quotient of V if and only if $\sigma \otimes \rho\chi$ is quotient of $\mathcal{C}_c^\infty(\omega\omega)$. This implies the proposition.

3. Group E_6

In this section we describe a reductive group G, whose quotient modulo its center is the split adjoint group of type E_6.

We recall from [Cx] that the algebra $\mathbb{O}$ is a non-associative division algebra of rank 8 over F

$$\begin{cases} F + Fe_1 + Fe_2 + Fe_3 + Fe_4 + Fe_5 + Fe_6 + Fe_7 \\ e_i^2 = -1 \quad \text{all } i \\ e_i \cdot (e_{i+1} \cdot e_{i+3}) = (e_i \cdot e_{i+1}) \cdot e_{i+3} \quad \text{all } i \text{ (mod 7)}. \end{cases}$$

By

$$\bar{e}_i = -e_i$$

one defines the standard F-linear anti-involution of $\mathbb{O}$. On $\mathbb{O}$, we have the trace

$$\begin{aligned} &\mathrm{Tr} : \mathbb{O} \to F, \\ &x \mapsto x + \bar{x}, \end{aligned}$$

which is F-linear, and the norm

$$\begin{aligned} &\mathbb{N}\colon \mathbb{O} \to F, \\ &x \mapsto x \cdot \bar{x} = \bar{x} \cdot x, \end{aligned}$$

which satisfies $\mathbb{N}(x \cdot y) = \mathbb{N}(x)\mathbb{N}(y)$. Although the multiplication is neither commutative nor associative, we have

$$\begin{aligned} &\mathrm{Tr}(x \cdot y) = \mathrm{Tr}(y \cdot x) \\ &\mathrm{Tr}(x \cdot (y \cdot z)) = \mathrm{Tr}((x \cdot y) \cdot z). \end{aligned}$$

We denote the latter rational number simply by $\mathrm{Tr}(xyz)$.

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The exceptional Jordan algebra is the vector space of 3×3 Hermitian symmetric matrices over the algebra of Octonions $\mathbb{O}$ over F

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix},$$

where a, b, c are in F and x, y, z are in $\mathbb{O}$. The multiplication in J is given by the formula

$$A \circ B = \tfrac{1}{2}(AB + BA),$$

where AB and BA stand for the ordinary multiplication of 3×3 matrices.

The determinant

$$\det(A) = abc + \mathrm{Tr}(xyz) - a\mathbb{N}(x) - b\mathbb{N}(y) - c\mathbb{N}(z).$$

gives an F-valued cubic form on J. The group G can be defined as the group of linear transformations g of J wich satisfy

$$\det(g(A)) = \lambda(g)\det(A)$$

for a similitude $\lambda(g)$ in $F^{\times}$ [A1]. The cubic form defines a symmetric trilinear form (A, B, C) on J (the Dickson form) normalized by

$$(A, A, A) = 6\det(A).$$

Let P be the maximal parabolic subgroup in G stabilizing the line through

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This parabolic subgroup is opposite to the one defined in the introduction, as the stabilizer of J_{10}, the 10-dimensional subspace of J consisting of all matrices in J such that the coefficients in the third row and the third column are zero. In particular, we have a decomposition $P = MN$, where the Levi factor M is defined as the stabilizer in P of J_{10}. Then $[M, M] = \mathrm{Spin}_{10}$ and $Q_D(X, Y) = (X, Y, D)$ is a Spin_{10}-invariant quadratic form on J_{10}. The unipotent radical N, can be identified with the space in J consisting of matrices

$$\begin{pmatrix} 0 & 0 & \bar{y} \\ 0 & 0 & x \\ y & \bar{x} & 0 \end{pmatrix} \tag{3.1}$$

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([A1], (4.7), (2)). The group G has three non-trivial orbits on J. Let Ω be the smallest orbit. It is the orbit of D and its dimension is 17. It can be characterized as the set of all non-zero matrices A in J such that

$$A^2 = \mathrm{Tr}(A)A,$$

or, in terms of the Dickson form,

$$(A, A, X) = 0 \quad \text{for all } \; X \in J.$$

On the other hand, M has two non-trivial orbits on N and the smaller (using the identification 3.1) is

$$\omega = \Omega \cap N.$$

4. Dual pair $G_2 \times \mathrm{PGL}_3$

Let G be the reductive group described in the previous section. We first describe a closed subgroup

$$G_2 \times \mathrm{GL}_3 \subset G.$$

The exceptional group G_2 is the automorphism group of the Octonion algebra $\mathbb{O}$, so the action of G_2 on the entries of matrices in J induces the inclusion $G_2 \subset G$.

On the other hand, GL_3 acts on J by

$$A \mapsto \det(g)^{-1}\, gAg^t,$$

where $\det(g)$ and g^t are the determinant and the transpose of g in GL_3. Note that this action of GL_3 is faithful, and

$$\lambda(g) = \det(g)^{-1}.$$

Since J is an irreducible G-module, the center of G consists of transformations $A \mapsto zA$, where $z \in F^\times$. Hence, it coincides with the center of GL_3, and we have a closed subgroup (dual pair)

$$G_2 \times \mathrm{PGL}_3$$

in the adjoint group of type E_6. However, we shall continue working with G, as it is more convenient.

Let $Q = LU = P \cap \mathrm{GL}_3$ be the corresponding maximal parabolic in GL_3. It consists of lower-triangular block matrices, and the Levi factor $L = \mathrm{GL}_2 \times \mathrm{GL}_1$. The unipotent radical $U \subset N$ can be identified with the space of all matrices (3.1) in J such that x and y are in F. In particular, if we identify N with pairs of Octonions $(y, \bar{x})$, the conjugation action of L on N is given by

$$l_1(y, \bar{x})l_2^{-1}, \in l_2 \times l_1\, \mathrm{GL}_2 \times \mathrm{GL}_1.$$

Using this identifications, Theorem 1.1 can be reformulated as:

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PROPOSITION 4.1. *Let V be the minimal representation of G. Let $f \in \mathcal{C}_c^\infty(\omega) \subset V$. Then*

$$(1) \qquad \Pi(g)f((y,\bar{x})) = f((g^{-1}y, g^{-1}\bar{x})), \quad g \in G_2.$$

$$(2) \qquad \Pi(l_2 \times l_1)f((y,\bar{x})) = \frac{|\det l_2|^2}{|l_1|^4} f(l_1^{-1}(y,\bar{x})l_2),$$
$$l_2 \times l_1 \in \mathrm{GL}_2 \times \mathrm{GL}_1,$$

where det *denotes the usual determinant of* 2×2 *matrices.*

Let π be an irreducible representation of GL_3. Assume that π is isomorphic to $\mathrm{Ind}_{\bar{Q}}^{\mathrm{GL}_3}(\tau)$ for some irreducible representation τ of L. Let σ be an irreducible representation of G_2. By the Frobenius reciprocity

$$\mathrm{Hom}_{G_2 \times \mathrm{GL}_3}(V, \sigma \otimes \mathrm{Ind}_{\bar{Q}}^{\mathrm{GL}_3}(\tau)) = \mathrm{Hom}_{G_2 \times L}(V_{\bar{U}}, \sigma \otimes \tau).$$

Hence, $\sigma \otimes \pi$ is a quotient of V if and only if $\sigma \otimes \tau$ is a quotient of $V_{\bar{U}}$. Since

$$0 \to \mathcal{C}_c^\infty(\omega)_{\bar{U}} \to V_{\bar{U}} \to V_{\bar{N}} \to 0$$

we need to understand $\mathcal{C}_c^\infty(\omega)_{\bar{U}}$. Let NN be the orthogonal complement of $\bar{U}$ in N with respect to the pairing between N and $\bar{N}$, induced by the Killing form. Since NN is the unique 14-dimensional G_2-invariant subspace, it is given by the space of all matrices (3.1) in J such that x and y are traceless Octonions, i.e. $\bar{x} = -x$ and $\bar{y} = -y$.

Let

$$\omega\omega = \omega \cap NN.$$

As in Lemma 2.2

$$\mathcal{C}_c^\infty(\omega)_{\bar{U}} = \mathcal{C}_c^\infty(\omega\omega),$$

and we have to understand the structure of $G_2 \times L$ orbits on $\omega\omega$.

