BULL. AUSTRAL. MATH. SOC. 13A15
VOL. 75 (2007) [417–429]

ON 2-ABSORBING IDEALS OF COMMUTATIVE RINGS

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Suppose that R is a commutative ring with $1 \neq 0$. In this paper, we introduce the concept of 2-absorbing ideal which is a generalisation of prime ideal. A nonzero proper ideal I of R is called a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It is shown that a nonzero proper ideal I of R is a 2-absorbing ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. It is shown that if I is a 2-absorbing ideal of R, then either $\mathrm{Rad}(I)$ is a prime ideal of R or $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime ideals of R that are minimal over I. Rings with the property that every nonzero proper ideal is a 2-absorbing ideal are characterised. All 2-absorbing ideals of valuation domains and Prüfer domains are completely described. It is shown that a Noetherian domain R is a Dedekind domain if and only if a 2-absorbing ideal of R is either a maximal ideal of R or M^2 for some maximal ideal M of R or M_1M_2 where M_1, M_2 are some maximal ideals of R. If R_M is Noetherian for each maximal ideal M of R, then it is shown that an integral domain R is an almost Dedekind domain if and only if a 2-absorbing ideal of R is either a maximal ideal of R or M^2 for some maximal ideal M of R or M_1M_2 where M_1, M_2 are some maximal ideals of R.

1. INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. Suppose that R is a ring. Then $T(R)$ denotes the total quotient ring of R, $\mathrm{Nil}(R)$ denotes the set of nilpotent elements of R , $Z(R)$ denotes the set of zerodivisors of R, and if I is a proper ideal of R, then $\mathrm{Rad}(I)$ denotes the radical ideal of I. We start by recalling some background material. A nonzero proper ideal I of a ring R is said to be Q-primal if $Z(R/I) = Q/I$ for some prime ideal Q of R containing I. A prime ideal P of a ring R is said to be a *divided prime ideal* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. An integral domain R is said to be a *divided domain* if every prime ideal of R is a divided prime ideal. An integral domain R is said to be a *valuation domain* if $x \mid y$ (in R) or $y \mid x$ (in R) for every nonzero $x, y \in R$. It is

Received 31st October, 2007
I would like to thank Professor David F. Anderson and the editors for their great effort in proofreading the manuscript.

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https://doi.org/10.1017/S0004972700039344 Published online by Cambridge University Press

known that a valuation domain is a divided domain. If I is a nonzero ideal of a ring R, then $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. An integral domain R is called a *Prüfer domain* if $II^{-1} = R$ for every nonzero finitely generated ideal I of R. An integral domain R is said to be a *Dedekind domain* if $II^{-1} = R$ for every nonzero ideal I of R. An integral domain R is called an almost Dedekind domain if R_M is a Dedekind domain for each maximal ideal M of R.

In this paper, we introduce the concept of 2-absorbing ideal which is a generalisation of prime ideal. A nonzero proper ideal I of R is called a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of k-absorbing ideals. We only state the definition of k-absorbing ideals. Suppose that k is a positive integer such that $k > 2$. A nonzero proper ideal I of R is called a *k-absorbing ideal* of R if whenever $a_1, a_2, \ldots, a_k \in R$ and $a_1a_2 \cdots a_k \in I$, then there are (k -1) of the a_i's whose product is in I. It is easily proved that a nonzero proper ideal I of a principal ideal domain R is a 2-absorbing ideal of R if and only if I is a prime ideal or $I = p^2R$ for some prime element p of R or $I = p_1p_2R$ where p_1, p_2 are distinct prime elements of R. Also, it is easily proved that if P and Q are some nonzero prime ideals of a ring R, then $P \cap Q$ is a 2-absorbing ideal of R. For nontrivial 2-absorbing ideals see Example 2.11, Example 2.12, Example 3.5, and Example 3.11.

Among many results in this paper, it is shown (Theorem 2.13) that a nonzero proper ideal I of R is a 2-absorbing ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. It is shown (Theorem 2.4) that if I is a 2-absorbing ideal of R, then either $\mathrm{Rad}(I)$ is a prime ideal of R or $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime ideals of R that are minimal over I. Rings with the property that every nonzero proper ideal is a 2-absorbing ideal are characterised in Theorem 3.4. It is shown (Corollary 2.7) that a 2-absorbing ideal of a ring R is a Q-primal ideal for some prime ideal Q of R. An example of a Q-primal ideal that is not a 2-absorbing ideal is illustrated in Example 3.12. For a valuation domain R, it is shown (Proposition 3.10) that a nonzero proper ideal I of R is a 2-absorbing ideal if and only $I = P$ or $I = P^2$ where $P = \mathrm{Rad}(I)$ is a prime ideal of R. For a Prüfer domain R, it is shown (Theorem 3.14) that a nonzero proper ideal I of R is a 2-absorbing ideal if and only if I is a prime ideal of R or $I = P^2$ is a P-primary ideal of R or $I = P_1 \cap P_2$ where P_1 and P_2 are nonzero prime ideals of R. It is shown (Corollary 3.16) that a Noetherian domain R that is not a field is a Dedekind domain if and only if a 2-absorbing ideal of R is either a maximal ideal of R or M^2 for some maximal ideal M of R or M_1M_2 where M_1, M_2 are some maximal ideals of R. If R_M is Noetherian for each maximal ideal M of an integral domain R, then it is shown (Proposition 3.17) that R is an almost Dedekind domain if and only if a 2-absorbing ideal of R is either a maximal ideal of R or M^2 for some maximal ideal M of R or M_1M_2 where M_1, M_2 are some maximal ideals of R. It is

https://doi.org/10.1017/S0004972700039344 Published online by Cambridge University Press

shown (Theorem 3.6) that if P is a divided prime ideal of a ring R and I is an ideal of R such that $\mathrm{Rad}(I) = P$, then I is a 2-absorbing ideal of R if and only if I is a P-primary ideal of R such that $P^2 \subseteq I$.

