STATES WHICH HAVE A TRACE-LIKE PROPERTY RELATIVE TO A C^*-SUBALGEBRA OF $B(H)$

by GUYAN ROBERTSON

(Received 8 May, 1975)

In what follows, $B(H)$ will denote the C^*-algebra of all bounded linear operators on a Hilbert space H. Suppose we are given a C^*-subalgebra A of $B(H)$, which we shall suppose contains the identity operator 1. We are concerned with the existence of states f of $B(H)$ which satisfy the following trace-like relation relative to A:

$$f(ab) = f(ba) \qquad (a \in A,\ b \in B(H)). \tag{*}$$

Our first result shows the existence of states f satisfying (*), when A is the C^*-algebra $C^*(x)$ generated by a normaloid operator x and the identity. This allows us to give simple proofs of some well-known results in operator theory. Recall that an operator x is *normaloid* if its operator norm equals its spectral radius.

PROPOSITION 1. *Let x be a normaloid operator on the Hilbert space H. Then there exists a pure state f of $B(H)$ such that*

$$f(ab) = f(ba) \qquad (a \in C\ \ (x),\ b \in B(H)).$$

Moreover the restriction $f|\,C^*(x)$ is a character (multiplicative linear functional) of $C^*(x)$ and satisfies $|f(x)| = 1$.

Proof. We may assume that $\|x\| = 1$. Since x is normaloid, there exists a complex number β of modulus one in the spectrum of x. There exists a pure state f of $C^*(x)$ such that $f(x) = \beta$. (This is because β is in the boundary of the spectrum of x, and hence $\beta - x$ is not left invertible. It follows that $\beta - x$ is in the left kernel of the pure state f of $C^*(x)$.) f may be extended to a pure state (also denoted by f) of $B(H)$. Then $f(x^*) = \bar{\beta}$ and $f(x^*x) = 1 = f(xx^*)$. Therefore

$$f((1-\bar{\beta}x)(1-\beta x^*)) = f(1-\beta x^* - \bar{\beta}x + xx^*) = 0.$$

Hence, for all $b \in B(H)$,

$$|f((\beta - x)b)|^2 = |f((1-\bar{\beta}x)\beta b)|^2 \leq f(b^*b)f((1-\bar{\beta}x)(1-\bar{\beta}x)^*) = 0,$$

which implies that $f(xb) = \beta f(b) = f(x)f(b)$.

Similarly $f(bx) = f(x)f(b)$, for all $b \in B(H)$. Thus $f(bx) = f(xb)$, for all $b \in B(H)$. In the same way we obtain $f(bx^*) = f(x^*b)$, for all $b \in B(H)$, and finally

$$f(ab) = f(ba) \qquad (a \in C^*(x),\ b \in B(H)).$$

It is now obvious that $f|\,C^*(x)$ is a character.

An immediate consequence of the above, concerning the distance of certain commutators from the identity, is a generalization of [**1**, Problem 185].

https://doi.org/10.1017/S0017089500002913 Published online by Cambridge University Press

COROLLARY 2. *In the notation of Proposition* 1, *we have*

$$\| ab-ba-1 \| \geqq 1 \qquad (a \in C^*(x),\ b \in B(H)).$$

This follows because $\|f\| = f(1) = 1$, and $f(ab-ba) = 0$. Similarly

$$\| ab-ba-x \| \geqq 1 \qquad (a \in C^*(x),\ b \in B(H)),$$

which is true because $|f(x)| = 1$.

COROLLARY 3. [**1**, Problem 188] *A positive self-commutator in* $B(H)$ *is not invertible.*

For a self-commutator is an element of $B(H)$ of the form $y = x^*x - xx^*$. If this is positive, x is normaloid, and so, by Proposition 1, there exists a state f of $B(H)$ such that $f(y) = 0$. This is impossible if y is positive and invertible.

More general results obviously follow in a similar way.

Our next result, which is analogous to Proposition 1, is in a more general setting.

PROPOSITION 4. *Let* A *be a* C^**-subalgebra of* $B(H)$ *and let* f *be a character of* A. *Then every state extension* $\bar{f}$ *of* f *to all of* $B(H)$ *satisfies*

$$\bar{f}(ab) = \bar{f}(ba) \qquad (a \in A,\ b \in B(H)).$$

Proof. We note that $|f(u)| = 1$, for each unitary u in A. Thus exactly the same method as in Proposition 1 shows that the required equality holds when a is a unitary in A and b is an operator in $B(H)$. The result now follows from the fact that A is the linear span of its unitary group.

