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A NOTE ON AUTOMORPHISMS OF FINITE p-GROUPS

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Abstract

Let p be an odd prime and let G be a finite p-group such that $xZ(G) \subseteq x^G$, for all $x \in G \setminus Z(G)$, where x^G denotes the conjugacy class of x in G. Then G has a noninner automorphism of order p leaving the Frattini subgroup $\Phi(G)$ elementwise fixed.

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1. Introduction

Let p be a prime number and let G be a nonabelian finite p-group. By a celebrated result of Gaschütz G admits noninner automorphisms of p-power order [4]. But the existence of a noninner automorphism of order p for G is a long-standing conjecture for which there is as yet no counterexample [8, Problem 4.13]. The validity of the conjecture, when G is a regular p-group, follows from a cohomological result of Schmid [9]. Abdollahi [2] has established it when $G/Z(G)$ is powerful. Deaconescu and Silberberg [3] proved that a finite p-group G satisfying the condition $C_G(Z(\Phi(G))) \neq \Phi(G)$ has a noninner automorphism of order p leaving the Frattini subgroup $\Phi(G)$ elementwise fixed. Liebeck [6] has shown the same result when G is an odd order p-group of class 2. In [1], Abdollahi has shown that every 2-group of class 2 has a noninner automorphism of order two fixing $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise.

Let G be a finite p-group and N be a nontrivial proper normal subgroup. Then (G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G \setminus N$, where x^G denotes the conjugacy class of x in G. The main result of this paper is the following theorem.

THEOREM 1.1. *Let p be an odd prime and G be a finite p-group such that $(G, Z(G))$ is a Camina pair. Then G has a noninner automorphism of order p leaving the Frattini subgroup $\Phi(G)$ elementwise fixed.*

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Before embarking on the proof, it may be worthy of remark that Yadav has verified the divisibility conjecture for automorphisms of p-groups, when $(G, Z(G))$ is a Camina pair [10]. This conjecture states that for every nonabelian finite p-group G, it follows that $|G|$ divides $|\mathrm{Aut}(G)|$.

2. Proof

Let G be a finite p-group. By $\mathrm{d}(G)$, $\mathcal{M}(G)$ and $\Omega_1(G)$ we denote the minimum number of generators of G, the set of all maximal subgroups of G and the subgroup of G generated by all elements of order p, respectively. For $x \in G$, $\{[x, G]\}$ denotes the set $\{[x, g] \mid g \in G\}$. Any other unexplained notation is standard and follows that of Gorenstein [5].

The following lemmas are well-known results and can be verified easily.

Lemma 2.1. *Let $n \in \mathbb{N}$, $x \in Z_2(G)$ and $y \in G$. Then:*

(i) $(xy)^n = x^n y^n [y, x]^{n(n-1)/2}$;
(ii) $[x^n, y] = [x, y]^n = [x, y^n]$.

Lemma 2.2. *Let G be a finite p-group, M be a maximal subgroup of G and $g \in G \setminus N$. Let $u \in Z(M)$ such that $(gu)^p = g^p$. Then the map α given by $g \mapsto gu$ and $m \mapsto m$, for all $m \in M$, can be extended to an automorphism of G and order p that fixes M elementwise.*

Proof of Theorem 1.1. Let $(G, Z(G))$ be a Camina pair and assume that G is a counterexample to the theorem.

First note that $Z(G) < Z(M)$ and $C_G(M) = Z(M)$, for all $M \in \mathcal{M}(G)$ [3, Remark 2].

Then we show that $Z_2(G)$ is abelian. It follows from [7, Theorem 2.2] that $Z_2(G)/Z(G)$ is elementary abelian. Therefore $x^p \in Z(G)$ whenever $x \in Z_2(G)$. Since $\Phi(G) = G^p G'$, Lemma 2.1 implies that $Z_2(G) \leq C_G(\Phi(G))$. On the other hand, by the main result of [3], $C_G(Z(\Phi(G))) = \Phi(G)$. Therefore $C_G(\Phi(G)) = Z(\Phi(G))$ and consequently $Z_2(G)$ is abelian.

Next, we claim that $|Z(G)| = p$ and $Z(M) \leq Z_2(G)$, for all $M \in \mathcal{M}(G)$. Let $M \in \mathcal{M}(G)$, $g \in G \setminus M$ and $x \in Z(M) \setminus Z(G)$. Since $g^p \in M$,

$$\{[x, G]\} = \{[x, \langle g \rangle M]\} = \{[x, g^i] \mid 1 \leq i \leq p\}.$$

Thus $\{[x, G]\}$ has at most p elements. By assumption $Z(G) \subseteq \{[x, G]\}$. Therefore $Z(G) = \{[x, G]\}$ and the claim follows.

After this, we prove that $\Omega_1(Z_2(G)) \setminus Z(G) \neq \emptyset$. It follows from [10, Theorem 3.1] that $d(Z_2(G)/Z(G)) = d(G)$. Since $Z_2(G)$ is abelian,

$$d(\Omega_1(Z_2(G))) = d(Z_2(G)) \geq d(Z_2(G)/Z(G)) \geq 2.$$

Now the assertion follows because $|Z(G)| = p$.

Finally, take $u \in \Omega_1(Z_2(G)) \setminus Z(G)$ and let $M = C_G(u)$. Then $M \in \mathcal{M}(G)$ and if $g \in G \setminus M$, it follows from Lemma 2.1 that $(gu)^p = g^p$. By Lemma 2.2, the map α

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given by $g \mapsto gu$ and $m \mapsto m$, for all $m \in M$, can be extended to an automorphism of order p. By assumption for some $x \in G$, $\alpha = \theta_x$, the inner automorphism induced by x. Since α is the identity on M, we must have $x \in C_G(M)$ and therefore $x \in Z(M) \leq Z_2(G)$. This means that $u = g^{-1}g^{\alpha} = [g, x] \in Z(G)$ and contradicts our choice of u. The proof is complete.

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