2. Basic properties of 2-absorbing ideals

Theorem 2.1. *Suppose that I is a 2-absorbing ideal of a ring R. Then $\mathrm{Rad}(I)$ is a 2-absorbing ideal of R and $x^2 \in I$ for every $x \in \mathrm{Rad}(I)$.*

Proof: Since I is a 2-absorbing ideal of R, observe that $x^2 \in I$ for every $x \in \mathrm{Rad}(I)$. Let $x, y, z \in R$ such that $xyz \in \mathrm{Rad}(I)$. Then $(xyz)^2 = x^2y^2z^2 \in I$. Since I is a 2-absorbing ideal, we may assume that $x^2y^2 \in I$. Since $(xy)^2 = x^2y^2 \in I$, $xy \in \mathrm{Rad}(I)$. ∎

We recall the following lemma.

Lemma 2.2. ([4, Theorem 2.1, p. 2]).

Let $I \subseteq P$ be ideals of a ring R, where P is a prime ideal. Then the following statements are equivalent:

(1) *P is a minimal prime ideal of I;*

(2) *For each $x \in P$, there is a $y \in R \setminus P$ and a nonnegative integer n such that $yx^n \in I$.*

Theorem 2.3. *Suppose that I is a 2-absorbing ideal of a ring R. Then there are at most two prime ideals of R that are minimal over I.*

Proof: Suppose that $J = \{P_i \mid P_i \text{ is a prime ideal of } R \text{ that is minimal over } I\}$ and suppose that J has at least three elements. Let $P_1, P_2 \in J$ be two distinct prime ideals. Hence there is an $x_1 \in P_1 \setminus P_2$, and there is an $x_2 \in P_2 \setminus P_1$. First we show that $x_1x_2 \in I$. By Lemma 2.2, there is a $c_2 \notin P_1$ and a $c_1 \notin P_2$ such that $c_2x_1^n \in I$ and $c_1x_2^m \in I$ for some $n, m \geqslant 1$. Since $x_1, x_2 \notin P_1 \cap P_2$ and I is a 2-absorbing ideal of R, we conclude that $c_2x_1 \in I$ and $c_1x_2 \in I$. Since $x_1, x_2 \notin P_1 \cap P_2$ and $c_2x_1, c_1x_2 \in I \subseteq P_1 \cap P_2$, we conclude that $c_2 \in P_2 \setminus P_1$ and $c_1 \in P_1 \setminus P_2$, and thus $c_1, c_2 \notin P_1 \cap P_2$. Since $c_2x_1 \in I$ and $c_1x_2 \in I$, we have $(c_1 + c_2)x_1x_2 \in I$. Observe that $c_1 + c_2 \notin P_1$ and $c_1 + c_2 \notin P_2$. Since $(c_1 + c_2)x_1 \notin P_2$ and $(c_1 + c_2)x_2 \notin P_1$, we conclude that neither $(c_1 + c_2)x_1 \in I$ nor $(c_1 + c_2)x_2 \in I$, and hence $x_1x_2 \in I$. Now suppose there is a $P_3 \in J$ such that P_3 is neither P_1 nor P_2. Then we can choose $y_1 \in P_1 \setminus (P_2 \cup P_3)$, $y_2 \in P_2 \setminus (P_1 \cup P_3)$, and $y_3 \in P_3 \setminus (P_1 \cup P_2)$. By the previous argument $y_1y_2 \in I$. Since $I \subseteq P_1 \cap P_2 \cap P_3$ and $y_1y_2 \in I$, we conclude that either $y_1 \in P_3$ or $y_2 \in P_3$ which is a contradiction. Hence J has at most two elements and that completes the proof. ∎

Theorem 2.4. *Let I be a 2-absorbing ideal of R. Then one of the following statements must hold:*

(1) $\mathrm{Rad}(I) = P$ *is a prime ideal of R such that $P^2 \subseteq I$.*

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(2) $\operatorname{Rad}(I) = P_1 \cap P_2$, $P_1P_2 \subseteq I$, *and* $\operatorname{Rad}(I)^2 \subseteq I$ *where* P_1, P_2 *are the only distinct prime ideals of* R *that are minimal over* I.

PROOF: By Theorem 2.3, we conclude that either $\operatorname{Rad}(I) = P$ is a prime ideal of R or $\operatorname{Rad}(I) = P_1 \cap P_2$, where P_1 P_2 are the only distinct prime ideals of R that are minimal over I. Suppose that $\operatorname{Rad}(I) = P$ is a prime ideal of R. Let $x, y \in P$. By Theorem 2.1, we have $x^2, y^2 \in I$. Now $x(x+y)y \in I$. Since I is a 2-absorbing ideal, we have $x(x+y) = x^2 + xy \in I$ or $(x+y)y = xy + y^2 \in I$ or $xy \in I$. It is easily proved that each case implies that $xy \in I$, and thus $P^2 \subseteq I$.

Now suppose that $\operatorname{Rad}(I) = P_1 \cap P_2$, where P_1 P_2 are the only distinct prime ideals of R that are minimal over I. Let $x, y \in \operatorname{Rad}(I)$. Then $xy \in I$ by the same argument given above, and hence $\operatorname{Rad}(I)^2 \subseteq I$. Now we show that $P_1P_2 \subseteq I$. First observe that $w^2 \in I$ for each $w \in \operatorname{Rad}(I)$ by Theorem 2.1. Let $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. Then $x_1x_2 \in I$ by the proof of Theorem 2.3. Let $z_1 \in \operatorname{Rad}(I)$ and $z_2 \in P_2 \setminus P_1$. Pick $y_1 \in P_1 \setminus P_2$. Then $y_1z_2 \in I$ by the proof of Theorem 2.3 and $z_1 + y_1 \in P_1 \setminus P_2$. Thus $z_1z_2 + y_1z_2 = (z_1 + y_1)z_2 \in I$, and hence $z_1z_2 \in I$. A similar argument will show that if $z_1 \in \operatorname{Rad}(I)$ and $z_2 \in P_1 \setminus P_2$, then $z_1z_2 \in I$. Hence $P_1P_2 \subseteq I$. ∎

THEOREM 2.5. *Let* I *be a 2-absorbing ideal of* R *such that* $\operatorname{Rad}(I) = P$ *is a prime ideal of* R *and suppose that* $I \neq P$. *For each* $x \in P \setminus I$ *let* $B_x = \{y \in R \mid yx \in I\}$. *Then* B_x *is a prime ideal of* R *containing* P. *Furthermore, either* $B_y \subseteq B_x$ *or* $B_x \subseteq B_y$ *for every* $x, y \in P \setminus I$.

