Proceedings of the Edinburgh Mathematical Society (2010) **53**, 31–38 ©
DOI:10.1017/S0013091508000692 Printed in the United Kingdom

PROPERTIES OF SOLUBLE GROUPS DETECTABLE AT THEIR METABELIAN SUBGROUPS

RAUL MOREIRA BEHS AND RUDOLF MAIER

Departamento de Matemática—IE, Universidade de Brasília, 70910-900 Brasília—DF, Brazil (raul.behs@hotmail.com; maierr@mat.unb.br)

(Received 24 June 2008)

Abstract In a soluble group G, if the metabelian subgroups have finite (or polycyclic, or Černikov) quotients by their centres, then so too does G.

Keywords: soluble groups; metabelian subgroups; centrequotient; finiteness conditions

2000 *Mathematics subject classification:* Primary 20F99

1. Introduction

For a group G and a group class (a group property) $\mathfrak{X}$, let $\mathfrak{X}G$ denote the set of its $\mathfrak{X}$-subgroups. Let G be a group belonging to a certain universe $\mathfrak{V}$ of groups. Let $\mathfrak{X}$ and $\mathfrak{T}$ be two further classes of groups. There is an interesting class of theorems in group theory that tell us how to verify the property $\mathfrak{X}$ for G by testing the same property at the members of a test family $\mathfrak{T}G$, which consists only of some 'small' subgroups of G. In the language of group classes, these theorems are of the form

$$G \in \mathfrak{V} \quad \text{and} \quad \mathfrak{T}G \subseteq \mathfrak{X}G \Longrightarrow G \in \mathfrak{X}$$

(i.e. if, in a $\mathfrak{V}$-group G, the $\mathfrak{T}$-subgroups are in $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group). For example, the *nilpotency* $\mathfrak{N}$ of any *finite* group G can be seen at its 2-*generated* subgroups (i.e. $G \in \mathfrak{F}$ and $\mathfrak{G}_2G \subseteq \mathfrak{N}G \Longrightarrow G \in \mathfrak{N}$). By definition, any local property $L\mathfrak{X}$ can be read off at the *finitely generated* subgroups. In a classic article [**2**], Baer studies properties of groups which can be detected at their *countable* subgroups. It is a good and well-known exercise that the *finiteness* of a soluble group is a consequence of the finiteness of its *abelian* subgroups, and the famous Hall–Kargapolov–Kulatilaka theorem [**6**, § 14.3.7, p. 432] shows that the same *finiteness* test also holds in the universe $\mathfrak{V} = L\mathfrak{F}$ of all locally finite (and hence in the universe of all soluble-by-locally finite) groups. Further classical theorems say that for a soluble group G the maximum (minimum) condition can likewise be tested at its *abelian* subgroups [**6**, § 15.2, p. 455].

Continuing with this philosophy, we investigate the classes

$$\mathfrak{X} = Z\mathfrak{Z}^{\infty}, \quad \mathfrak{X} = Z\mathfrak{F} \quad \text{and} \quad \mathfrak{X} = Z\check{\mathfrak{C}}$$

in the universe $\mathfrak{V} = \mathfrak{A}^\infty$ of all soluble groups. That is, the properties of having *polycyclic*, *finite* and *Černikov* quotients by their centres, respectively. Our result is that the set $\mathfrak{T}G = \mathfrak{A}^2 G$ of the *metabelian* subgroups is a test family for all these properties. In other words, we prove the following.

Theorem 1.1. *Let G be a soluble group. Suppose the metabelian subgroups $X \leqslant G$ have polycyclic quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is polycyclic.*

Theorem 1.2. *Let G be a soluble group. Suppose the metabelian subgroups $X \leqslant G$ have finite quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is finite.*

Theorem 1.3. *Let G be a soluble group. Suppose the metabelian subgroups $X \leqslant G$ have Černikov quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is a Černikov group.*

We may state these results symbolically as follows:

- $G \in \mathfrak{A}^\infty$ and $\mathfrak{A}^2 G \subseteq (Z\mathfrak{Z}^\infty)G \Longrightarrow G \in Z\mathfrak{Z}^\infty$;
- $G \in \mathfrak{A}^\infty$ and $\mathfrak{A}^2 G \subseteq (Z\mathfrak{F})G \Longrightarrow G \in Z\mathfrak{F}$;
- $G \in \mathfrak{A}^\infty$ and $\mathfrak{A}^2 G \subseteq (Z\check{\mathfrak{C}})G \Longrightarrow G \in Z\check{\mathfrak{C}}$.

