THE RANGE OF A CONTINUOUS LINEAR FUNCTIONAL OVER A CLASS OF FUNCTIONS DEFINED BY SUBORDINATION

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Introduction. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and $H(\Delta)$ the set of analytic functions on Δ. We recall the definition of subordination between two functions, say f and g, analytic on Δ: this means that $f(0) = g(0)$ and there is a function $\rho \in H(\Delta)$ such that $\rho(0) = 0$, $|\rho(z)| < 1$ if $z \in \Delta$, and $f(z) \equiv g(\rho(z))$. Subordination between f and g will be denoted by $f \prec g$. The Hadamard product (or convolution) of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H(\Delta)$ is the function $f * g \in H(\Delta)$ defined as $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Let $F \in H(\Delta)$ be univalent, convex and bounded; it follows that $F(\Delta)$ is a bounded convex domain and F extends to a homeomorphism between the closed unit disc and the closure of $F(\Delta)$. We define $s(F) = \{f \in H(\Delta) \mid f \prec F\}$ and for $\theta \in \mathbb{R}$, $s(F, \theta) = \left\{f \in s(F) \,\middle|\, \lim_{z \to 1} f(z) = F(e^{i\theta})\right\}$. In this paper any limit as $z \to 1$ is understood to be a non-tangential limit. Let $G \in H(\Delta)$ be convex and univalent; our main result is a description of the range of the linear functional

$$I(f) = \int_0^1 f * G(x)\, dx$$

over the set $s(F, \theta)$. We shall prove the following result.

THEOREM 1. *For each* $\theta \in \mathbb{R}$,

$$\{I(f) \mid f \in s(F, \theta)\} = \left\{\frac{1}{z}\int_0^z F * G(x)\, dx \mid z \in \Delta \text{ or } z = e^{i\theta}\right\}.$$

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$$I(f) = \frac{1}{e^{i\theta}} \int_0^{e^{i\theta}} F * G(x)\, dx \Leftrightarrow f(z) \equiv F(e^{i\theta} z).$$

This result appears to be useful because it gives a complete description of the range of a linear operator over a class of functions defined by subordination and a constraint, namely the value of the angular limit at some point of the unit circle. Our conclusion will include some applications of our theorem.

REMARK 1. It is clear that any bounded function in $H(\Delta)$ is integrable over $[0, 1]$. For any $G \in H(\Delta)$ convex and univalent we have [2]

$$G(z) = G(0) + G'(0) \int_{\partial\Delta} \frac{z}{1 - \xi z}\, d\mu(\xi),$$

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where μ is a probability measure on the unit circle. Therefore, for any $f \in s(F)$,

$$f * G(z) = f(0)G(0) + G'(0)\int_{\partial\Delta} \frac{f(\xi z) - f(0)}{\xi}\, d\mu(\xi)$$

and

$$|f * G(z)| \leq |f(0)|\, |G(0)| + 2\,|G'(0)| \max_{|\xi| \leq 1} |F(\xi)| \quad (z \in \Delta).$$

This shows that the functionnal I is well-defined on $s(F)$. It is also easy to establish that I is a continuous functional on $s(F)$ endowed with the topology of uniform convergence on compact subsets of Δ.

REMARK 2. There exists a nice theory dealing with continuous linear functionals on subsets of $H(\Delta)$ defined by subordination. In a book by Hallenbeck and MacGregor **[5]** we find many precise results about extreme points and support points of $s(F)$. However the classes $s(F, \theta)$ are not compact (this follows from Lemma 3 below) and it seems difficult to apply directly the results contained in **[5]** to prove Theorem 1.

Our method to obtain Theorem 1 lies closer to the method of Hallenbeck and Ruscheweyh in **[6]** where they more or less established the range over $s(F)$ of the functional

$$I'(f) = \gamma z^{-\gamma} \int_0^z x^{\gamma - 1} f(x)\, dx$$

with F convex univalent, z a fixed complex number in Δ and γ a complex parameter in the right half-plane.

A special case of Theorem 1 was obtained in **[4]**. In our conclusion we shall point out some consequences of our result.

Preliminary Lemmas. In this section we quote several lemmas needed for our proof of Theorem 1. The first one is a convolution property of convex functions due to Ruscheweyh and Sheil-Small **[7]**.