PROOF: Let $x \in P \setminus I$. Since $P^2 \subseteq I$ (by Theorem 2.4), we conclude that $P \subseteq B_x$. Suppose that $P \neq B_x$ and $yz \in B_x$ for some $y, z \in R$. Since $P \subset B_x$, we may assume that $y \notin P$ and $z \notin P$, and thus $yz \notin I$. Since $yz \in B_x$, we have $yzx \in I$. Since I is a 2-absorbing ideal of R and $yz \notin I$, we conclude that either $yx \in I$ or $zx \in I$, and thus either $y \in B_x$ or $z \in B_x$. Hence B_x is a prime ideal of R containing P.

Let $x, y \in P \setminus I$ and suppose that $z \in B_x \setminus B_y$. Since $P \subseteq B_y$, $z \in B_x \setminus P$. We show that $B_y \subset B_x$. Let $w \in B_y$. Since $P \subseteq B_x$, we may assume that $w \in B_y \setminus P$. Since $z \notin P$ and $w \notin P$, we conclude that $zw \notin I$. Since $z(x+y)w \in I$ and $zw, zy \notin I$, we conclude that $(x+y)w \in I$. Hence $wx \in I$ since $(x+y)w \in I$ and $wy \in I$. Thus $w \in B_y \subseteq B_x$. ∎

THEOREM 2.6. *Let* I *be a 2-absorbing ideal of* R *such that* $I \neq \operatorname{Rad}(I) = P_1 \cap P_2$ *where* P_1 *and* P_2 *are the only nonzero distinct prime ideals of* R *that are minimal over* I. *Then for each* $x \in \operatorname{Rad}(I) \setminus I$, $B_x = \{y \in R \mid xy \in I\}$ *is a prime ideal of* R *containing* P_1 *and* P_2. *Furthermore, either* $B_y \subseteq B_x$ *or* $B_x \subseteq B_y$ *for every* $x, y \in \operatorname{Rad}(I) \setminus I$.

PROOF: Let $x \in \operatorname{Rad}(I) \setminus I$. Since $P_1P_2 \subseteq I$ by Theorem 2.4, we conclude that $xP_1 \subseteq I$ and $xP_2 \subseteq I$. Thus $P_1 \subset B_x$ and $P_2 \subset B_x$. Suppose $yz \in B_x$ for some $y, z \in R$. Since $P_1 \subset B_x$ and $P_2 \subset B_x$, we may assume that $y, z \notin P_1$ and $y, z \notin P_2$, and thus $yz \notin I$. Since $yz \in B_x$, we have $yzx \in I$. Since I is a 2-absorbing ideal of R and $yz \notin I$, we conclude that either $yx \in I$ or $zx \in I$, and thus either $y \in B_x$ or $z \in B_x$. Hence B_x

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is a prime ideal of R. By using an argument similar to that in the proof of Theorem 2.5, one can easily complete the proof. $\square$

Recall that a nonzero proper ideal I of a ring R is said to be Q-primal if $Z(R/I) = Q/I$ for some prime ideal Q of R containing I.

COROLLARY 2.7. *Suppose that I is a 2-absorbing ideal of R such that $I \neq \operatorname{Rad}(I)$. Then I is a Q-primal ideal of R where $Q = \cup_{x\in\operatorname{Rad}(I)\setminus I} B_x$ (recall that $B_x = \{y \in R \mid yx \in I\}$).*

PROOF: Let $a, b \in R \setminus I$ such that $ab \in I$. We show that $a, b \in B_f$ for some $f \in \operatorname{Rad}(I) \setminus I$. By Theorem 2.3, we conclude that either $\operatorname{Rad}(I) = P$ is a prime ideal of R or $\operatorname{Rad}(I) = P_1 \cap P_2$, where P_1, P_2 are the only distinct prime ideals of R that are minimal over I. Suppose that $\operatorname{Rad}(I) = P$ is a prime ideal of R. Hence either $a \in P \setminus I$ or $b \in P \setminus I$, and thus either $a, b \in B_a$ or $a, b \in B_b$. Since $I \neq \operatorname{Rad}(I)$, $D = \{B_x \mid x \in \operatorname{Rad}(I) \setminus I\}$ is a set of linearly ordered (prime) ideals of R by Theorem 2.5. Thus $Z(R/I) = \cup_{B_x\in D}(B_x/I)$ is an ideal of R/I.

Now suppose that $\operatorname{Rad}(I) = P_1 \cap P_2$, where P_1, P_2 are the only distinct prime ideals of R that are minimal over I. Since $ab \in \operatorname{Rad}(I)$, without loss of generality we may conclude that either $a \in \operatorname{Rad}(I) \setminus I$ or $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. If $a \in \operatorname{Rad}(I) \setminus I$, then $a, b \in B_a$. Suppose that $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Since $I \neq \operatorname{Rad}(I)$, there is a $d \in \operatorname{Rad}(I) \setminus I$. Since $P_1 \subset B_d$ and $P_2 \subset B_d$ by Theorem 2.6, we have $a, b \in B_d$. Again, since $I \neq \operatorname{Rad}(I)$, $D = \{B_x \mid x \in \operatorname{Rad}(I) \setminus I\}$ is a set of linearly ordered (prime) ideals of R by Theorem 2.6. Thus $Z(R/I) = \cup_{B_x\in D}(B_x/I)$ is an ideal of R/I. $\square$

In Section 3, we give an example (see Example 3.12) of a Q-primal ideal I of R such that $\operatorname{Rad}(I) = P$ is a prime ideal of R and $P^2 \subset I$, but I is not a 2-absorbing ideal of R.

THEOREM 2.8. *Suppose that I is an ideal of R such that $I \neq \operatorname{Rad}(I)$ and $\operatorname{Rad}(I)$ is a prime ideal of R. Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

(2) *$B_x = \{y \in R \mid yx \in I\}$ is a prime ideal of R for each $x \in \operatorname{Rad}(I) \setminus I$.*

PROOF: (1) $\Rightarrow$ (2). This is clear by Theorem 2.5.