These three results can easily be extended as follows.

Consequence 1.4. Let G be a soluble-by-Noetherian group. Suppose the metabelian subgroups $X \leqslant G$ have polycyclic quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is Noetherian.

Consequence 1.5. Let G be a soluble-by-finite group. Suppose the metabelian subgroups $X \leqslant G$ have finite quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is finite.

Consequence 1.6. Let G be a soluble-by-finite group. Suppose the metabelian subgroups $X \leqslant G$ have Černikov quotients by their centres $X/\zeta(X)$. Then $G/\zeta(G)$ is a Černikov group.

For the convenience of the reader, we recall some definitions. A group X is *metabelian* if $X'' = 1$, i.e. X is soluble of derived length less than or equal to 2. A group G is *Noetherian* if it satisfies the maximal condition on its subgroups. A soluble-by-finite group is Noetherian if and only if it is polycyclic-by-finite. A group is a *Černikov* group if it is a finite extension of an abelian group with minimal condition on its subgroups. Černikov groups are exactly the soluble-by-finite groups with minimal condition. The finite residual G_0 (the intersection of all finite index subgroups) of a Černikov group G is an abelian divisible characteristic subgroup with finite quotient [**6**, p. 156].

For some similar results within the same framework, see [**3**].

2. Proofs of the theorems

2.1. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *Let A be an abelian normal subgroup of a group G.*

(a) *If G/A is Noetherian and if $X/\zeta(X)$ is Noetherian (i.e. polycyclic) for all metabelian subgroups X of G, then $G/\zeta(G)$ is Noetherian.*

https://doi.org/10.1017/S0013091508000692 Published online by Cambridge University Press

(b) *If G/A is finite and if $X/\zeta(X)$ is finite for all metabelian subgroups X of G, then $G/\zeta(G)$ is finite.*

Proof. We have $A \trianglelefteq G$ and G/A is Noetherian under hypothesis (a) and finite under (b). Let G/A be generated by $\{Ag_1, Ag_2, \dots, Ag_r\}$. For $i = 1, 2, \dots, r$, the subgroups $X_i = A\langle g_i \rangle$ are metabelian. Therefore, the $X_i/\zeta(X_i)$ are polycyclic (respectively, finite) by hypothesis and the same holds for $A/A \cap \zeta(X_i) \cong A\zeta(X_i)/\zeta(X_i) \leqslant X_i/\zeta(X_i)$. Hence, for

$$D = A \cap \bigcap_{i=1}^{r} \zeta(X_i)$$

we have $D \leqslant \zeta(G)$ and A/D is polycyclic (respectively, finite). So $G/\zeta(G)$ is Noetherian (respectively, finite). □

This allows us to conclude Consequences 1.4 and 1.5 from Theorems 1.1 and 1.2.

Proof of Consequence 1.4. Let G be soluble-by-Noetherian, satisfying the hypothesis that the metabelian subgroups of G have polycyclic quotients by their centres. Let $S \trianglelefteq G$ be such that S is soluble and G/S is Noetherian. By Theorem 1.1, $S/\zeta(S)$ is polycyclic. So $A = \zeta(S)$ is an abelian normal subgroup of G with Noetherian quotient $G/\zeta(S)$. By Lemma 2.1 (a), $G/\zeta(G)$ is Noetherian. □

The proof of Consequence 1.5 from Theorem 1.2 is the same, using Lemma 2.1 (b).