PROPOSITION 4.2. (1) $\omega\omega = \{(y,\bar{x}) \neq (0,0) \mid \bar{x} = -x, \bar{y} = -y; x^2 = y^2 = x \cdot y = 0\}$.

(2) *Let AA and BB be the subsets of $\omega\omega$ consisting all pairs $(y,\bar{x})$ such that the space $Fx + Fy$ has dimension* 2 *and* 1 *respectively. Clearly,*

$$\omega\omega = AA \cup BB \ (BB \text{ is contained in the closure of } AA).$$

Moreover, AA and BB are $G_2 \times \mathrm{GL}_2$-orbits.

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Proof. Let n be an element in N. As before, represent it as

$$n = \begin{pmatrix} 0 & 0 & \bar{y} \\ 0 & 0 & x \\ y & \bar{x} & 0 \end{pmatrix}.$$

Now, n is in NN precisely when x and y are traceless. Furthermore, since n is traceless, it is in Ω if and only if $n^2 = 0$. But this is equivalent to $x^2 = y^2 = x{\cdot}y = 0$. The first part of the proposition is proved.

We go on to observe that G_2 has three orbits on the set of spaces of traceless Octonions with the property that the Octonion multiplication is trivial. These are characterized by their dimension; the possible choices being $0, 1, 2$. The stabilizers of the nontrivial spaces are the maximal parabolics of G_2.

Let z be a traceless Octonion such that $z^2 = 0$. Let P_1 be the maximal parabolic subgroup of G_2 stabilizing the line Fz. The Levi factor of P_1 is 'spanned' by a long root. Consider

$$B = \{(az, 0) \mid a \in F \text{ and } a \neq 0\}.$$

Let QQ be the maximal parabolic subgroup of $\mathrm{GL}_2 \subset L$ stabilizing B. Then $P_1 \times QQ$ acts transitively on B and

$$BB = (G_2 \times \mathrm{GL}_2) \times_{(P_1 \times QQ)} B.$$

Let x and y be two traceless and linearly independent Octonions such that $x^2 = y^2 = x \cdot y = 0$. Let P_2 be the maximal parabolic subgroup of G_2 stabilizing the space $Fx + Fy$. The Levi factor of P_2 is 'spanned' by a short root. Consider

$$A = \{(ax + by, cx + dy) \mid a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Then $P_2 \times \mathrm{GL}_2$ acts transitively on A and since G_2 acts transitively on the set of all two-dimensional spaces of traceless Octonions with trivial multiplication,

$$AA = G_2 \times_{P_2} A.$$

The proposition is proved.

We can now summarize the structure of $V_{\bar{U}}$ as a $G_2 \times \mathrm{GL}_2$-module in the following theorem (compare [Ku]. Here GL_2 is the first factor of $L = \mathrm{GL}_2 \times \mathrm{GL}_1$.

THEOREM 4.3. *$V_{\bar{U}}$ has a filtration with succesive quotients:*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB), \quad \textit{and}\ V_{\bar{N}},$$

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where $\mathcal{C}_c^\infty(AA)$ *is a submodule, and* $V_{\bar{N}}$ *a quotient. Moreover*

(1) $\mathcal{C}_c^\infty(AA) = \mathrm{ind}_{P_2}^{G_2}(\mathcal{C}_c^\infty(A)) \otimes |\det|^2,$

(2) $\mathcal{C}_c^\infty(BB) = \mathrm{ind}_{P_1\times QQ}^{G_2\times GL_2}(\mathcal{C}_c^\infty(B)) \otimes |\det|^2,$

(3) $V_N \cong V_M \otimes |\det| + 1 \otimes |\det|^2,$

as $G_2 \times \mathrm{GL}_2$*-modules.*

Proof. We have

$$0 \to \mathcal{C}_c^\infty(\omega\omega) \to V_{\bar{U}} \to V_{\bar{N}} \to 0$$

and

$$0 \to \mathcal{C}_c^\infty(AA) \to \mathcal{C}_c^\infty(\omega\omega) \to \mathcal{C}_c^\infty(BB) \to 0.$$

Parts (1) and (2) follow from the description of AA and BB given in the proof of Proposition 4.2. The theorem is proved.

We now give the first application. Namely, we show the following.

THEOREM 4.4. *Let* $\Phi\colon \mathrm{SL}_3(\mathbb{C}) \to G_2(\mathbb{C})$ *be the standard inclusion of the dual groups of* PGL_3 *and* G_2*;* $\mathrm{SL}_3(\mathbb{C})$ *is generated by the long root spaces of* $G_2(\mathbb{C})$*. Let* π *be a tempered spherical representation of* PGL_3*. Let* $s \in \mathrm{SL}_3(\mathbb{C})$ *be its Satake parameter. Let* π' *be the tempered spherical representation of* G_2 *whose Satake parameter is* $s' = \Phi(s)$*. The representation* π' *is also called the Langlands lift of* π*. Then*

$$\Theta(\pi) = \{\pi'\}.$$

Proof. We first describe the Langlands lift from PGL_3 to G_2 of a spherical tempered representation π. Write

$$\pi = \mathrm{Ind}_{\bar{Q}}^{\mathrm{GL}_3}(\tau).$$

Note that there are up to three different choices for τ. Since π is a representation of PGL_3, the representation τ is completely determined by its restriciton to GL_2 (the first factor of L). Henceforth, we think of τ as a representation of GL_2, and let $(\chi_1|\cdot|^{1/2}, \chi_2|\cdot|^{1/2})$ be its parameter, where χ_1 and χ_2 are unitary characters; $|\cdot|^{1/2}$ comes from the normalization of the parabolic induction, χ_1 and χ_2 are unitary because π is tempered.

Let U_2 be the unipotent radical of P_2. It is a Heisenberg group. Let Z be the center of U_2. The Levi factor GL_2 of P_2 acts on Z via the character det, and its

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action on U_2/Z is isomorphic to $S^3(F^2) \otimes \det^{-1}$. It follows that the normalization of the parabolic induction in this case is given by $\rho_{U_2} = |\det|^{3/2}$.

Let τ' be a spherical representation of GL_2 with the parameter $(\chi_1^{-1}|\cdot|^{3/2}, \chi_2^{-1}|\cdot|^{3/2})$, and let

$$\pi' = \mathrm{Ind}_{P_2}^{G_2}(\tau').$$

The representation π' is tempered and, thus, irreducible by a result of Keys [Ke].

The representation π' is the Langlands lift of π. Indeed, the Satake parameter of π' is

$$\begin{pmatrix} \chi_1(\varpi) & 0 \\ 0 & \chi_2(\varpi) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \subset G_2(\mathbb{C}),$$

where $\mathrm{GL}_2(\mathbb{C})$ is the Levi factor of the parabolic subgroup $P_1(\mathbb{C})$ ('spanned' be a long root). Since $\mathrm{SL}_3(\mathbb{C})$ is 'spanned' by long roots of $G_2(\mathbb{C})$, π' must be a lift of a representation of PGL_3 induced from Q: π or π^*. Note that replacing the pair (χ_1, χ_2) by $(\chi_1^{-1}, \chi_2^{-1})$ does not change π' but replaces π by π^*.

We now proceed with the proof of the Theorem. As we have remarked earlier, $\sigma \otimes \pi$ is a quotient of V if and only if $\sigma \otimes \tau$ is a quotient of $V_{\bar{U}}$. We need the following lemma.

