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AMENABILITY AND WEAK AMENABILITY OF TENSOR ALGEBRAS AND ALGEBRAS OF NUCLEAR OPERATORS

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Abstract

Let E and F constitute a Banach pairing. We prove that the algebra of F-nuclear operators on E, $\mathscr{N}_F(E)$, is amenable if and only if E is finite dimensional and is weakly amenable if and only if $\dim K_F \leq 1$, and the trace on $E\widehat{\otimes}F$ is injective on K_F. Here K_F is the kernel of the canonical map $E\widehat{\otimes}F \to \mathscr{N}_F(E)$. On the route we find the corresponding statements for the associated tensor algebra, $E\widehat{\otimes}F$.

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1. Introduction

In [5] B. E. Johnson developed a cohomology theory for Banach algebras. Special focus was put on the cohomology groups with coefficients in dual modules of Banach modules over the Banach algebra in question. The significant class of amenable Banach algebras, for which these invariants vanish, was singled out, and some important examples, notably groups algebras, were computed.

It has appeared to be, perhaps surprisingly, difficult to calculate specific instances of cohomology groups, and except in the C^*-algebra case no substantial enlargement of the class of amenable Banach algebras has been discovered since Johnson's original paper.

Much more recent the notion of weak amenability has been introduced

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([1], [2], and [6]). Here focus is on the invariant $H^1(\mathfrak{A}, \mathfrak{A}^*)$. When $H^1(\mathfrak{A}, \mathfrak{A}^*) = (0)$ the algebra $\mathfrak{A}$ is called weakly amenable. In the non-commutative case the list of classes of Banach algebras known to be weakly amenable, rapidly runs out (cf. [2] and [6]), and it is pertinent to the clarification of the importance of the concept to demonstrate weak amenability in other concrete cases.

In this paper we shall look at algebras of nuclear operators. We show that amenability only occurs in the finite dimensional case and that weak amenability is closely related to the Grothendieck approximation property. Simultaneously we answer amenability/weak amenability questions for the tensor algebras defining the algebras of nuclear operators.

We have made a point out of using proofs that are based on homological concepts, but are aware that other and perhaps more direct proofs can be given. The reader is referred to [4] for details of a general homology theory for Banach algebras.

If $\mathfrak{A}$ is a Banach algebra, then $\mathfrak{A}_+$ denotes the Banach algebra obtained by formal adjunction of a unit.

2. The results

Let $(E, F, \langle\cdot,\cdot\rangle_F)$ be a Banach pairing, that is E and F are Banach spaces and $\langle\cdot,\cdot\rangle_F : E \times F \to \mathbb{C}$ is a continuous bilinear form so that $\langle x, F\rangle_F = \{0\}$ implies $x = 0$ and $\langle E, f\rangle_F = \{0\}$ implies $f = 0$. On the projective tensor product $E\widehat{\otimes}F$ we define an algebra multiplication by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1\rangle_F x_1 \otimes y_2 \qquad (x_i \in E, y_i \in F).$$

Then $E\widehat{\otimes}F$ becomes a Banach algebra. In [8] it is shown that $E\widehat{\otimes}F$ is (the model example of) a biprojective Banach algebra. In the usual way we shall represent the conjugate space of $E\widehat{\otimes}F$ as $B(F, E^*)$, the set of bounded operators from F to E^*. The space of F-nuclear operators on E, $\mathcal{N}_F(E)$, is defined as the range of $\tau : E\widehat{\otimes}F \to B(E)$ given by

(1) $$\tau(x \otimes y)(\xi) = \langle \xi, y\rangle_F x, x(\xi \in E, y \in F).$$

With the norm and multiplication inherited from $E\widehat{\otimes}F$, $\mathcal{N}_F(E)$ becomes a continuously embedded Banach subalgebra of $B(E)$. To each F-nuclear map S we associate two dual maps

$$S' \in B(F) \text{ and } S^* \in B(E^*),$$

where E^* is the conjugate space of E. The map S^* is the usual adjoint of $S \in B(E)$ and S' is given by

(2) $$S^* \circ i = i \circ S',$$

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where $i\colon F \to E^*$ is the natural inclusion.

The canonical module action of $E\widehat{\otimes}F$ on $B(F, E^*)$ is given by

$$(3)\qquad u.T = T \circ \tau(u)' \qquad T.u = \tau(u)^* \circ T$$

where $u \in E\widehat{\otimes}F$, $T \in B(F, E^*)$.