Lemma 2.2. *Let A be a soluble automorphism group of a polycyclic group. Then A is polycyclic.*

Proof. See [**5**, § 3.27, p. 82]. □

Lemma 2.3. *Let $\mathfrak{X}$ be a group property closed under subgroups, quotients and finite direct products. Let G be a nilpotent group of class less than or equal to 2, such that $G' \in \mathfrak{X}$ and suppose that M is maximal among the abelian subgroups of G. If $M/\zeta(G)$ is finitely generated, then $G/M \in \mathfrak{X}$.*

Proof. For fixed $g \in G$, the map $\varphi_g : G \to G'$, defined by $\varphi_g(x) = [x, g]$ for all $x \in G$, is an endomorphism with $\operatorname{Im}\varphi_g \leqslant G'$ and $\operatorname{Ker}\varphi_g = \boldsymbol{C}_G(g)$. Therefore, $G/\boldsymbol{C}_G(g) \in \mathfrak{X}$.

Let $M/\zeta(G) = \langle \zeta(G)g_1, \zeta(G)g_2, \dots, \zeta(G)g_r \rangle$. It follows that

$$G \Big/ \bigcap_{k=1}^{r} \boldsymbol{C}_G(g_k) \leqslant G/\boldsymbol{C}_G(g_1) \times G/\boldsymbol{C}_G(g_2) \times \cdots \times G/\boldsymbol{C}_G(g_r) \in \mathfrak{X}.$$

But $M = \boldsymbol{C}_G(M) = \bigcap_{k=1}^{r} \boldsymbol{C}_G(g_k)$ and therefore $G/M \in \mathfrak{X}$. □

Corollary 2.4. *Let G be a nilpotent group of class less than or equal to 2 and suppose that G' is polycyclic. If $M/\zeta(G)$ is polycyclic for some maximal abelian subgroup M of G, then $G/\zeta(G)$ is polycyclic.*

Proof of Theorem 1.1. Let G be soluble, satisfying the hypothesis that $X/\zeta(X)$ is polycyclic for all $X \in \mathfrak{A}^2 G$, which of course is inherited by every subgroup of G. We prove the assertion by induction on the derived length of G. We may assume that the derived group G' already has polycyclic quotient by its centre $G'/\zeta(G')$. For every subgroup $V \leqslant G$, the quotient $V'/\zeta(V')$, being isomorphic to a section of $G'/\zeta(G')$, is polycyclic.

To prove that $G/\zeta(G)$ is polycyclic, it suffices, by Lemma 2.1 (a), to exhibit in G a normal abelian subgroup A with G/A polycyclic.

(a) *There is a* $V \trianglelefteq G$ *with* G/V *polycyclic and such that* $[V', V] \leqslant \zeta(V')$.

Let $V = \boldsymbol{C}_G(G'/\zeta(G'))$. We have $\zeta(G') \leqslant V \trianglelefteq G$, and G/V, the automorphism group induced by G on $G'/\zeta(G')$, is polycyclic by Lemma 2.2, as $G'/\zeta(G')$ is polycyclic. Moreover, $V \cap G' = \zeta_2(G')$, where $\zeta_2(G')/\zeta(G') = \zeta(G'/\zeta(G'))$. Now $[\zeta_2(G'), V] \leqslant \zeta(G')$, i.e. $\zeta_2(G')/\zeta(G')$ is a V-central factor. Therefore, since $V' \leqslant \zeta_2(G')$, $V'\zeta(G')/\zeta(G')$ is also V-central. By operator isomorphism, $V'/V' \cap \zeta(G')$ is also V-central. Since $\zeta(G') \cap V' \leqslant \zeta(V')$, $V'/\zeta(V')$ is also V-central. i.e. $[V', V] \leqslant \zeta(V')$.

We now abbreviate $N = \zeta(V')$.

(b) V/N *is nilpotent of class less than or equal to* 2.

By (a), we have $(V/N)' = V'/N \leqslant \zeta(V/N)$.

We set $Z/N = \zeta(V/N)$ and let M/N be a maximal among the abelian subgroups of V/N. Clearly, $\zeta(V/N) \leqslant M/N$, i.e. $Z \leqslant M$ and $M \trianglelefteq V$.