LEMMA 4.5. *Let σ be a representation of G_2, and τ the representation of* GL_2 *defined above. Then $\sigma \otimes \tau$ is a quotient of $V_{\bar{U}}$ if and only if it is a quotient of $\mathcal{C}_c^\infty(AA)$.*

Proof. We again use Lemma 2.5, so we need to construct appropriate operators. Recall that the component of the Bernstein center of GL_2 acting non-trivially on representations generated by their Iwahori-fixed vectors is isomorphic to

$$\mathbb{C}[x, x^{-1}, y, y^{-1}]^W$$

where $W = \{1, w\}$, $w(x) = y$ and $w(y) = x$ is the Weyl group of GL_2. Let I be the Iwahori subgroup of GL_2. Let ϖ be the uniformizing element in F. Then any unramified character χ is determined by its value on ϖ. If E is a subquotient of an induced representation with the parameter (χ_1, χ_2) then

$$(x + y) = \chi_1(\varpi) + \chi_2(\varpi) \quad \text{and} \quad xy = \chi_1(\varpi)\chi_2(\varpi)$$

on E. Let

$$T_1 = q^{1/2}(x^{-1} + y^{-1}) - q^{-1}(xy)^{-1},$$

where $q = |\varpi|^{-1}$. On τ, T_1 acts as the scalar

$$z = q(\chi_1(\varpi)^{-1} + \chi_2(\varpi)^{-1}) - \chi_1(\varpi)^{-1}\chi_2(\varpi)^{-1}.$$

We need the following:

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LEMMA 4.6. *Let* $z_1 = a_1 + ib_1$ *and* $z_2 = a_2 + ib_2$ *be two complex numbers of norm* 1. *Let* z *be the complex number*

$$z = q(z_1 + z_2) - z_1 z_2.$$

Then $\Re(z) < 2q$, *where* $\Re(z)$ *is the real part of* z.

Proof. The lemma follows from the sequence of inequalities:

$$\begin{aligned}\Re(z) &= q(a_1 + a_2) - a_1 a_2 + b_1 b_2 \leqslant q(a_1 + a_2) - a_1 a_2 + \frac{b_1^2 + b_2^2}{2} \\ &= q(a_1 + a_2) - a_1 a_2 + \frac{(1 - a_1^2) + (1 - a_2^2)}{2} \\ &= (q - 1)(a_1 + a_2) + \frac{3}{2} - \frac{(1 - a_1 - a_2)^2}{2} \\ &< (q - 1)2 + 2 = 2q.\end{aligned}$$

Since $\mathcal{C}_c^\infty(B)$ is the regular representation of GL_1, the GL_2-module $\mathcal{C}_c^\infty(BB)$ consists of induced representations whose inducing parameters are $(|\cdot|^{3/2}, \chi)$. Such an induced representation has an Iwahori-fixed vector only when χ is unramified, and then, T_1 acts as

$$q^2 = q^{1/2}(q^{3/2} + \chi(\varpi)^{-1}) - q^{1/2}\chi(\varpi)^{-1}.$$

Since $2q \leqslant q^2$, the eigenvalue q^2 of T_1 on the Iwahoric component of $\mathcal{C}_c^\infty(BB)$ is different from the eigenvalue z of T_1 on τ.

Let

$$T_2 = xy.$$

Then T_2 acts on τ as $|\varpi|\chi_1(\varpi)\chi_2(\varpi)$ which is different from $|\varpi|^2$ and $|\varpi|^4$, the eigenvalues of T_2 on the Iwahoric component of $V_{\bar{N}}$.

Lemma 4.5 follows from Lemma 2.5 applied to T_1 and T_2.

We can now finish the proof of the theorem. Note that $\mathcal{C}_c^\infty(A)$ is the regular representation of GL_2. After taking into account the twist with $|\det|^2$, it follows that $\sigma \otimes \tau$ is a quoteint of

$$\mathrm{Ind}_{P_2}^{G_2}(\tau') \otimes \tau = \pi' \otimes \tau.$$

Therefore $\sigma \cong \pi'$, and the theorem is proved.

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5. Dual pair $G_2 \times \mathrm{PGSp}_6$

Let G be the split adjoint group of type E_7. Let $P = MN$ be the maximal parabolic subgroup of G defined in the Section 1. Then M is the group introduced in Section 3, i.e. it is the group of isogenies of the cubic form on J. The unipotent radical N is commutative, and isomorphic to J as an M-module. Let $G_2 \times \mathrm{GL}_3$ be the dual pair in M, described in Section 4. The centralizer of G_2 in G is $\mathrm{PGSp}(6)$. This can be easily seen on the level of Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G. Then

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n},$$

where $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ is the Lie algebra of P. Since G_2 is contained in M, we can write its centralizer in $\mathfrak{g}$ as

$$C_{\mathfrak{g}}(G_2) = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u},$$

where $\mathfrak{l} \subset \mathfrak{m}$, $\bar{\mathfrak{u}} \subset \bar{\mathfrak{n}}$ and $\mathfrak{u} \subset \mathfrak{n}$. Obviuosly, $\mathfrak{l} = \mathrm{gl}(3)$, and $\mathfrak{u} \subset \mathfrak{n}$ corresponds to the inclusion $J_6 \subset J$ of the subalgebra consisiting of 3×3 symmetric matrices, since $J^{G_2} = J_6$. Therefore $C_{\mathfrak{g}}(G_2) = \mathrm{sp}(6)$ whose the Siegel parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$.

Let $Q = LU = \mathrm{PGL}_6 \cap P$ be the Siegel parabolic subgroup of PGSp_6, corresponding to the Lie algebra $\mathfrak{q}$. Remarkably, the group L is isomorphic to GL_3: Recall that the Levi factor of the Siegel parabolic in Sp_{2n} is GL_n. Let $\mathbb{Z}^n$ be the standard co-character lattice for GL_n. Then

$$\Lambda_n = \mathbb{Z}(\tfrac{1}{2}, \dots, \tfrac{1}{2}) + \mathbb{Z}^n \subset \mathbb{R}^n$$

is a co-character lattice of the Levi factor in PGSp_{2n}. For $n = 3$, however, these two lattices are isomorphic

$$T: \mathbb{Z}^3 \to \Lambda_3,$$

where T is given by the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Since the isomorphism T commutes with the action of S_3, the Weyl group of GL_3, we have

$$L \cong \mathrm{GL}_3.$$

With this identification, the conjugation action of L on $N \cong J$ is given by

$$gAg^{-1} = \det(g)^{-1}\, gAg^t.$$

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Again, as in Sections 2 and 4, we want to compute $V_{\bar{U}}$. Since

$$0 \to \mathcal{C}_c^\infty(\omega)_{\bar{U}} \to V_{\bar{U}}, \to V_{\bar{N}} \to 0,$$

we need to understand $\mathcal{C}_c^\infty(\omega)_{\bar{U}}$. Let NN be the orthogonal complement of $\bar{U}$ in N with respect to the form $\langle\ ,\ \rangle$. Since $\bar{U}$ can be identified with J_6, and the pairing with $\mathrm{Tr}(AB)$, it follows that

$$NN = \left\{ \begin{pmatrix} 0 & z & \bar{y} \\ \bar{z} & 0 & x \\ y & \bar{x} & 0 \end{pmatrix} \mid \bar{x} = -x, \bar{y} = -y \quad \text{and} \ \bar{z} = -z \right\}.$$

Let $\omega\omega = \omega \cap NN$. As in Lemma 2.2

$$\mathcal{C}_c^\infty(\omega)_{\bar{U}} = \mathcal{C}_c^\infty(\omega\omega)$$

and we have

PROPOSITION 5.1. *Identify NN with the set of triples of traceless Octonions (x, y, z). Let $f \in \mathcal{C}_c^\infty(\omega\omega) \subset V_{\bar{U}}$. Then*

$$(1) \quad \Pi_{\bar{U}}(g) f((x, y, z)) = f((g^{-1}x, g^{-1}y, g^{-1}z)), \quad g \in G_2.$$

$$(2) \quad \Pi_{\bar{U}}(g) f((x, y, z)) = |\det g|^2 f((x, y, z)g), \quad g \in \mathrm{GL}_3.$$

Proof. This is a reformulation of Theorem 1.1. Note, however, that in this case we are already describing the action on $\mathcal{C}_c^\infty(\omega\omega)$.

We have to understand the structure of $G_2 \times \mathrm{GL}_3$ orbits on $\omega\omega$.

PROPOSITION 5.2. (1) $\omega\omega = \{(x, y, z) \mid \bar{x} = -x, \bar{y} = -y, \bar{z} = -z; x^2 = y^2 = z^2 = x \cdot y = y \cdot z = z \cdot x = 0\}$.

(2) *Let AA and BB be the subsets of $\omega\omega$ consisting of all triples (x, y, z) such that the space $Fx + Fy + Fz$ has dimension* 2 *and* 1 *respectively. Then*

$$\omega\omega = AA \cup BB.$$

Moreover, AA and BB are $G_2 \times \mathrm{GL}_2$-orbits.