LEMMA 1. *Let K_F be the kernel of the map $\tau\colon E\widehat{\otimes}F \to B(E)$. Then $(E\widehat{\otimes}F)K_F = K_F(E\widehat{\otimes}F) = 0$.*

PROOF. This is an immediate consequence of (3) and the Hahn-Banach Theorem.

LEMMA 2. *The space of bounded traces on $E\widehat{\otimes}F$ is one-dimensional, spanned by $t(\sum x_i \otimes y_i) = \sum \langle x_i, y_i\rangle_F$.*

PROOF. Choose $x_0 \in E$ and $y_0 \in F$ such that $\langle x_0, y_0\rangle_F = 1$. Let $x \otimes y$ be an arbitrary elementary tensor. Then $x \otimes y = (x \otimes y_0)(x_0 \otimes y)$. Since $(x_0 \otimes y)(x \otimes y_0) = \langle x, y\rangle_F x_0 \otimes y_0$, the assertion follows.

We shall now determine when $\mathcal{N}_F(E)$ is weakly amenable. First note that for any Banach pairing, E, F, the tensor algebra $E\widehat{\otimes}F$ is biprojective [6, Lemma 2.2]. Hence $(E\widehat{\otimes}F)^*$ is biinjective so from the general formula $H^n(\mathfrak{A}, \mathfrak{A}^*) = \operatorname{Ext}^n_{\mathfrak{A}^{env}}(\mathfrak{A}_+, \mathfrak{A}^*)$ it is thus obvious that $E\widehat{\otimes}F$ is weakly amenable.

THEOREM 3. *$\mathcal{N}_F(E)$ is weakly amenable if and only if the following two conditions hold.*

(a) $\dim K_F \leq 1$.

(b) *t is injective on K_F,*

where t is the trace on $E\widehat{\otimes}F$.

PROOF. Since $E\widehat{\otimes}F$ is weakly amenable we see from [2, Proposition 2.6] that $\mathcal{N}_F(E)$ is weakly amenable if and only if the canonical map of cocycles $Z^0(E\widehat{\otimes}F, (E\widehat{\otimes}F)^*) \to Z^0(E\widehat{\otimes}F, K_F^*)$ is surjective. By Lemma 1,

$$Z^0(E\widehat{\otimes}F, K_F^*) = K_F^*,$$

and, by Lemma 2,

$$\dim(Z^0(E\widehat{\otimes}F, (E\widehat{\otimes}F)^*)) = 1.$$

Hence (a) and (b) are clearly necessary, and since $t(K_F) = \{0\}$ if and only if $K_F = (0)$ when (b) holds, the result follows.

REMARK. Recall that, if E or F has the approximation property, then $K_F = (0)$, so weak amenability of $\mathscr{N}_F(E)$ is apparently slightly weaker than E having the approximation property. The author does not know, whether the case $\dim K_F = 1$ can occur, and, if it does, whether necessarily $t(K_F) \neq (0)$.

According to one of the characterizations of the approximation property in [3], the condition (b) is automatically satisfied when $F = E^*$. More generally we have:

COROLLARY 4. (i) *Suppose that either* $B(E)$ *or* $B(F)$ *is a total space of functionals on* $E\hat{\otimes}F$. *Then* $\mathscr{N}_F(E)$ *is weakly amenable if and only if* $\dim K_F \leq 1$.

(ii) *Suppose that either* $B(E)$ *or* $B(E)$ *contains no closed two-sided ideals of co-dimension 1. Then* $\mathscr{N}_F(E)$ *is weakly amenable if and only if* $\mathscr{N}_F(E) = E\hat{\otimes}F$.

PROOF. (i) The action of $B(F)$ on $E\hat{\otimes}F$ is given by

$$\langle u, B\rangle = t((1_E \otimes B)u).$$

Since $\langle \tau((1_E \otimes B)u)x, f\rangle_F = \langle x, B \circ \tau(u)'(f)\rangle_F \quad (u \in E\hat{\otimes}F, B \in B(F), x \in E, f \in F)$ we clearly have $(1_E \otimes B)K_F \subseteq K_F \quad (B \in B(F))$. Hence, when $B(F)$ is total on $E\hat{\otimes}F$, we have $t(K_F) \neq (0)$ if $K_F \neq (0)$. (This is essentially Grothendieck's argument in [3].) A similar proof works if $B(E)$ is total.