(2) $\Rightarrow$ (1). Suppose that $xyz \in I$ for some $x, y, z \in R$. Since $\operatorname{Rad}(I)$ is a prime ideal of R, we may assume that $x \in \operatorname{Rad}(I)$. If $x \in I$, then $xy \in I$ and we are done. Hence assume that $x \in \operatorname{Rad}(I) \setminus I$. Thus $yz \in B_x$. Since B_x is a prime ideal of R by Theorem 2.5, we conclude that either $yx \in I$ or $zx \in I$. Thus I is a 2-absorbing ideal of R. $\square$

THEOREM 2.9. *Let I be an ideal of R such that $I \neq \operatorname{Rad}(I) = P_1 \cap P_2$ where P_1 and P_2 are nonzero distinct prime ideals of R that are minimal over I. Then the following statement are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

(2) *$P_1P_2 \subseteq I$ and $B_x = \{y \in R \mid yx \in I\}$ is a prime ideal of R for each $x \in \operatorname{Rad}(I) \setminus I$.*

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(3) $B_x = \{y \in R \mid yx \in I\}$ *is a prime ideal of* R *for each* $x \in (P_1 \cup P_2) \setminus I$.

PROOF: (1) $\Rightarrow$ (2). This is clear by Theorems 2.4 and 2.6.

(2) $\Rightarrow$ (3). Let $x \in P_1 \setminus P_2$. It is clear that $yx \in I$ if and only if $y \in P_2$. Since $P_1P_2 \subseteq I$, we conclude that $B_x = P_2$ is a prime ideal of R. Let $z \in P_2 \setminus P_1$. By a similar argument as before we conclude that $B_z = P_1$ is a prime of R. Since B_d is a prime ideal of R for each $d \in \text{Rad}(I) \setminus I$, we are done.

(3) $\Rightarrow$ (1). Let $xyz \in I$. We may assume that $x \in (P_1 \cup P_2) \setminus I$. Thus $yz \in B_x$. Since B_x is a prime ideal of R by Theorem 2.6, we conclude that either $yx \in I$ or $zx \in I$, and hence I is a 2-absorbing ideal of R. ∎

THEOREM 2.10. *Let* I *be a 2-absorbing ideal of a ring* R *such that* $I \neq \text{Rad}(I)$. *For each* $x \in \text{Rad}(I) \setminus I$, *let* $B_x = \{y \in R \mid yx \in I\}$. *Then :*

(1) *If* $x \in \text{Rad}(I) \setminus I$ *and* $y \in R$ *such that* $yx \notin I$, *then* $B_{yx} = B_x$.

(2) *If* $x, y \in \text{Rad}(I) \setminus I$ *and* B_x *is properly contained in* B_y, *then* $B_{fx+dy} = B_x$ *for every* $f, d \in R$ *such that* $fd \notin B_x$. *In particular, if* $x, y \in \text{Rad}(I) \setminus I$ *and* B_x *is properly contained in* B_y, *then* $B_{x+y} = B_x$.

PROOF: (1) Let $x, y \in \text{Rad}(I) \setminus I$. Since $B_x \subset B_y$, it is clear that $B_x \subseteq B_{yx}$. Let $c \in B_{yx}$. Since $cyx \in I$, we conclude that $cy \in B_x$. Since B_x is a prime ideal of R by Theorems 2.5, 2.6 and $y \notin B_x$ because $yx \notin I$, we have $cx \in I$. Hence $c \in B_x$, and thus $B_x = B_{yx}$.

(2) Let $x, y \in \text{Rad}(I) \setminus I$. Since $B_x \subset B_y$, it is clear that $B_x \subseteq B_{fx+dy}$. Suppose that $B_x \neq B_{fx+dy}$. Since B_x, B_{fx+dy}, B_y are linearly ordered by Theorems 2.5, 2.6 and B_x is properly contained in B_y, there is a $z \in B_y \cap B_{fx+dy}$ such that $z \notin B_x$. Since $zy \in I$ and $z(fx + dy) \in I$, we conclude that $zfx \in I$. Hence $zf \in B_x$, a contradiction since neither $z \in B_x$ nor $f \in B_x$. Thus $B_x = B_{fx+dy}$. ∎

EXAMPLE 2.11. Suppose that $R = \mathcal{Z}[x, y]$ where $\mathcal{Z}$ is the ring of integers and x, y are indeterminates, $P_1 = (x, 2)R, P_2 = (y, 2)R$ are prime ideals of R, and let $I = P_1P_2 = (4, 2x, 2y, xy)R$. Then $\text{Rad}(I) = P_1 \cap P_2 = (2, xy)R$. Since $B_2 = \{z \in R \mid 2z \in I\} = (2, x, y)R$ is a (maximal) prime ideal of R, it is easy to see that $B_d = B_2$ for each $d \in \text{Rad}(I) \setminus I$. Hence I is a 2-absorbing ideal of R by Theorem 2.9.

EXAMPLE 2.12. Suppose that $R = \mathcal{Z}[x, y, z]$ where x, y, z are indeterminates, $P = (2, x)R$ is a prime ideal of R, and $I = (4, 2x, 2y, xy, xz, x^2)R$. Then $P^2 \subset I$ and $\text{Rad}(I) = P$. Now $B_2 = (2, x, y)R$ is a prime ideal of R, $B_x = (2, x, y, z)R$ is a (prime) maximal ideal of R, and $B_{2+x} = B_2$. It is easy to see that if $d \in P \setminus I$, then either $B_d = B_2$ or $B_d = B_x$. Thus I is a 2-absorbing ideal of R by Theorem 2.8. Observe that I is not a primary ideal.

Part of this paper was presented at a commutative ring conference in Cortona, Italy (June, 2004). During the conference, Bruce Olberding asked the author the following

https://doi.org/10.1017/S0004972700039344 Published online by Cambridge University Press

question: Let I be a 2-absorbing ideal of a ring R and suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, does it follow that $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$? The answer to the question is yes as in the following result.