Proof. Let

$$n = \begin{pmatrix} 0 & z & \bar{y} \\ \bar{z} & 0 & x \\ y & \bar{x} & 0 \end{pmatrix} \in \omega\omega.$$

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Since n is in NN, x, y and z are traceless. Furthermore, it is a traceless matrix in Ω, hence $n^2 = 0$. But this is equivalent to $x^2 = y^2 = z^2 = x \cdot y = y \cdot z = z \cdot x = 0$. The first part of the proposition is proved.

Again, recall that G_2 has three orbits on the set of spaces of traceless Octonions with the property that the Octonion multiplication is trivial. These are characterized by their dimension; the possible choices being $0, 1, 2$. The stabilizers of the nontrivial spaces are the maximal parabolics of G_2. It follows that x, y and z are linearly dependent. Hence $\omega\omega = AA \cup BB$. It remains to show that AA and BB are single orbits. The proof is analogous to the proof of Proposition 4.2.

Let z be a traceless Octonion such that $z^2 = 0$. Let P_1 be the maximal parabolic subgroup of G_2 stabilizing the line Fz. Consider

$$B = \{(az, 0, 0) \mid a \in F \text{ and } a \neq 0\}.$$

Let Q_1 be the maximal parabolic of GL_3 stabilizing B. Then $P_1 \times Q_1$ acts transitively on B and

$$BB = (G_2 \times \mathrm{GL}_3) \times_{(P_1 \times Q_1)} B.$$

Let x and y be two traceless and linearly independent Octonions such that $x^2 = y^2 = y \cdot z = 0$. Let P_2 be the maximal parabolic subgroup of G_2 stabilizing the space $Fx + Fy$. Consider

$$A = \{(ax + by, cx + dy, 0) \mid a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Let Q_2 be the maximal parabolic subgroup of GL_3 stabilizing A. Then $P_2 \times Q_2$ acts transitively on A and

$$AA = (G_2 \times \mathrm{GL}_3) \times_{(P_2 \times Q_2)} A.$$

The proposition is proved.

We can now summarize the structure of $V_{\bar{U}}$ as a $G_2 \times \mathrm{GL}_3$-module.

THEOREM 5.3. *$V_{\bar{U}}$ has a filtration with succesive quotients*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB), \quad \textit{and}\ V_{\bar{N}},$$

where $\mathcal{C}_c^\infty(AA)$ is a submodule, and $V_{\bar{N}}$ a quotient. Moreover

$$(1) \quad \mathcal{C}_c^\infty(AA) = \mathrm{ind}_{P_2 \times Q_2}^{G_2 \times \mathrm{GL}_3}(\mathcal{C}_c^\infty(A)) \otimes |\det|^2,$$

$$(2) \quad \mathcal{C}_c^\infty(BB) = \mathrm{ind}_{P_1 \times Q_1}^{G_2 \times \mathrm{GL}_3}(\mathcal{C}_c^\infty(B)) \otimes |\det|^2,$$

$$(3) \quad V_{\bar{N}} \cong V(M) \otimes |\det| + 1 \otimes |\det|^2$$

as $G_2 \times \mathrm{GL}_3$-modules. Here det *denotes the usual determinant of* 3×3 *matrices.*

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We are now ready to state and prove a result about Θ-correspondence.

THEOREM 5.4. *Let* $\Phi\colon G_2(\mathbb{C}) \to \mathrm{Spin}_7(\mathbb{C})$ *be the standard inclusion of the dual groups of* G_2 *and* PGSp_6*;* $G_2(\mathbb{C})$ *fixes a non-zero vector in the* 8*-dimensional spin representation of* $\mathrm{Spin}_7(\mathbb{C})$*. Let* π' *be a tempered spherical representation of* PGSp_6*. Then* $\Theta(\pi')$ *is not empty only if the Satake parameter of* π' *is* $s' = \Phi(s)$ *for some* s*, a Satake parameter of a tempered spherical representation* π *of* G_2*. In that case*

$$\Theta(\pi') = \{\pi\}.$$

Proof. Let π' be a spherical tempered representation of PGSp_6. Every tempered spherical representation of PGSp_6 is fully induced (see [T1] Theorem 7.5), so we can write

$$\pi' = \mathrm{Ind}_{\bar{Q}}^{\mathrm{PGSp}_6}(\tau \otimes |\det|),$$

where τ is a tempered spherical representation of GL_3 (note that $\rho_{\bar{U}} = |\det|$).

Assume now that the parameter of π' is $\Phi(s)$. This means that τ can be taken to be a tempered representation of PGL_3. Moreover, the representation π of G_2 with the parameter s is the Langlands lift of τ. By Theorem 5.3 the minimal representation of E_6 (twisted by $|\det|$) is a quotient of $V_{\bar{U}}$, so it follows from Theorem 4.4, and the Frobenius reciprocity that

$$\{\pi\} \subseteq \Theta(\pi').$$

The rest of the theorem follows from the knowledge of $V_{\bar{U}}$. Indeed, let σ be in $\Theta(\pi')$. Then, by the Frobenius reciprocity, $\sigma \otimes (\tau \otimes |\det|)$ is a quotient of $V_{\bar{U}}$, i.e. it is a quotient of one of the three pieces in Theorem 5.3. For example, if it is a quotient of $V_{\bar{N}}$, then $\sigma \cong \pi$, by Theorem 4.4. We leave the details of the other two cases to the reader to check. The reader can also consult [GS] where the map Φ is described, and it is shown that a spherical representation of PGSp_6 (not necessarily tempered) is a quotient of V only when its parameter is of the form $\Phi(s)$.

6. Heisenberg parabolic of G

In this section we prove a variant of Theorem 1.1 for the maximal parabolic subgroup P of G, whose unipotent radical N is a Heisenberg group. We call this parabolic subgroup the Heisenberg parabolic subgroup.

Let $\mathfrak{g}$ be a simple split exceptional Lie algebra of rank $\geqslant 4$, over F. For our purposes, this algebras can be best desribed in terms of a $\mathbb{Z}/3\mathbb{Z}$-gradation (see [HPS]). Let Δ be the Dynkin diagram of $\mathfrak{g}$. We shall identify it with a set of simple

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roots. Let $\tilde{\alpha}$ be the highest positive root. Let α be the unique simple root not perpendicular to $\tilde{\alpha}$. Let

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$$

be the maximal parabolic subalgebra corresponding the simple root α. Extend Δ by adding $-\tilde{\alpha}$. Let β be the unique simple root not perpendicular to α. Remove the vertex corresponding to the simple root β. The extended diagram breaks into several pieces, one of which is an A_2 diagram corresponding to $\{\alpha, -\tilde{\alpha}\}$. Let $\mathfrak{l} \subset \mathfrak{g}$ be the semi-simple subalgebra, corresponding to the rest of the diagram. Under the adjoint action of sl(3) $\oplus$ $\mathfrak{l}$, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathrm{sl}(3) \oplus \mathfrak{l} \oplus W \otimes I \oplus (W \otimes I)^*,$$

where W is the standard 3-dimensional representation of $sl(3)$. The irreducible $\mathfrak{l}$-module I has unique (up to normalization) $\mathfrak{l}$-invariant symmetric trilinear form on I.

As in [HPS], this $\mathbb{Z}/3\mathbb{Z}$-gradation can be used to construct the dual pair

$$\mathfrak{g}_2 \times \mathfrak{h} \subset \mathfrak{g}.$$

Indeed, choose an element e in I such that $(e, e, e) = 6$ (rescale the form, if needed). The algebra $\mathfrak{h}$ is the centralizer in $\mathfrak{l}$ of e:

$$\mathfrak{h} = C_{\mathfrak{l}}(e).$$

Since the centralizer of $\mathfrak{h}$ in I is Fe, it follows that

$$C_{\mathfrak{g}}(\mathfrak{h}) = \mathrm{sl}(3) \oplus W \oplus W^* \cong \mathfrak{g}_2$$

(for the last isomorphism see [FH], p. 361). The possible cases are given by the Freudenthal's magic square:

I	$\mathfrak{g}$	$\mathfrak{m}$	$\mathfrak{l}$	$\mathfrak{h}$
J_F	F_4	C_3	A_2	A_1
J_9	E_6	A_5	$A_2 \times A_2$	A_2
J_{15}	E_7	D_6	A_5	C_3
J	E_8	E_7	E_6	F_4

Where J_6 is the vector space over F of 3×3-symmetric matrices, J_9 is the vector space over F of all 3×3-matrices, and J_{15} is the vector space over F of 6×6-skew symmetric matrices.