As a complement to the above result and to Proposition 1, we have the following result.

PROPOSITION 5. *Let* A *be a* C^**-subalgebra of* $B(H)$ *and let* f *be a pure state of* $B(H)$ *which satisfies the trace-like property* (*) *relative to* A. *Then the restriction* $f|A$ *is a character of* A.

Proof. Let h be an element of A, with $0 \leqq h \leqq 1$. Define a bounded linear functional f_h on $B(H)$ by $f_h(x) = f(xh) - f(x)f(h)$. For each positive element x of $B(H)$,

$$(f+f_h)(x) = f(x)(1-f(h)) + f(h^{\frac{1}{2}}xh^{\frac{1}{2}}) \geqq 0.$$

Also $(f+f_h)(1) = 1$, and so $f+f_h$ is a state of $B(H)$. Similarly $f-f_h$ is a state of $B(H)$. Thus, since f is a pure state of $B(H)$, we have $f_h = 0$.

It follows that, for all $a \in A$ and $b \in B(H)$, $f(ab) = f(a)f(b)$. This gives the result.

REMARK. All the above results are clearly also true with $B(H)$ replaced by an arbitrary C^*-algebra B. In applications this often does not provide a real increase in generality, so we have restricted ourselves to the statements given.

We now show that the conclusion of Proposition 4 need not hold when f is merely a tracial state of A. We restrict ourselves to the case when A is a von Neumann algebra.

PROPOSITION 6. *Let* M *be a finite von Neumann factor acting on* H. *Then a necessary and*

https://doi.org/10.1017/S0017089500002913 Published online by Cambridge University Press

sufficient condition for the existence of a state f of B(H) satisfying

$$f(bx) = f(xb) \qquad (x \in M,\ b \in B(H))$$

is that there exists a projection of norm one from B(H) onto M.

Proof. Suppose there exists a projection π of norm one from $B(H)$ onto M. According to Tomiyama [**3**, Theorem 1], π satisfies $\pi(xby) = x\pi(b)y$, for $x, y \in M$, $b \in B(H)$. Hence, if t denotes the (unique) tracial state of M, $f = t\pi$ is the required state of $B(H)$.

Conversely, suppose there exists a state f of $B(H)$ which satisfies our trace-like property relative to M. We then have $f|M = t$, the unique (normal, faithful) tracial state of M.

Let h be an operator in $B(H)$, with $0 \leqq h \leqq 1$. Define a bounded linear functional f^h on M by $f^h(x) = f(hx)$. Then, for $y \in M$, $y \geqq 0$, we have

$$f^h(y) = f(y^{\frac{1}{2}}hy^{\frac{1}{2}}) \geqq 0.$$

Hence f^h is a positive linear functional on M. Also f^h is normal on M. For if (m_α) is a net in M which increases to m, then

$$\begin{aligned} |f(h(m-m_\alpha))|^2 &\leqq f((m-m_\alpha)^*h^2(m-m_\alpha)) \\ &\leqq \|h\|^2 f((m-m_\alpha)^*(m-m_\alpha)) \to 0, \end{aligned}$$

since $f|M = t$ is normal.

Now f^h is dominated by t. Hence, by Sakai's Radon-Nikodym Theorem [**2**, Theorem 1.24.3], there exists a positive element k of M such that, for each $x \in M$, $f^h(x) = t(kx)$. It is easy to see that k is the unique positive element of M satisfying the above equality for all x in M. Define $\pi(h) = k$. Exactly as in the proof of [**2**, 4.4.23], we may now extend π to a well-defined linear mapping from $B(H)$ onto M, which turns out to be the required projection of norm one, and incidentally satisfies $f = t\pi$.

REMARKS. 1. The above result clearly establishes a one–one correspondence between projections of norm one from $B(H)$ onto M and state extensions of t to $B(H)$ which have the trace-like property.

2. If M is the (finite) von Neumann algebra of the free group on two generators then no such projections π exist, and hence no state of $B(H)$ satisfies the trace-like property relative to M.

3. Suppose M is a type II_1 factor. Then the state f of Proposition 6 (if it exists) cannot be normal. For if it were then the corresponding projection π would also be normal. According to [**4**, Proposition 1.1] this cannot happen.

Finally, the author is grateful for the financial support of a W. G. S. Brown research fellowship.

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