THEOREM 2.13. *Suppose that I is a nonzero proper ideal of a ring R. The following statements are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

(2) *If $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.*

PROOF: Since (2) $\Rightarrow$ (1) is trivial, we only need to show that (1) $\Rightarrow$ (2). Suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R. By Theorem 2.4, we conclude that $\mathrm{Rad}(I)$ is a prime ideal of R or $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1 and P_2 are nonzero distinct prime ideals of R that are minimal over I. If $I = \mathrm{Rad}(I)$, then it is easily proved that $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. Hence assume that $I \neq \mathrm{Rad}(I)$. We consider two cases.

Case I. Suppose that $\mathrm{Rad}(I)$ is a prime ideal of R. Then we may assume that $I_1 \subseteq \mathrm{Rad}(I)$ and $I_1 \not\subseteq I$. Let $x \in I_1 \setminus I$. Since $xI_2I_3 \subseteq I$, we conclude that $I_2I_3 \subseteq B_x$. Since B_x is a prime ideal of R by Theorem 2.8, we conclude that either $I_2 \subseteq B_x$ or $I_3 \subseteq B_x$. If $I_2 \subseteq B_d$ and $I_3 \subseteq B_d$ for each $d \in I_1 \setminus I$, then $I_1I_2 \subseteq I$ (and $I_1I_3 \subseteq I$) and we are done. Hence assume that that $I_2 \subseteq B_y$ and $I_3 \not\subseteq B_y$ for some $y \in I_1 \setminus I$. Since $\{B_w \mid w \in I_1 \setminus I\}$ is a set of prime ideals of R that are linearly ordered by Theorem 2.5 and $I_2 \subseteq B_y$ and $I_3 \not\subseteq B_y$, we conclude that $I_2 \subseteq B_z$ for each $z \in I_1 \setminus I$, and thus $I_1I_2 \subseteq I$.

Case II. Suppose that $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1 and P_2 are nonzero distinct prime ideals of R that are minimal over I. We may assume that $I_1 \subseteq P_1$. If either $I_2 \subseteq P_2$ or $I_3 \subseteq P_2$, then either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ because $P_1P_2 \subseteq I$ by Theorem 2.4. Hence assume that $I_1 \subseteq \mathrm{Rad}(I)$ and $I_1 \not\subseteq I$. By an argument similar to that one given in case I and Theorem 2.5, we are done. ∎

3. ON 2-ABSORBING IDEALS IN PARTICULAR CLASSES OF RINGS

THEOREM 3.1. *Suppose that I is a P-primary ideal of a ring R. Then I is a 2-absorbing ideal of R if and only if $P^2 \subseteq I$. In particular, M^2 is a 2-absorbing ideal of R for each maximal ideal M of R.*

PROOF: Suppose that I is a 2-absorbing ideal of a ring R. Then $P^2 \subseteq I$ by Theorem 2.4(1). Conversely, suppose that $P^2 \subseteq I$ and $xyz \in I$. If either $x \in I$ or $yz \in I$, then there is nothing to prove. Hence assume that neither $x \in I$ nor $yz \in I$. Since I is a P-primary ideal of R, we conclude that $x \in P$ and $yz \in P$. Thus $x, y \in P$ or $x, z \in P$. Since $P^2 \subseteq I$, we conclude that $xy \in I$ or $xz \in I$. ∎

COROLLARY 3.2. *Suppose that P is a nonzero prime ideal of R. Then $P^{(2)} = P^2R_P \cap R$ is a 2-absorbing ideal of R.*

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PROOF: It is well-known that $P^{(2)}$ is a P-Primary ideal of R. Since $P^2 \subseteq P^{(2)}$, $P^{(2)}$ is a 2-absorbing ideal of R by Theorem 3.1. ∎

The following lemma is useful in the proof of our next result.

LEMMA 3.3. *Suppose that R is a zero-dimensional ring with exactly two distinct maximal ideals such that $\mathrm{Nil}(R) \neq \{0\}$, $\mathrm{Nil}(R)^2 = \{0\}$ and $\mathrm{Nil}(R) = wR$ for each nonzero $w \in \mathrm{Nil}(R)$. Then R is ring-isomorphic to $R/M_1^2 \oplus R/M_2$ where M_1 is a maximal ideal of R such that $M_1^2 \neq M_1$ and M_2 is a maximal of R such that $M_2^2 = M_2$. Furthermore, each nonzero proper ideal of R is a 2-absorbing ideal of R.*

PROOF: Let M_1, M_2 be the two distinct maximal ideals of R. Since $\mathrm{Nil}(R) = M_1M_2$ and $\mathrm{Nil}(R)^2 = \{0\}$, we conclude that $M_1^2M_2^2 = \{0\}$. Since M_1^2, M_2^2 are co-maximal, R is ring-isomorphic to $D = R/M_1^2 \oplus R/M_2^2$. Since $\mathrm{Nil}(R) \neq \{0\}$, we conclude that at least one of the maximal ideals of R is a non-idempotent ideal. Hence we may assume that $M_1^2 \neq M_1$, and thus there is an element $m_1 \in M_1$ such that $m_1 \notin M_1^2$. Now suppose that $M_2^2 \neq M_2$. Then there is an element $m_2 \in M_2$ such that $m_2 \notin M_2^2$. Since $(0, m_2 + M_2^2), (m_1 + M_1^2, 0)$ are nonzero nilpotent elements of D, $(0, m_2 + M_2^2) \in (m_1 + M_1^2, 0)D$ by hypothesis, which is impossible. Thus $M_2^2 = M_2$. Hence $\mathrm{Nil}(D) = \{0\} \oplus (M_1/M_1^2)$. Since $wD = \mathrm{Nil}(D)$ for each nonzero $w \in \mathrm{Nil}(D)$, we conclude that $\mathrm{Nil}(D)$ is the only proper non-maximal ideal of D. Thus every nonzero proper ideal of D is a 2-absorbing ideal of D, and hence every nonzero proper ideal of R is a 2-absorbing ideal of R. ∎

Recall that an element $x \in R$ is said to be a π-regular element of R if there is a positive integer n and an element $y \in R$ such that $x^{2n}y = x^n$. If every element of R is a π-regular element, then R is called a π-regular ring. It is well-known [4, Theorem 3.1] that a ring R is a π-regular ring if and only if R is a zero-dimensional ring.