LEMMA 1. *Let g, $h \in H(\Delta)$ be convex univalent. Then $g * h$ is also convex univalent and*

$$f \in s(g) \Rightarrow f * h \in s(g * h).$$

We say that $f \in H(\Delta)$ is *properly subordinate* to $g \in H(\Delta)$ if $f(z) = g(\rho(z))$ where $\rho \in H(\Delta)$ and $|\rho(z)| < |z|$ if $z \neq 0$; we shall denote this by $f \prec_p g$. In fact Ruscheweyh and Sheil-Small proved in **[7]**

$$f * h \prec_p g * h \text{ if } f \prec_p g \text{ and } h, g \text{ are convex univalent.} \tag{1}$$

We shall also need the following classical result, due to Julia (see [**1**, Section 1.4])

LEMMA 2. *Let $v \in H(\Delta)$ with $|v(z)| \leq |z|$ for all $z \in \Delta$. Then $\lim_{z \to 1} \frac{1 - v(z)}{1 - z}$ exists and is either ≥ 1 or infinite. Moreover if $\lim_{z \to 1} \frac{1 - v(z)}{1 - z} := l < \infty$, then $1 \leq \lim_{z \to 1} v'(z) = l$ and equality is possible only if $v(z) \equiv z$.*

The next lemma appears in **[4]**.

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LEMMA 3. *Let $\alpha, \beta \in \mathbb{R}$. There exists a sequence of functions $\{\rho_k\}$ analytic in the closed unit disc such that $|\rho_k(z)| \le |z|$, $\rho_k(1) = e^{i\alpha}$ and $\lim_{k\to\infty} \rho_k(z) = e^{i\beta}z$ uniformly on compact subsets of Δ.*

If E is a subset of $\mathbb{C}$ let $\bar{E}$ designate the closure of E. We shall finally need the following result, a proof of which, due to St. Ruscheweyh, is given below.

LEMMA 4. *Let $f \in H(\Delta)$ be univalent, convex and bounded; let*

$$g(z) \equiv \frac{1}{z} \int_0^z f(x)\, dx.$$

Then $\overline{g(\Delta)} \subset f(\Delta)$.

The fact that $U(z) \equiv -z \log(1-z)$ is convex univalent in Δ will be used at several stages of our work.

Proof of Lemma 4. We may assume that $f(0) = f'(0) - 1 = 0$. Under this normalization it is known that (see [3, Chapter 2])

$$|f(z)| \le |z|/1 - |z| \quad (z \in \Delta), \tag{2}$$

$$f(\Delta) \text{ contains a disc of radius } \tfrac{1}{2}, \text{ centered at the origin.} \tag{3}$$

The representation $g(z) = f(z) * -z^{-1} \log(1-z)$ and Lemma 1 show that g is convex univalent; the representation $g(z) = \int_0^1 f(tz)\, dt$ shows that $g(\Delta)$ lies in the closed convex hull of $f(\Delta)$. Therefore $\overline{g(\Delta)} \subseteq \overline{f(\Delta)}$ and g is continuous on $\bar{\Delta}$. Clearly the proof will be completed if we can prove that $g(\Delta)$ and $f(\Delta)$ share no boundary point.

Let $w_0 \in \partial g(\Delta) \cap \partial f(\Delta)$. There must exist a supporting line through w_0 such that $\overline{f(\Delta)}$ lies in one of the closed half-planes determined by this line. There must also exist an ε, with $|\varepsilon| = 1$, such that

$$\max_{|z|\le 1} \operatorname{Re}(\varepsilon g(z)) = \max_{|z|\le 1} \operatorname{Re}(\varepsilon f(z)) = \operatorname{Re}(\varepsilon w_0) > 0. \tag{4}$$