EXAMPLE: Let $v_1, \ldots, v_6$ be a standard basis of a 6-dimensional vector space over F. Then $J_{15} = \wedge^2 F^6$ with a basis

$$x_{ij} = v_i \wedge v_j \qquad 1 \leqslant i < j \leqslant 6.$$

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The $\mathrm{sl}(6)$-invariant trilinear form on J_{15} is given by

$$(\wedge^2 F^6) \wedge (\wedge^2 F^6) \wedge (\wedge^2 F^6) \to \wedge^6 F^6 \cong F.$$

Then $(e, e, e) = 6$ for

$$e = x_{16} + x_{25} + x_{34},$$

and the centralizer of e in $\mathrm{sl}(6)$ is $\mathrm{sp}(6)$.

The trilinear form on I can be used to define a structure of Jordan algebra of rank 3, with identity e, on I. For example,

$$2\mathrm{Tr}(a) = (a, e, e) \quad \text{and}$$
$$\mathrm{Tr}(ab) = -(a, b, e) + \mathrm{Tr}(a)\mathrm{Tr}(b).$$

Conversly,

$$\begin{aligned}(a, b, c) = {} & 2\mathrm{Tr}(abc) - \mathrm{Tr}(a)\mathrm{Tr}(bc) - \mathrm{Tr}(b)\mathrm{Tr}(ac) \\ & -\mathrm{Tr}(c)\mathrm{Tr}(ab) + \mathrm{Tr}(a)\mathrm{Tr}(b)\mathrm{Tr}(c).\end{aligned}$$

Now, it is a simple matter to check that the following two are equivalent

(1) $a^2 = \mathrm{Tr}(a)a$.
(2) $(a, a, x) = 0$ for all x in I.

These elements are also called rank-one, and they are highest weight vectors in the irreducible $\mathfrak{l}$-module I. Finally, note that the bilinear form $\mathrm{Tr}(ab)$ gives an $\mathfrak{h}$-invariant identification of I and I^*.

Let $\mathfrak{t} \subset \mathrm{sl}(3)$ be the maximal Cartan subalgebra consisiting of diagonal matrices. Let

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{t}.$$

Define

$$\mathfrak{g}(k) = \{x \in \mathfrak{g} \mid [h, x] = kx\}.$$

Since the eigenvalues of h on the standard 3-dimensional representation W of $\mathrm{sl}(3)$ are $-1, 0, 1$, $\mathfrak{g}(k) \neq 0$ for $k = -2, -1, 0, 1, 2$. Also, one easily checks that the maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ is given by

$$\begin{cases} \mathfrak{m} = \mathfrak{g}(0) = I^* \oplus \mathfrak{t} \oplus \mathfrak{l} \oplus I, \\ \mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2). \end{cases}$$

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The nilpotent radical $\mathfrak{n}$ is a Heisenberg Lie algebra, with the center $\mathfrak{z} = \mathfrak{g}(2)$. Furthermore, under the action of $\mathfrak{t} \oplus \mathfrak{l} \subset \mathfrak{m}$, we have a direct sum decomposition

$$\mathfrak{n}/\mathfrak{z} \cong \mathfrak{g}(1) = F \oplus I \oplus I^* \oplus F^*.$$

Let $P = MN$ be the maximal parabolic subgroup of G, with Lie algebra $\mathfrak{p}$. Let Z be the center of N. Let $\bar{P} = M\bar{N}$ be the parabolic subgroup opposite to P, and $\bar{Z}$ the center of $\bar{N}$. The Killing form on $\mathfrak{g}$, defines a non-degenerate pairing $\langle\ ,\ \rangle$ between N/Z and $\bar{N}/\bar{Z}$. Let Ω be the smallest non-trivial M-orbit in N/Z. It is simply the orbit of a highest weight vector.

THEOREM 6.1. ($p \neq 2$ *if* $G = E_8$) *Let* (Π, V) *be the minimal representation of* G. *Let* $\bar{Z}$ *be the center of* $\bar{N}$ *as above. Let* $V_{\bar{Z}}$ *and* $V_{\bar{N}}$ *be the maximal* $\bar{Z}$*-invariant and* $\bar{N}$*-invariant quotients of* V. *Then*

$$0 \to C_c^\infty(\Omega) \to V_{\bar{Z}} \to V_{\bar{N}} \to 0,$$

where $C_c^\infty(\Omega)$ *denotes the space of locally constant, compactly supported functions on* Ω.

(1) *The action of* $\bar{P}$ *on* $C_c^\infty(\Omega)$ *is given by*

$$\Pi_{\bar{Z}}(\bar{n})f(x) = \psi(\langle x, \bar{n}\rangle)f(x), \quad \bar{n} \in \bar{N}$$
$$\Pi_{\bar{Z}}(m)f(x) = |\det(m)|^{s/d} f(m^{-1}xm), \quad m \in M,$$

(2)

$$V_{\bar{N}} \cong V(M) \otimes |\det|^{t/d} + |\det|^{s/d}$$

where $V(M)$ *is the minimal representation of* M *(center acting trivially).*

Here det *is the determinant of the representation of* M *on* $\bar{N}/\bar{Z}$, d *is the dimension of* N/Z. *The values of* s *and* t *are given by the following table.*

G	s	t	d
E_6	4	3	20
E_7	6	4	32
E_8	10	6	56

Proof. Part (2) is Proposition 4.1 [S1]. Next, every element $x \in N/Z$ defines a character

$$\psi_x(y) = \psi(\langle x, y\rangle)$$

of $\bar{N}$.

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LEMMA 6.2. *($p \neq 2$ if $G = E_8$) Let $x \in N/Z$, $x \neq 0$. Then*

$$\dim V_{\bar{N},\psi_x} \leqslant 1,$$

and it is one if and only if $x \in \Omega$.

Proof. If $p \neq 2$, then the character expansion of V and [MW] imply that $\dim V_{\bar{N},\psi_x} = 0$ or 1 and it is one if and only if $x \in \Omega$. Now assume that $G \neq E_8$. Let $P' = M'N'$ be the maximal parabolic subgroup of G as in Theorem 1.1. Then, by Theorem 1.1,

$$0 \to \mathcal{C}_c^\infty(\omega') \to V \to V_{\bar{N}'} \to 0,$$

where ω' is the minimal M'-orbit in N'. Assume that P' and P are in a standard position, i.e. $P' \cap P$ contains a Borel subgroup of G (in particular, Z is the highest root group). Obviously, $\dim V_{\bar{N},\psi_x}$ is constant along M-orbits in N/Z. Since N/Z is an irreducible M-module, in each non-trivial M-orbit in N/Z we can choose x such that the restriction of ψ_x to $\bar{N}' \cap \bar{N}$ is non-trivial. Hence

$$V_{\bar{N},\psi_x} = \mathcal{C}_c^\infty(\omega')_{\bar{N},\psi_x},$$

and $\dim V_{\bar{N},\psi_x}$ clearly does not depend on p. The lemma follows.