THEOREM 3.4. *Every nonzero proper ideal of a ring R is a 2-absorbing ideal of R if and only if R is zero-dimensional (that is, R is a π-regular ring) and one of the following statements hold:*

(1) *R is quasi-local with maximal ideal $M = \mathrm{Nil}(R) \neq \{0\}$ such that $M^2 \subseteq xR$ for each nonzero $x \in M$.*
(2) *R has exactly two distinct maximal ideals such that either R is ring-isomorphic to $F_1 \oplus F_2$ where F_1 and F_2 are fields or $\mathrm{Nil}(R)^2 = \{0\}$ and $\mathrm{Nil}(R) = wR$ for each nonzero $w \in \mathrm{Nil}(R)$.*
(3) *R is ring-isomorphic to $F_1 \oplus F_2 \oplus F_3$ where F_1, F_2, F_3 are fields.*

PROOF: Suppose that R is quasi-local with maximal ideal $M = \mathrm{Nil}(R) \neq \{0\}$ such that $M^2 \subseteq xR$ for each nonzero $x \in M$. Since every nonzero proper ideal I of R is an M-primary ideal of R and $M^2 \subseteq I$, we conclude that every nonzero proper ideal of R is a 2-absorbing ideal of R by Theorem 3.1. Suppose that R is zero-dimensional and the second condition holds. If $\mathrm{Nil}(R) = \{0\}$, then it is easily proved that every nonzero proper ideal of R is a 2-absorbing ideal of R. If $\mathrm{Nil}(R) \neq \{0\}$, then every nonzero proper

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ideal of R is a 2-absorbing ideal of R by Lemma 3.3. Suppose that R is ring-isomorphic to $D = F_1 \oplus F_2 \oplus F_3$ where F_1, F_2, F_3 are fields. Since every nonzero proper ideal of D is either a maximal ideal of D or a product(intersection) of two distinct maximal ideals of D, we conclude that every nonzero proper ideal of D is a 2-absorbing ideal of D, and hence every nonzero proper ideal of R is a 2-absorbing ideal of R.

Conversely, suppose that every nonzero proper ideal of R is a 2-absorbing ideal of R. We show that R is a zero-dimensional ring. Let $w \in R$. If w is a unit of R or a nilpotent of R, then w is a π-regular element of R. Hence assume that w is a nonunit non-nilpotent element of R. Then w^4R is a nonzero proper ideal of R, and hence it is a 2-absorbing ideal of R. Since $w^4 \in w^4R$, we conclude that $w^2 \in w^4R$, and thus w is a π-regular element of R. Hence R is a π-regular ring, and thus R is a zero-dimensional ring.

Next we show that R has at most three distinct maximal ideals. Suppose that M_1, M_2, M_3 are distinct maximal ideals of R. Then $I = M_1M_2M_3 = M_1 \cap M_2 \cap M_3 = \{0\}$, for if $I \neq \{0\}$, then $I = \mathrm{Rad}(I)$ is a 2-absorbing ideal of R which is impossible by Theorem 2.4. Since $M_1M_2M_3 = \{0\}$, R has at most three distinct maximal ideals.

Now suppose that R has exactly three distinct maximal ideal M_1, M_2, M_3. Since $M_1M_2M_3 = \{0\}$, we conclude that R is ring-isomorphic to $R/M_1 \oplus R/M_2 \oplus R/M_3$, and thus the third condition holds.

Suppose that R has exactly two distinct maximal ideals M_1, M_2. If $\mathrm{Nil}(R) = M_1M_2 = \{0\}$, then R is ring-isomorphic to $R/M_1 \oplus R/M_2$. Hence assume that $\mathrm{Nil}(R) = M_1M_2 \neq \{0\}$. Suppose that $\mathrm{Nil}(R)^2 \neq \{0\}$. Then there are nonzero elements w_1, $w_2 \in \mathrm{Nil}(R)$ such that $w_1w_2 \neq 0$. Since w_1w_2R is a 2-absorbing ideal of R, we conclude that $w_1 \in M1M_2 = \mathrm{Nil}(R) \subseteq w_1w_2R$ by Theorem 2.4. Hence $w_1 = w_1w_2k$ for some nonzero $k \in R$, and thus $w_1(1 - w_2k) = 0$. Hence $w_1 = 0$ since $1 - w_2k$ is a unit of R, a contradiction. Thus $\mathrm{Nil}(R)^2 = \{0\}$. Suppose that w is a nonzero nilpotent element of R. Since wR is a 2-absorbing ideal of R, we conclude that $\mathrm{Nil}(R) = M_1M_2 \subseteq wR$ by Theorem 2.4, and hence the second condition holds.

Finally suppose that R is a quasi-local ring with maximal ideal $\mathrm{Nil}(R) \neq \{0\}$. Suppose that w is a nonzero element of $\mathrm{Nil}(R)$. Since wR is a 2-absorbing ideal of R, we conclude that $\mathrm{Nil}(R)^2 \subseteq wR$ by Theorem 2.4. Thus the first condition holds. □

EXAMPLE 3.5.

(a) Let $\mathcal{Z}$ be the ring of integers, $R = \mathcal{Z}_8$, and $D = \mathcal{Z}_{p^2} \oplus F$ where p is a prime number of $\mathcal{Z}$ and F is a field. Then every nonzero proper ideal of R is a 2-absorbing ideal and every nonzero proper ideal of D is a 2-absorbing ideal.

(b) Let $\mathcal{R}$ be the ring of all real numbers and X, Y be indeterminates. Set $R = \mathcal{R}[[X, Y]]/(XY, X^2 - Y^2, X^3, Y^3)$. Then every nonzero proper ideal of R is a 2-absorbing ideal.

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Recall that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \backslash P$.

THEOREM 3.6. *Suppose that P is a nonzero divided prime ideal of R and I is an ideal of R such that* $\mathrm{Rad}(I) = P$. *Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

(2) *I is a P-primary ideal of R such that $P^2 \subseteq I$.*

PROOF: (1) $\Rightarrow$ (2). Suppose that I is a 2-absorbing ideal of R. Since $\mathrm{Rad}(I) = P$ is a nonzero prime ideal of R, $P^2 \subseteq I$ by Theorem 2.4(1). Now let $xy \in I$ for some $x, y \in R$ and suppose that $y \notin P$. Since $x \in P$ and P is a divided ideal of R, we conclude that $x = yk$ for some $k \in R$. Hence $xy = y^2k \in I$. Since $y^2 \notin I$ and I is a 2-absorbing ideal of R, we conclude that $yk = x \in I$. Thus I is a P-primary ideal of R.