By (3) and (4) we clearly have $\operatorname{Re}(\varepsilon w_0) \ge \frac{1}{2}$. On the other end, for any $x \in (0, 1)$, by (2) and (4),

$$\begin{aligned}
\operatorname{Re}(\varepsilon w_0) &= \max_{|z|\le 1} \operatorname{Re}(\varepsilon g(z)) \\
&= \max_{|z|\le 1} \int_0^1 \operatorname{Re}(\varepsilon f(tz))\, dt \\
&\le \max_{|z|\le 1} \int_0^x \operatorname{Re}(\varepsilon f(z))\, dt + \max_{|z|\le 1} \int_x^1 \operatorname{Re}(\varepsilon f(tz))\, dt \\
&\le \max_{|z|\le 1} \int_0^x \frac{t\,|z|}{1 - t\,|z|}\, dt + (1-x)\operatorname{Re}(\varepsilon w_0) \\
&\le \frac{x^2}{1-x} + (1-x)\operatorname{Re}(\varepsilon w_0).
\end{aligned}$$

It leads to $\frac{1}{2} \le \operatorname{Re}(\varepsilon w_0) \le \dfrac{x}{1-x}$ and this is impossible if x is small enough. Therefore $g(\Delta)$ and $f(\Delta)$ share no boundary point.

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Proof of Theorem 1. We first establish the formula

$$\{I(f) \mid f \in s(F)\} = \left\{\frac{1}{z}\int_0^z F * G(x)\,dx \mid |z| \le 1\right\}. \tag{5}$$

In fact Lemma 1 yields, for $f \in s(F)$,

$$f * G \prec F * G \text{ and } (f * G(z)) * \frac{-\log(1-z)}{z} \prec (F * G(z)) * \frac{-\log(1-z)}{z}$$

and the last statement is simply

$$\frac{1}{z}\int_0^z f * G(x)\,dx \prec \frac{1}{z}\int_0^z F * G(x)\,dx. \tag{6}$$

By letting $z \to 1$, we obtain

$$\{I(f) \mid f \in s(F)\} \subseteq \left\{\frac{1}{z}\int_0^z F * G(x)\,dx \mid |z| \le 1\right\}.$$

The reversed inclusion follows from the fact that $F(\xi z) \prec F(z)$ for any $\xi \in \bar{\Delta}$ and

$$\int_0^1 F(\xi x) * G(x)\,dx = \frac{1}{\xi}\int_0^\xi F * G(x)\,dx.$$

Let $f \in s(F, \theta)$; we define

$$k(z) \equiv \frac{1}{z}\int_0^z f * G(x)\,dx \quad \text{and} \quad K(z) \equiv \frac{1}{z}\int_0^z F * G(x)\,dx.$$

We have, by (6), $k(z) = K(\rho(z))$ where $\rho \in H(\Delta)$ and $|\rho(z)| \le |z|$, $z \in \Delta$. We shall prove that

$$\lim_{z \to 1} \rho(z) \text{ exists,} \tag{7}$$

$$\sup_{z \in \Delta} |\rho'(z)| < \infty. \tag{8}$$

In fact $f * g$ is integrable over $[0, 1]$ and $\lim_{z\to 1} k(z) = \lim_{z \to 1} (1/z)\int_0^z f * G(x)\,dx$ exists. Also K is convex univalent (it can be written as the convolution of three convex functions), it extends to a homeomorphism and, since $\rho(z) = K^{-1}(k(z))$, we obtain (7). On the other hand

$$|\rho'(z)| = |k'(z)|/|K'(\rho(z))| \quad (z \in \Delta),$$

where $|K'(\rho(z))|$ is bounded below on Δ by a strictly positive constant (since K is convex univalent) and $|k'(z)|$ is bounded above on Δ because

$$|zk'(z)| = \left|\int_0^z (f * G(z) - f * G(x))\,dx\right| \le 2 \sup_{\xi \in \Delta} |f * G(\xi)|.$$

This establishes (8).

We now prove that $\rho(1) := \lim_{z\to 1} \rho(z)$ belongs to Δ unless $f(z) \equiv F(e^{i\theta}z)$, in which case we shall obtain $\rho(1) = e^{i\theta}$. This will mean, in view of (5) and the identity

$$I(f) = \frac{1}{\rho(1)} \int_0^{\rho(1)} F * G(x)\, dx,$$

that

$$\left\{\frac{1}{e^{i\theta}} \int_0^{e^{i\theta}} F * G(x)\, dx\right\} \subset \{I(f) \mid f \in s(F, \theta)\} \subseteq \left\{\frac{1}{z} \int_0^{z} F * G(x)\, dx \mid z \in \Delta \text{ or } z = e^{i\theta}\right\} \quad (9)$$