(ii) Suppose $\dim K_F = 1$. Then we may write $K_F = \mathbb{C}u$ for some $u \in E\hat{\otimes}F\backslash(0)$. Define a bounded linear functional $\varphi: B(F) \to \mathbb{C}$ so that $\varphi(B)u = (1_E \otimes B)u$. Since $(1_E \otimes B)K_F \subseteq K_F$, this definition is meaningful and one easily checks that φ is multiplicative. If $B(F)$ has no co-dimension 1 closed bi-ideals, we have $\varphi = 0$. In particular $u = (1_E \otimes 1_F)u = \varphi(1_F)u = 0$. This contradiction shows that if $\mathscr{N}_F(E)$ is weakly amenable then $\mathscr{N}_F(E) = E\hat{\otimes}F$.

A similar reasoning applies to the statement involving $B(E)$.

We now turn to the question of amenability. Since $E\hat{\otimes}F$ is biprojective and hence biflat, $E\hat{\otimes}F$ is amenable if and only if it has a bounded approximate identity ([4, Theorem II. 21]).

THEOREM 5. *Let* E, F *be a Banach pairing. Then* $E\hat{\otimes}F$ *has a bounded approximate identity, and consequently* $E\hat{\otimes}F$ *is amenable, if and only if* $\dim E < \infty$.

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PROOF. By [4, Theorem II. 31], $E\widehat{\otimes}F$ has a bounded approximate identity if and only if the short exact sequence of dual bimodules

$$0 \to \mathbb{C}^* \to (E\widehat{\otimes}F)^*_+ \to (E\widehat{\otimes}F)^* \to 0$$

splits as Banach bi-modules over $E\widehat{\otimes}F$. Suppose it does, and let $\rho: (E\widehat{\otimes}F)^*_+ \to \mathbb{C}^*$ be a splitting module homomorphism. We represent $(E\widehat{\otimes}F)^*_+$ as $B(F, E^*) \times \mathbb{C}$. Then the canonical module action of $E\widehat{\otimes}F$ is given by

$$u.(T, \lambda) = (u.T, \langle u, T\rangle) \qquad (T, \lambda).u = (T.u, \langle u, T\rangle),$$

$(u \in E\widehat{\otimes}F, T \in B(F, E^*), \lambda \in \mathbb{C})$. Since the module multiplication on $\mathbb{C}^*$ is trivial we get

$$\rho((u.T, 0)) = \rho((T.u, 0)) = -\langle u, T\rangle \tag{4}$$

$(u \in E\widehat{\otimes}F, T \in B(F, E^*))$.

Hence, if we define $m \in B(E)^*$ by

$$m(U) = \rho((U^* \circ i, 0)) \quad (U \in B(E)),$$

where $i: F \to E^*$ is the natural injection, we have

$$m(US - SU) = 0 \quad (S \in \mathcal{N}_F(E), U \in B(E)),$$

since $S^* \circ i = i \circ S'$ (cf. (2) and (3)). As in the proof of Lemma 2 we see that

$$m(x \otimes x^*) = \langle x, x^*\rangle_{E^*} m(x_0 \otimes f_0) \quad (x \in E, x^* \in E^*),$$

where $x_0 \in E$ and $f_0 \in F$ is any choice of vectors satisfying $\langle x_0, f_0\rangle_F = 1$.

Let Q be a finite rank projection. We may write $Q = \sum_{i=1}^n x_i \otimes x_i^*$ with $\langle x_i, x_j^*\rangle = \delta_{ij}(i, j \in \{1, \dots, n\})$ so that $m(Q) = m(x_0 \otimes f_0)n$. By [7, Theorem 1.14] there is to each finite-dimensional subspace X of E a projection Q onto X with $\|Q\| \le \sqrt{\dim X}$. Hence, if $\dim E = \infty$ we must have $m(x_0 \otimes f_0) = 0$. By (3) and (4) we have the contradiction $\langle x, f\rangle_F = 0$ for all $x \in E$ and $f \in F$.

THEOREM 6. *If $\mathcal{N}_F(E)$ is amenable, then E is finite dimensional.*

PROOF. Suppose that $\mathcal{N}_F(E)$ is amenable. Then $\dim K_F \le 1$ by Theorem 3, so by [4, Proposition II.31] the short exact sequence

$$0 \to K_F \to E\widehat{\otimes}F \to \mathcal{N}_F(E) \to 0$$

splits as Banach bi-modules over $\mathcal{N}_F(E)$. In particular

$$K_F = \mathcal{N}_F(E).K_F = (0),$$

by Lemma 2, and continuity of the splitting homomorphism. The result now follows from Theorem 5.

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