(2) $\Rightarrow$(1). This is clear by Theorem 3.1. ∎

THEOREM 3.7. *Suppose that* $\mathrm{Nil}(R)$ *and P are divided prime ideals of a ring R such that $P \neq$* $\mathrm{Nil}(R)$. *Then P^2 is a 2-absorbing ideal of R.*

PROOF: First we observe that $\mathrm{Nil}(R) \subset P^2$ since $P \neq \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is divided. By Theorem 3.6 it suffices to show that P^2 is a P-primary ideal of R. Suppose that $xy = p_1q_1 + \cdots + p_nq_n \in P^2$ where the p_i's and the q_i's are in P, and suppose that $y \notin P$. Since P is a divided ideal of R, we conclude that $xy = yc_1q_1 + \cdots + yc_nq_n \in P^2$ where the c_i's are in P. Hence $y(x - c_1q_1 - \cdots - c_nq_n) = 0 \in \mathrm{Nil}(R)$. Since $y \notin \mathrm{Nil}(R)$ (because $y \notin P$) and $\mathrm{Nil}(R)$ is a prime ideal of R, we conclude that $x - c_1q_1 - \cdots - c_nq_n = w \in \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R) \subset P^2$, we conclude that $x = c_1q_1 + \cdots c_nq_n + w \in P^2$, and thus P^2 is a P-primary ideal of R. ∎

If R is an integral domain, then $\mathrm{Nil}(R) = \{0\}$ is a divided prime ideal of R. Hence we have the following corollary.

COROLLARY 3.8. *Suppose that P is a a nonzero divided prime ideal of an integral domain R. Then P^2 is a 2-absorbing ideal of R.*

The following is an example of a prime ideal P of an integral domain R such that P^2 is not a 2-absorbing ideal of R.

EXAMPLE 3.9. Suppose that $R = \mathcal{Z} + 6x\mathcal{Z}[x]$ and $P = 6x\mathcal{Z}[x]$ (where $\mathcal{Z}$ is the ring of integers and x is an indeterminate). Then P is a prime ideal of R. Since $6x^2 \in P \setminus P^2$ and $B_{6x^2} = \{y \in R \mid 6x^2y \in P^2\} = 6\mathcal{Z} + 6x\mathcal{Z}[x]$ is not a prime ideal of R, P^2 is not a 2-absorbing ideal of R by Theorem 2.8.

PROPOSITION 3.10. *Suppose that R is a valuation domain and I is a nonzero proper ideal of R. Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

(2) *I is a a P-primary ideal of R such that $P^2 \subseteq I$;*

(3) *$I = P$ or $I = P^2$ where $P =$* $\mathrm{Rad}(I)$ *is a prime ideal of R.*

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PROOF: (1) $\Rightarrow$ (2). Suppose that I is a 2-absorbing ideal of R. Then $\mathrm{Rad}(I) = P$ is a prime ideal of R. Since R is a divided domain, I is a P-primary ideal of R such that $P^2 \subseteq I$ by Theorem 3.6.

(2) $\Rightarrow$ (3). Suppose that I is a P-primary ideal of R such that $P^2 \subseteq I$. Since R is a valuation domain, we conclude that either $I = P$ or $I = P^2$ by [**5**, Theorem 5.11, p. 106].

(3) $\Rightarrow$ (1). Suppose that either $I = P$ or $I = P^2$ where $P = \mathrm{Rad}(I)$ is a prime ideal of R. If $I = P$, then I is a 2-absorbing ideal of R. If $I = P^2$, then I is a 2-absorbing ideal of R by Corollary 3.8. ∎

The following is an example of a prime ideal P of an integral domain R such that P^2 is a 2-absorbing ideal of R, but P^2 is not a P-primary ideal of R.

EXAMPLE 3.11. Suppose that $R = \mathcal{Z} + 3x\mathcal{Z}[x]$ (where $\mathcal{Z}$ is the ring of integers and x is an indeterminate) and let $P = 3x\mathcal{Z}[x]$ be a prime ideal of R. Since $3(3x^2) \in P^2$, we conclude that P^2 is not a P-primary ideal of R. It is easy to verify that if $d \in P \setminus P^2$, then either $B_d = \{y \in R \mid yd \in I\} = P$ or $B_d = 3\mathcal{Z} + 3x\mathcal{Z}[x]$ is a prime ideal of R. Hence P^2 is a 2-absorbing ideal by Theorem 2.8.

Next we show that for each $n \geqslant 2$, there is a valuation domain R with maximal ideal M and Krull dimension n that admits an M-primal ideal I such that $\mathrm{Rad}(I) = P$ is a prime ideal of R , $P^2 \subset I$, and the Krull dimension of R/I is $n-1$, but I is not a 2-absorbing ideal of R.

EXAMPLE 3.12. Suppose that $n \geqslant 2$ and D be a valuation domain with quotient field K and Krull dimension $n-1$. Let X be an indeterminate and set $R = D + XK\big[[X]\big]$. Then R is a valuation domain with Krull dimension n. Let $P = XK\big[[X]\big]$ be a prime ideal of R and let Q be a nonzero prime ideal of R such that $Q \neq P$. Then it is clear that $P \subset Q$. Set $I = XR_Q$. Then I is an ideal of R such that $\mathrm{Rad}(I) = P$ and $Z(R/I) = Q/I$ by [**1**, Proposition 2.1]. Hence I is not a primary ideal of R. Since R is a valuation domain and $X \in P \setminus P^2$, we have $P^2 \subset I$ and I is not a 2-absorbing ideal of R by Proposition 3.10. By construction it is clear that the Krull dimension of R/I is $n-1$.

Before we state our next theorem, the following lemma is needed.