Let E be the kernel of the projection of $V_{\bar{Z}}$ onto $V_{\bar{N}}$. Then by Lemma 6.2 $\dim E_{\bar{N},\psi_x} = 0$ or 1 and it is one if and only if $x \in \Omega$. Let $x \in \Omega$. Let M_x be the stabilizer of x in M and δ the character of M_x describing the action of M_x on $E_{\bar{N},\psi_x}$. By the Frobenius reciprocity there exists a non-trivial $\bar{P}$-homomorphism

$$T\colon E \to \mathrm{Ind}_{M_x\bar{N}}^{\bar{P}}(\delta \otimes \psi_x).$$

Let $\mathcal{C}^\infty(\Omega)$ denote the space of locally constant functions on Ω. Note that we have an inclusion

$$\mathrm{Ind}_{M_x\bar{N}}^{\bar{P}}(\delta \otimes \psi_x) \subseteq \mathcal{C}^\infty(\Omega).$$

Let $w \in E$ and $f = T(w)$. We need to show that f is a compactly supported function on Ω. Let $\bar{N}(k)$, $k \in \mathbb{Z}$ be a chain of lattices in $\bar{N}/\bar{Z}$ such that $\cup_k \bar{N}(k) = \bar{N}/\bar{Z}$ and $\cap_k \bar{N}(k) = 0$. Let $N(k)$ be their dual lattices in N/Z. Since E is a smooth module, there exists an integer k_1 depending on w such that $\Pi_{\bar{Z}}(\bar{n})w = w$ for all $n \in \bar{N}(k_1)$. This implies that f is supported inside $N(k_1)$. Since $E_{\bar{N}} = 0$ there exists an integer k_2 depending on w such that

$$\int_{\bar{N}(k_2)} \Pi_{\bar{Z}}(\bar{n})w\,d\bar{n} = 0$$

(see 2.33 [BZ]). This implies that f is supported outside $N(k_2)$. Since Ω is locally closed and the boundary is $\{0\}$, it follows that $f \in \mathcal{C}_c^\infty(\Omega)$. Let ind denote smooth

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induction with compact support. By the Bernstein–Zelevinsky analogue of Mackey Theory (see [BZ], pages 46–47)

$$\mathrm{ind}_{M_x \bar{N}}^{\bar{P}}(\delta \otimes \psi_x) = \mathcal{C}_c^\infty(\Omega)$$

is an irreducible $\bar{P}$-module. Hence $T(E) = \mathcal{C}_c^\infty(\Omega)$. Let E' be the kernel of T. Since

$$\dim E_{\bar{N},\psi_x} = \dim \mathcal{C}_c^\infty(\Omega)_{\bar{N},\psi_x}$$

for any $x \in N/Z$, it follows that $E'_{\bar{N},\psi_x} = 0$ for any $x \in N/Z$ (2.35 [BZ]). Therefore $E' = 0$ by 5.14 [BZ].

Note that the inclusion $M_x \to M$ induces an isomorphism

$$M_x/[M_x, M_x] \cong M/[M, M].$$

This can be easily checked by choosing x to be in the root space $\mathfrak{g}_\alpha$. Hence, δ is a character of M and to finish the proof we have to show that

$$\delta(m) = |\det(m)|^{s/d} \quad m \in M.$$

Furthermore, $P_2 = G_2 \cap P$ is a Heisenberg maximal parabolic subgroup P_2 of G_2. Its Levi factor is isomorphic to GL_2, and the inclusion of $\mathrm{GL}_2 \times H$ into M induces in isomorphism

$$\mathrm{GL}_2/\mathrm{SL}_2 \times H/[H, H] \cong M/[M, M].$$

Therefore it suffices to find the restriction of δ to $\mathrm{GL}_2/\mathrm{SL}_2 \times H/[H, H]$. In Section 8 we shall use the information on correspondences obtained in previous sections to find the character.

7. Jacquet functor for G_2 - Heisenberg parabolic

This section continues the notation and hypotheses of Section 6. In particular, $p \neq 2$ if $G = E_8$. Let

$$P_2 = G_2 \cap P \quad \text{and} \quad \bar{P}_2 = G_2 \cap \bar{P}$$

be the Heisenberg parabolic of G_2 and its opposite parabolic subgroup. Note that $Z \subset P_2$, and $\bar{Z} \subset \bar{P}_2$. Identifying I^* with I via the trace form on I, we obtain

$$N/Z \cong \bar{N}/\bar{Z} \cong F \oplus I \oplus I \oplus F,$$

with the pairing $\langle\ ,\ \rangle$ given by

$$\langle (x, u, v, y), (\bar{x}, \bar{u}, \bar{v}, \bar{y}) \rangle = x\bar{x} + \mathrm{Tr}(u\bar{u}) + \mathrm{Tr}(v\bar{v}) + y\bar{y}.$$

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Let U_2 and $\bar{U}_2$ be the unipotent radicals of P_2 and $\bar{P}_2$. Then

$$U_2/Z \cong \bar{U}_2/\bar{Z} \cong F \oplus Fe \oplus Fe \oplus F.$$

Hence the orhogonal complement of $\bar{U}_2/\bar{Z}$ in N/Z is

$$NN \cong I^0 \oplus I^0 \subset F \oplus I \oplus I \oplus F,$$

where I^0 is the set of traceless elements in I.

By Theorem 6.1

$$0 \to C_c^\infty(\Omega)_{\bar{U}_2} \to V_{\bar{U}_2} \to V_{\bar{N}} \to 0.$$

Let $\Omega\Omega = \Omega \cap NN$. Then, as in Lemma 2.2,

$$\mathcal{C}_c^\infty(\Omega)_{\bar{U}_2} = \mathcal{C}_c^\infty(\Omega\Omega),$$

and Theorem 6.1 implies:

PROPOSITION 7.1. *Identify the Levi factor of* $\bar{P}_2$ *with* GL_2 *so that it acts on* $\bar{Z}$ *via the character* det, *and the action on the quotient* $\bar{U}_2/\bar{Z}$ *is isomorphic to* $S^3(F^2)\otimes\det^{-1}$. *Identify* NN *with pairs of elements in* I^0. *Let* $f \in \mathcal{C}_c^\infty(\Omega\Omega) \subset V_{\bar{U}_2}$. *Then*

(1) $\Pi_{\bar{U}_2}(g)f(y,z) = f(g^{-1}y, g^{-1}z), \quad g \in H.$

(2) $\Pi_{\bar{U}_2}(g)f(y,z) = |\det(g)|^s f((y,z)g), \quad g \in \mathrm{GL}_2.$

Here det *is the usual determinant of* 2×2-*matrices, and* s *is* $2, 3$ *and* 5, *respectively.*

Again, we need to describe $\mathrm{GL}_2 \times H$-orbits on $\Omega\Omega$. We say that a subspace S of I^0 is singular, if the Jordan multiplication is trivial on S. In terms of the trilinear form, this is eqivalent to $S \subseteq x\Delta$, for every x in S, where

$$x\Delta = \{u \in I \mid (x,u,v) = 0 \text{ for all } v \in I\}.$$

The group H acts transitively on singular points. We need to understand H-orbits of singular two-dimensional subspaces in I^0. We have two different cases.

J_9 is the Jordan algebra of all 3×3-matrices with coefficients in F and $H = \mathrm{PGL}_3$ acts by conjugation. In this case, singular points in I^0 are nilpotent rank-one matrices. There are two PGL_3-orbits of singular two-dimensional spaces in I^0. Indeed, let $Fx + Fy$ be a singular space. Then either the images,

$$\mathrm{Im}(x) = \mathrm{Im}(y)$$

or the kernels

$$\ker(x) = \ker(y)$$

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of these two linear maps on F^3 coincide. If we fix S^+ and S^- two non-conjugated singular subspaces, then their stabilizers in PGL_3 are two non-conjugated maximal parabolic subgroups Q^+ and Q^-.

In the other two cases, a stabilizer of a singular two-dimensional space is a parabolic subgroup only if the space is 'amber'. This notion is due to Aschbacher [A1]. For an element x in I, one defines

$$\mu(x) = \{u \in x\Delta \cap I^0 \mid (e, u, v) = 0 \quad \text{for all} \quad v \in x\Delta \cap I^0\}.$$

DEFINITION 7.2. Let $S \subset I^0$ be a singular space. We say that S is amber if $S \subset \mu(x)$ for every nonzero $x \in S$.

PROPOSITION 7.3. *If $I = J_{15}$ or J, then the group H acts transitively on the set of amber, singular two-dimensional subspaces of I^0.*

Proof. If $I = J$ this is a result of Aschbacher, 9.3-5 [A1]. We now give a proof for J_{15}. Fix e and the trilinear form on

$$J_{15} = \wedge^2 F^6 =< x_{ij} > \quad 1 \leqslant i < j \leqslant 6.$$

as in Section 6, and let GSp_6 be the subgroup of all g in GL_6 such that $g(e) = \lambda(g)e$ for a scalar $\lambda(g)$ in $F^\times$. Then

$$\bigwedge^2 F^6 \otimes \lambda^{-1}$$

defines a faithfull action of $H = \mathrm{PGSp}_6$ on J_{15}, fixing e.