If $\lim_{z\to 1} \rho(z) = e^{i\psi}$, where $\psi \in \mathbb{R}$, we obtain

$$\sup_{z\in\Delta} \left|\frac{1 - \rho(z)/e^{i\psi}}{1 - z}\right| < \infty$$

because, by (8),

$$\begin{aligned}\left|\frac{1 - \rho(z)/e^{i\psi}}{1 - z}\right| &= \lim_{x\to 1} \left|\frac{\rho(x)/e^{i\psi} - \rho(z)/e^{i\psi}}{x - z}\right| \\ &= \lim_{x\to 1} \left|\frac{1}{x - z} \int_z^x \rho'(t)\, dt\right| \\ &\le \sup_{|\xi|<1} |\rho'(\xi)| < \infty.\end{aligned}$$

It follows from Lemma 2 that

$$1 \le \lim_{z\to 1} \frac{\rho'(z)}{e^{i\psi}} = \lim_{z\to 1} \frac{z\rho'(z)}{\rho(z)} := \frac{1}{\tau},$$

where $0 < \tau \le 1$, and $\tau = 1$ if and only if $\rho(z) \equiv e^{i\psi}z$. Upon differentiation of (6), we obtain for any $z \in \Delta$,

$$F * G(\rho(z)) = \left(1 - \frac{\rho(z)}{z\rho'(z)}\right) \frac{1}{\rho(z)} \int_0^{\rho(z)} F * G(x)\, dx + \frac{\rho(z)}{z\rho'(z)} f * G(z).$$

This shows first $\lim_{z\to 1} f * G(z)$ exists. Then, by letting $z \to 1$, we deduce that

$$F * G(e^{i\psi}) = (1 - \tau)e^{-i\psi} \int_0^{e^{i\psi}} F * G(x)\, dx + \tau f * G(1). \quad (10)$$

Note that, by Lemma 4, $e^{-i\psi} \int_0^{e^{i\psi}} F * G(x)\, dx$ belongs to the interior of the convex set $F * G(\bar{\Delta})$; we can interpret (10) as follows: a boundary point of $F * G(\bar{\Delta})$, namely $F * G(e^{i\psi})$, is a convex combination of one of its interior points and of one of its points (namely $f * G(1)$, because $f * G \prec F * G$); clearly then $\tau = 1$ and $\rho(z) \equiv e^{i\psi}z$. Let us assume that $f \prec_p F$; we obtain, from (1),

$$\frac{1}{z} \int_0^z f * G(x)\, dx \prec_p \frac{1}{z} \int_0^z F * G(x)\, dx$$

which is impossible because $\rho(z) \equiv e^{i\psi}z$. Therefore f is not properly subordinate to F and since $f \in s(F, \theta)$ we have $f(z) \equiv F(e^{i\theta}z)$ because F extends to a homemorphism on $\bar{\Delta}$. Hence we have obtained (9).

We can now complete the proof of Theorem 1. If the conclusion were false it would follow from (9) that

$$E_1 := \left\{\int_0^1 f * G(x)\,dx \mid f \in s(F, \theta)\right\} \subsetneqq E_2 := \left\{z^{-1}\int_0^z F * G(x)\,dx \mid |z| < 1 \text{ or } z = e^{i\theta}\right\}. \quad (11)$$

The sets E_1 and E_2 are convex and bounded. One boundary point of E_2 belongs to E_1, namely $e^{-i\theta}\int_0^{e^{i\theta}} F * G(x)\,dx$. From (9) and (11), some interior point of E_2 does not belong to E_1; there must exist an open half-plane P such that $\bar{E}_1 \cap P$ is empty and P contains some boundary point of E_2, say $e^{-i\beta}\int_0^{e^{i\beta}} F * G(x)\,dx$, with $e^{i\beta} \neq e^{i\theta}$. By Lemma 3, there exists a sequence $\{\rho_k\} \subset H(\Delta)$ such that $|\rho_k(z)| \leq |z|$, $\rho_k(1) = e^{i\theta}$ and $\lim_{k\to\infty} \rho_k(z) = e^{i\beta}z$, uniformly on compact subsets of Δ. Clearly $\{F(\rho_k)\} \subset s(F, \theta)$ and since

$$\lim_{k\to\infty} \int_0^1 F(\rho_k) * G(x)\,dx = \int_0^1 F * G(e^{i\beta}x)\,dx = e^{-i\beta}\int_0^{e^{i\beta}} F * G(x)\,dx,$$

we have $e^{-i\beta}\int_0^{e^{i\beta}} F * G(x)\,dx \in \bar{E}_1 \cap P$. This contradicts the emptiness of $\bar{E}_1 \cap P$ and the conclusion of Theorem 1 must hold.