LEMMA 3.13. *Suppose that I is a 2-absorbing ideal of a ring R and let S be a multiplicatively closed subset of R. If $IR_S \neq \{0\}$, then IR_S is a 2-absorbing ideal of R_S.*

PROOF: Suppose that $xyz \in IR_S$ for some $x, y, z \in R_S$. Then there are elements $s \in S$, and $x_1, x_2, x_3 \in R$ such that $xyz = (x_1/s)(x_2/s)(x_3/s) = x_1x_2x_3/s^3 \in IR_S$. Thus, $x_1x_2x_3 \in I$. Since I is a 2-absorbing ideal of R, we have $x_1x_2 \in I$ or $x_1x_3 \in I$ or $x_2x_3 \in I$, and thus $xy \in IR_S$ or $xz \in IR_S$ or $yz \in IR_S$. ∎

THEOREM 3.14. *Suppose that R is a Prüfer domain and I is a nonzero ideal of R. Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of R;*

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(2) *I is a prime ideal of R or $I = P^2$ is a P-primary ideal of R or $I = P_1 \cap P_2$ where P_1 and P_2 are nonzero prime ideals of R.*

PROOF: Suppose that I is a nonzero 2-absorbing ideal of R. Then either $\mathrm{Rad}(I) = P$ is a prime ideal of R or $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1, P_2 are the only minimal prime ideals of R over I by Theorem 2.4. Suppose that $\mathrm{Rad}(I) = P$ is a prime ideal of R and $I \neq P$. Then I is a Q-primal ideal of R by Corollary 2.7, and $P \subseteq Q$ because $P^2 \subseteq I$ by Theorem 2.4. Since IR_Q is a 2-absorbing ideal of R_Q by Lemma 3.13 and R_Q is a valuation domain, we conclude that IR_Q is a PR_Q-primary ideal of R_Q by Proposition 3.10. Hence $IR_Q \cap R$ is a P-primary ideal of R by [**5**, Corollary 3.10, p. 68]. It is easy to verify that $IR_Q \cap R = I$ (for a proof see [**2**, Lemma 1.3]). Hence $I = IR_Q \cap R$ is a P-primary ideal of R. Since $P^2 \subseteq I$ by Theorem 2.4 and $I \neq P$, we conclude that $I = P^2$ by [**5**, Proposition 6.9(4), p. 132].

Next suppose that $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1, P_2 are the only minimal prime ideals of R over I. Assume that $I \neq \mathrm{Rad}(I)$. Then I is a Q-primal ideal of R by Corollary 2.7. Since $P_1 \subset Q$ and $P_2 \subset Q$ and R_Q is a valuation domain, either $P_1R_Q \subset P_2R_Q$ or $P_2R_Q \subset P_1R_Q$, which is impossible. Thus $I = \mathrm{Rad}(I) = P_1 \cap P_2$.

For the converse, just observe that if $I = P^2$ is a P-primary ideal of R, then I is a 2-absorbing ideal of R by Theorem 3.1. ∎

Recall that an integral domain R is said to be a Dedekind domain if every nonzero ideal of R is invertible.

THEOREM 3.15. *Let R be a Noetherian domain that is not a field. The following statements are equivalent:*

(1) *R is a Dedekind domain;*

(2) *If I is a 2-absorbing ideal of R, then I is a maximal ideal of R or $I = M^2$ for some maximal ideal M of R or $I = M_1M_2$ where M_1, M_2 are some maximal ideals of R;*

(3) *If I is a 2-absorbing ideal of R, then I is a prime ideal of R or $I = P^2$ for some prime ideal P of R or $I = P_1 \cap P_2$ where P_1, P_2 are some prime ideals of R.*

PROOF: (1) $\Rightarrow$ (2). Since R is a one-dimensional ring, every nonzero prime ideal of R is maximal. Suppose that I is a 2-absorbing ideal of R. Then either $\mathrm{Rad}(I) = M$ is a a maximal ideal of R or $\mathrm{Rad}(I) = M_1 \cap M_2 = M_1M_2$ for some distinct maximal ideals M_1, M_2 of R by Theorem 2.4.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). Let M be a maximal ideal of R. Since every ideal between M^2 and M is an M-Primary ideal and hence a 2-absorbing ideal of R by Theorem 3.1, the hypothesis in (3) implies that there are no ideals properly between M^2 and M. Hence R is a Dedekind domain by [**3**, Theorem 39.2, p. 470]. ∎

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Recall that an integral domain R is said to be an almost Dedekind domain if R_M is a Dedekind domain for each maximal ideal M of R (that is, R_M is a Noetherian valuation domain for each maximal ideal M of R and hence R is a one-dimensional ring.) The following result is a characterisation of an almost Dedekind domain in terms of 2-absorbing ideals. The proof of the following result is similar to the proof of Theorem 3.15, and hence it is left to the reader.

PROPOSITION 3.16. *Let R be an integral domain that is not a field and suppose that R_M is Noetherian for each maximal ideal M of R. The following statements are equivalent:*

(1) *R is an almost Dedekind domain;*

(2) *If I is a 2-absorbing ideal of R, then I is a maximal ideal of R or $I = M^2$ for some maximal ideal M of R or $I = M_1M_2$ where M_1, M_2 are some maximal ideals of R;*

(3) *If I is a 2-absorbing ideal of R, then I is a prime ideal of R or $I = P^2$ for some prime ideal P of R or $I = P_1 \cap P_2$ where P_1, P_2 are some prime ideals of R.*

REFERENCES

[1] A. Badawi, 'On ϕ-pseudo-valuation rings, II', *Houston J. Math.* **26** (2000)), 473–480.

[2] L. Fuchs, W. Heinzer, and B. Olberding, 'Commutative ideal theory without finiteness conditions: primal ideals', *Tran. Amer. Math. Soc.* (to appear).

[3] R. Gilmer, *Multiplicative ideal theory*, Queen's Papers Pure Appl. Math. **90** (Queen's University, Kingston, 1992).

[4] J. Huckaba, *Commutative rings with zero divisors* (Marcel Dekker, New York, Basel, 1988).

[5] M.D. Larson and P.J. McCarthy, *Multiplicative theory of ideals* (Academic Press, New York, London, 1971).

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https://doi.org/10.1017/S0004972700039344 Published online by Cambridge University Press