Let $Fx + Fy$ be an amber space in I^0. Since H acts transitively on the set of singular points, we can assume that $x = x_{12}$. A simple computation shows that

$$x_{12}\Delta = \langle x_{12}, x_{1,i}, x_{2,j} \rangle \quad i, j \neq 1, 2,$$

and

$$\mu(x_{12}) = \langle x_{12}, x_{13}, x_{14}, x_{23}, x_{24} \rangle.$$

Let Q_1 be the parabolic subgroup stabilizing the line through the singular point x_{12}. Its Levi factor $L_1 = \mathrm{GL}_2 \times \mathrm{GL}_2/\Delta F^\times$ acts on the 4-dimensional space $\mu(x)/Fx_{12}$ as on the space of 2×2-matrices. So it has two non-trivial orbits, the smaller being the orbit of the singular x_{13}. The proposition is proved.

PROPOSITION 7.4. (1) $\Omega\Omega = \{(x, y) \neq (0, 0) \mid x, y \in I^0$, *the space* $Fx + Fy$ *is singular and amber*$\}$.

(2) *Let AA and BB be the subsets of $\Omega\Omega$ consisting of all pairs (x, y) such that the space $Fx + Fy$ has dimension* 2 *and* 1 *respectively. Then* $\mathrm{GL}_2 \times H$ *acts transitively on BB. It also acts transitively on AA if $I \neq J_9$. If $I = J_9$ then then AA is a union of two* $\mathrm{GL}_2 \times \mathrm{PGL}_3$*-orbits.*

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Proof. As before, write

$$\mathfrak{n}/\mathfrak{z} = F \oplus I \oplus I \oplus F.$$

Then the maximal parabolic subalgebra

$$\mathfrak{q} = (\mathfrak{t} \oplus \mathfrak{l}) \oplus I \subset \mathfrak{m} = I^* \oplus (\mathfrak{t} \oplus \mathfrak{l}) \oplus I,$$

stabilizes the partial flag

$$F \oplus I \oplus I \oplus F \supset I \oplus I \oplus F \supset I \oplus F \supset F.$$

More precisely, let $u \in I$, be in the unipotent radical of $\mathfrak{q}$, and $(a, x, y, b) \in \mathfrak{n}/\mathfrak{z}$. Then

$$u(a, x, y, b) = (0, au, u \times x, \mathrm{Tr}(uy)),$$

where $u \times x$, the cross product, is the element of I such that

$$\mathrm{Tr}((u \times x)v) = (x, u, v)$$

for all $v \in I$.

LEMMA 7.5. *Let $Q \subset M$ be the corresponding maximal parabolic subgroup. Then Q has 4 orbits on Ω. Their representatives are*

$$v_1 = (1, 0, 0, 0), \qquad v_2 = (0, z, 0, 0),$$

$$v_3 = (0, 0, z, 0), \qquad v_4 = (0, 0, 0, 1),$$

where z is any non-zero element in I such that $z^2 = \mathrm{Tr}(z)z$.

Proof. Note that $\mathbb{P}(\Omega) = M/Q$. We have to compute $Q\backslash M/Q$ which is the same as $W_L\backslash W_M/W_L$, here W_M and W_L denote the Weyl groups of M and L ($L \subset Q$ corresponds to $\mathfrak{l} \subset \mathfrak{q}$). Since N/Z is a miniscule representation of M, its weight vectors are all contained in one W_M-orbit, it follows that they are parametrized by W_M/W_L. On the other hand, I and I^* are miniscule representations of L so $W_L\backslash W_M/W_L$ has four orbits.

The group L acts transitively on the set of elements such that $z^2 = \mathrm{Tr}(z)z$. It is simply the orbit of a highest weight vector, and hence of any weight vector, since the representation is miniscule. Hence, the vectors v_i, $(1 \leqslant i \leqslant 4)$ clearly represent 4 different orbits, so the lemma is proved.

Let $(0, x, y, 0) \in \Omega\Omega$. If $x \neq 0$ then Lemma 7.5 implies that it is in the Q-orbit of v_2. Hence x is in the L-orbit of z, so x is singular. Since the action of GL_2, the Levi factor of $\bar{P}_2$, is

$$(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy),$$

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the same argument implies that any element of $Fx + Fy$ is singular. Hence

$$x^2 = (x+y)^2 = y^2 = 0,$$

and

$$2x \circ y = (x+y)^2 - x^2 - y^2 = 0.$$

We have shown that S is a singular space.

Furthermore, $(0, x, y, 0)$ is Q-conjugated to $(0, x, 0, 0)$. But this two elements can be conjugated only by an element of the unipotent radical of Q. Hence

$$(0, x, y, 0) = \exp(u)(0, x, 0, 0),$$

for some u in I. This implies that $y = u \times x$. Since $\mathrm{Tr}((u \times x)v) = (u, x, v)$ for all v in I, and $(e, y, v) = -\mathrm{Tr}(yv)$ for any v in I^0 (see Section 6), it follows that

$$(e, y, v) = -(u, x, v) = 0$$

if $v \in x\Delta \cap I^0$. Hence $y \in \mu(x)$. Since the same argument can be repeated for any linear combination of x and y, the first part of the proposition follows.

Let $z \in I^0$ such that $z^2 = 0$. Let Q_1 be the parabolic subgroup of H stabilizing the line Fz. Consider

$$B = \{(az, 0) \mid a \in F \text{ and } a \neq 0\}.$$

Let $QQ \subset \mathrm{GL}_2$ be the Borel subgroup stabilizing the line B. Then $QQ \times Q_1$ acts transitively on B and

$$BB = (\mathrm{GL}_2 \times H) \times_{(QQ \times Q_1)} B.$$

Assume now that $I = J_{15}$ or J. Let $Fx + Fy \subseteq I^0$ be a 2-dimensional singular, amber space. Let Q_2 be the parabolic subgroup of H stabilizing the space $Fx + Fy$. Consider

$$A = \{(ax + by, cx + dy) \mid a, b, c, d \in F \text{ and } ad - bc \neq 0\}.$$

Then $\mathrm{GL}_2 \times Q_2$ acts transitively on A and

$$AA = H \times_{Q_2} A.$$

In J_9 we have two orbits of singular two-dimensional spaces. Let $Fx^+ + Fy^+$ and $Fx^- + Fy^-$ be their representatives and Q^+ and Q^- their stabilizers in PGL_3. One can define A^+ and A^- as above, hence

$$AA = \mathrm{PGL}_3 \times_{Q^+} A^+ \cup \mathrm{PGL}_3 \times_{Q^-} A^-.$$

The proposition is proved.

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THEOREM 7.6. *V_{U_2} has a filtration with three succesive quotients*

$$\mathcal{C}_c^\infty(AA),\ \mathcal{C}_c^\infty(BB),\quad \text{and}\ \ V_{\bar{N}},$$

where $\mathcal{C}_c^\infty(AA)$ is a submodule, and $V_{\bar{N}}$ a quotient. Moreover, as $\mathrm{GL}_2 \times H$-modules,

$$(1)\quad \mathcal{C}_c^\infty(AA) = |\det|^s \otimes \mathrm{ind}_{Q_2}^H(\mathcal{C}_c^\infty(A)) \quad \text{if}\ \ H \neq \mathrm{PGL}_3,$$

$$\mathcal{C}_c^\infty(AA) = |\det|^s \otimes \mathrm{ind}_{Q^+}^{\mathrm{PGL}_3}(\mathcal{C}_c^\infty(A^+)) + |\det|^s \otimes \mathrm{ind}_{Q^-}^{\mathrm{PGL}_3}(\mathcal{C}_c^\infty(A^-)).$$

$$(2)\quad \mathcal{C}_c^\infty(BB) = |\det|^s \otimes \mathrm{ind}_{QQ\times Q_1}^{\mathrm{GL}_2\times H}(\mathcal{C}_c^\infty(B)),$$

$$(3)\quad V_{\bar{N}} \cong |\det|^t \otimes V(M) + |\det|^s \otimes 1,$$

where $V(M)$ is the minimal representation of M (center acting trivially).