Conclusion. 1° The assumptions made on the function F of our theorem (that is, boundedness and convexity) may seem too strong but they are in some sense necessary. For example let us put $F(z) \equiv G(z) \equiv 1/(1-z)$. If the point at infinity is considered as an admissible boundary point, it is easily seen that (5) is valid, i.e.,

$$\left\{\int_0^1 f(x)\,dx \mid f \in s(F)\right\} = \left\{z^{-1}\int_0^z \frac{1}{1-\xi}\,d\xi \mid |z| \leq 1\right\}.$$

However any function as $f_n(z) \equiv 1/(1-z^n)$ belongs to $s(F, 0)$ if n is a positive integer and

$$\int_0^1 f_n(x)\,dx = \int_0^1 F(x)\,dx = \infty,$$

so that Theorem 1 is not valid for this choice of F. A similar counterexample can be constructed where F is a starlike function mapping Δ into Δ minus a "radial" slit.

2° We consider, for $\lambda \in [0, 1]$ the class of normalized univalent functions

$$S_\lambda = \{h \in H(\Delta) \mid h(0) = h'(0) - 1 = 0 \text{ and } |h'(z) - 1| \leq \lambda,\ z \in \Delta\}.$$

We define the order of starlikeness of S_λ as $\inf_{h \in S_\lambda, z \in \Delta} \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right)$. Observe that

$$\begin{aligned} h \in S_\lambda &\Leftrightarrow h'(z) = 1 + \lambda w(z) \quad (z \in \Delta) \\ &\Leftrightarrow \frac{zh'(z)}{h(z)} = \frac{1 + \lambda w(z)}{1 + \lambda z^{-1}\int_0^z w(x)\,dx} \end{aligned}$$

where w is an arbitrary function in $H(\Delta)$ subordinate to the identity function. It follows clearly from Theorem 1 (with $F(z) \equiv z$ and $G(z) \equiv z/(1-z)$) that

$$\inf_{\substack{h \in S_\lambda \\ z \in \Delta}} \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) = \inf_{\substack{\theta \in \mathbb{R} \\ \psi \in \mathbb{R}}} \operatorname{Re}\left(\frac{1+\lambda e^{i\theta}}{1+\dfrac{\lambda}{2}e^{i\psi}}\right)$$

and some simple computations would lead to the following result.

Corollary 1. *Let $\lambda \in [0, 1]$ and ρ_λ denote the order of starlikeness of S_λ. Then*

$$\rho_\lambda = \begin{cases} \dfrac{(1-\lambda)(1-\lambda/2)}{1-\lambda^2/4}, & \text{if} \quad 0 \le \lambda \le \frac{2}{3}, \\ \dfrac{\frac{1}{2}(1-5\lambda^2/4)}{1-\lambda^2/4}, & \text{if} \quad \frac{2}{3} \le \lambda \le 1. \end{cases}$$

3° Finally we remark that Theorem 1 can be used to obtain a sharp variability region for certain combinations of the Taylor coefficients of functions $f \in s(F, \theta)$. For example if $F(z) \equiv \sum_{n=1}^{\infty} A_n z^n \in H(\Delta)$ is convex univalent and bounded, we obtain the following result as an immediate consequence of Theorem 1, by choosing

$$G(z) \equiv z + n^{-2} z^n.$$

Corollary 2. *For any $\theta \in \mathbb{R}$, we have*

$$\left\{\frac{a_1}{2} + \frac{a_n}{(n+1)n^2} \,\middle|\, f(z) \equiv \sum_{n=1}^{\infty} a_n z^n \in s(F, \theta)\right\} = \left\{\frac{A_1}{2}\xi + \frac{A_n}{(n+1)n^2}\xi^n \,\middle|\, |\xi| < 1 \text{ or } \xi = e^{i\theta}\right\}.$$

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