In the above formulas det *is the usual determinant of* 2×2 *matrices, and s and t are given by the following table*

G	s	t
E_6	2	3/2
E_7	3	2
E_8	5	3

Proof. This follows from Theorem 6.1, Proposition 7.1 and 7.4.

8. Θ-lifts from G_2

In this section we compute Θ-lifts of spherical tempered representations of G_2 in all three cases. In the process we also compute the normalizing factors (i.e. coefficients s) in Theorem 6.1 and 7.6.

We study the dual pair $G_2 \times F_4$ in a simple group G of type E_8 first. Let π be a spherical tempered representation of G_2. Write

$$\pi = \mathrm{Ind}_{\bar{P}_2}^{G_2}(\tau)$$

where τ is a spherical representation of GL_2 with the parameter $(\chi_1^{-1}|\cdot|^{3/2}, \chi_2^{-1}|\cdot|^{3/2})$. As before, χ_1 and χ_2 must be unitary characters. Let $Q_2 = L_2V_2$ be a maximal parabolic subgroup of F_4 stabilizing a singular, amber two-dimensional space in J. The action of L_2 on the corresponding amber line gives an exact sequence

$$1 \to \mathrm{SL}_3 \to L_2 \to \mathrm{GL}_2 \to 1,$$

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where SL_3 is 'spanned' by two simple long roots. One checks that $\rho_{V_2} = |\det|^7$, where det is the usual determinant on GL_2, the quotient of L_2 by SL_3. Let τ' be a spherical representation of GL_2 with the parameter $(\chi_1|\cdot|^{7/2}, \chi_2|\cdot|^{7/2})$. Pull τ' back to L_2. Let

$$\pi' = \mathrm{Ind}_{Q_2}^{F_4}(\tau').$$

Note that π' is a unitarizable representation of F_4. It is quite possible that π' is always irreducible but we do not know.

THEOREM 8.1. $(p \neq 2)$ *Let π be the spherical tempered representation of G_2 and π' the representation of F_4, defined above. Assume, for simplicity, that π' is irreducible. Then $\Theta(\pi) = \{\pi'\}$. Let $s \in G_2(\mathbb{C})$ be the Satake parameter of π. Then the Satake parameter of π' is $\Psi(s \times \rho)$ where*

$$\Psi: G_2(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C}) \to F_4(\mathbb{C})$$

is the embedding of the dual pair $G_2(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C})$ in $F_4(\mathbb{C})$, and $\rho \in \mathrm{SO}_3(\mathbb{C})$ is the Satake parameter of the trivial representation of SL_2.

Proof. Let $P = MN$ be the Heisenberg parabolic subgroup of G. In Section 6, we described an embedding of the dual pair $G_2 \times F_4$ in G such that $G_2 \cap P$ is the Heisenberg maximal parabolic subgroup P_2.

Yet another embedding of the dual pair $G_2 \times F_4$ is given by the inclusion of Jordan algebras $J_6 \to J$ (use the $\mathbb{Z}/3\mathbb{Z}$-gradation of the exceptional Lie algebras given in Section 6). In this case,

$$G_2 \subset M \quad \text{and} \quad F_4 \cap P = Q_4 = L_4V_4,$$

the Heisenberg maximal parabolic subgroup of F_4. The Levi component L_4 is isomorphic to GSp_6. Note that the inclusion $\mathrm{GSp}_6 \to M$ induces an isomorphism $\mathrm{GSp}_6/\mathrm{Sp}_6 \cong M/[M, M]$. This is easily seen by considering the action of GSp_6 and M on Z, the center of both, V_4 and N.

Let V be the minimal representation of G. Let V_N be the maximal N-invariant quotient of V. Obviously, it is a $G_2 \times \mathrm{GSp}_6$-module. By Proposition 4.1 of [S1],

$$V_N \cong V(M) \otimes |\det|^3 + 1 \otimes |\det|^5,$$

where $V(M)$ is the minimal representation of M, with center acting trivially, and det denotes the usual determinant of GSp_6. Note that the quotient of M by its center is the adjoint group of type E_7. So, if σ is the Langlands lift of π to PGSp_6, by Theorem 5.4, $\pi \otimes \sigma$ is a quotient of $V(M)$, and of V_N. Hence, by the Frobenius reciprocity, $\pi \otimes \sigma'$ is a quotient of V for some subquotient σ' of

$$\mathrm{Ind}_{Q_4}^{F_4}(\sigma \otimes |\det|^3).$$

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In particular, $\Theta(\pi)$ is not empty.

On the other hand, by the Frobenius reciprocity, $\tau \otimes \sigma'$ is a quotient of $V_{\bar{U}_2}$. So, for a generic choice of χ_1 and χ_2, $\tau \otimes \sigma'$ will be a quotient of $\mathcal{C}_c^\infty(AA)$ in Theorem 7.6. Recall that

$$\mathcal{C}_c^\infty(AA) = \delta(\det) \otimes \mathrm{ind}_{Q_2}^H(\mathcal{C}_c^\infty(A))$$

for a certain character δ which we shall now determine. Since $\mathcal{C}_c^\infty(A)$ is a regular representation of GL_2 twisted by δ, $\tau \otimes \sigma'$ must be a quotient of

$$\tau \otimes \mathrm{Ind}_{Q_2}^{F_4}(\tau')$$

where τ' is the representation of L_2 pulled back from a representation of GL_2 with a parameter $(\chi_1\delta\,|\cdot|^{-3/2}, \chi_2\delta|\cdot|^{-3/2})$. We get that σ' is a subquotient of both,

$$\mathrm{Ind}_{Q_4}^{F_4}(\sigma \otimes |\det|^3) \quad \text{and} \quad \mathrm{Ind}_{Q_2}^{F_4}(\tau').$$

This immediately implies that

$$\delta(\det) = |\det|^5 \quad \text{and} \quad \mathrm{Ind}_{Q_2}^{F_4}(\tau') = \pi'.$$

Moreover, the knowledge of δ implies that any τ, with χ_1 and χ_2 unitary, is a quotient of $\mathcal{C}_c^\infty(AA)$ only. Hence, by the Frobenius reciprocity, a Θ-lift of π must be a quotient of π'.

It remains to check the statement about Satake parameters. The dual Langlands group of F_4 is $F_4(\mathbb{C})$. Let $Q_3 = L_3V_3$ be the maximal parabolic subgroup of F_4 such that $L_3(\mathbb{C})$ is the dual group of L_2. In particular, it fits into the exact sequence

$$1 \to \mathrm{GL}_2(\mathbb{C}) \to L_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C}) \to 1.$$

Let $s \in \mathrm{GL}_2(\mathbb{C})$ be the parameter (χ_1, χ_2). The Satake parameter of π' is

$$s \times \rho \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C}) \subset L_3(\mathbb{C}).$$

On the other hand, the centralizer of $\mathrm{SO}_3(\mathbb{C})$ in $F_4(\mathbb{C})$ is $G_2(\mathbb{C})$. Since

$$L_3(\mathbb{C}) \cap G_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}),$$

and s is the Satake parameter of π (see the proof of Theorem 4.4), the theorem follows.

Theorem 7.6 can be used, in a similar way, to prove converses of Theorems 4.4 and 5.4. We state results without giving details of proofs.

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THEOREM 8.2. *Let* $\Phi\colon \mathrm{SL}_3(\mathbb{C}) \to G_2(\mathbb{C})$ *be the standard inclusion of the dual groups of* PGL_3 *and* G_2. *Let* π *be a tempered spherical representation of* G_2. *The Satake parameter of* π *is* $\Phi(s)$ *for some* s, *the Satake parameter of a tempered spherical representation* σ *of* PGL_3. *Note that* $\Phi(s) = \Phi(s^*)$ *where* s^* *is the Satake parameter of* σ^*, *the dual of* σ. *Then*

$$\Theta(\pi) = \{\sigma, \sigma^*\}.$$

THEOREM 8.3. *Let* $\Phi\colon G_2(\mathbb{C}) \to \mathrm{Spin}_7(\mathbb{C})$ *be the standard inclusion of the dual groups of* G_2 *and* PGSp_6. *Let* π *be a tempered spherical representation of* G_2. *Then*

$$\Theta(\pi) = \{\pi'\},$$

where π' *is the spherical tempered representation of* PGSp_6 *whose Satake parameter is* $\Phi(s)$.

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