(c) *The quotient* $(M/N)/\zeta(V/N)$ *is polycyclic.*

M is a metabelian subgroup of V. Therefore, $M/\zeta(M)$ is polycyclic by hypothesis. Since $\zeta(M)$ centralizes M and $V' \leqslant M$, we see that $\zeta(M)$ centralizes V'. Therefore, $[\zeta(M), V] \leqslant V' \cap \zeta(M) \leqslant V' \cap \boldsymbol{C}_V(V') = N$. This means that $N\zeta(M)/N \leqslant \zeta(V/N) = Z/N$. Thus, $\zeta(M) \leqslant Z$ and we see that $(M/N)/\zeta(V/N) \cong M/Z$ is polycyclic.

(d) V/M *is polycyclic.*

$(V/N)'$ is polycyclic and V/N is nilpotent of class less than or equal to 2. Since $(M/N)/\zeta(V/N)$ is polycyclic by (c), so is $(V/N)/(M/N)$ by Corollary 2.4, applied to V/N. Thus, V/M is polycyclic.

(e) *Conclusion of the proof.*

V/M is polycyclic by (d) and $M/\zeta(M)$ is polycyclic by hypothesis. Thus, $V/\zeta(M)$, being an extension of two polycyclic groups, is also polycyclic. Applying Lemma 2.1 (a) to V with $A = \zeta(M) \trianglelefteq V$, it follows that $V/\zeta(V)$ is polycyclic. Since $\zeta(V) \trianglelefteq G$ and $G/\zeta(V)$ is polycyclic, we see once more with Lemma 2.1 (a) now applied to G and $A = \zeta(V) \trianglelefteq G$ that $G/\zeta(G)$ must be polycyclic.

□

Proof of Theorem 1.2. Let $X/\zeta(X)$ be finite for all $X \in \mathfrak{A}^2G$. By Theorem 1.1 we already know that $G/\zeta(G)$ is polycyclic. We prove the assertion by induction on the length $r = r(G)$ of a shortest subnormal chain of $G/\zeta(G)$ with cyclic factors to show that $G/\zeta(G)$ is finite. Let $\zeta(G) = C_0 \lhd C_1 \lhd \cdots \lhd C_{r-1} = C \lhd C_r = G$ such that C_i/C_{i-1} is cyclic for $i = 1, 2, \dots, r$. Since $\zeta(G) \leqslant \zeta(C)$, we have $r(C) \leqslant r(G) - 1$, so that $C/\zeta(C)$ is finite by induction. Let $y \in G$ such that $G/C = \langle yC \rangle$. Now $X = \zeta(C)\langle y \rangle$ is metabelian, so $X/\zeta(X)$ is finite. Moreover, $|G : X| = |C\langle y \rangle : X| = |CX : X| = |C : C \cap X| \leqslant |C/\zeta(C)| < \infty$. Also $|G : \zeta(X)| < \infty$ and therefore $G/(\zeta(X))_G$ is finite, where

$$(\zeta(X))_G = \bigcap_{g \in G} (\zeta(X))^g.$$

With $A = (\zeta(X))_G$ in Lemma 2.1 (b), we see that $G/\zeta(G)$ must be finite. □

2.2. Proof of Theorem 1.3

The proof of Theorem 1.3 is somewhat more involved and not completely analogous to that of Theorem 1.1. One reason is that we do not have at our disposal a result corresponding to Lemma 2.2; namely, a soluble automorphism group of a Černikov group need not be Černikov.

Lemma 2.5 (Polovickii). *The class $\check{\mathfrak{C}}$ of all Černikov groups is a Schur class, i.e. for every group G, we have the implication*

$$G/\zeta(G) \in \check{\mathfrak{C}} \implies G' \in \check{\mathfrak{C}}.$$

Proof. See [**5**, § 4.23, p. 115] and [**4**]. □

Lemma 2.6 (Kurosh). *Let G be an abelian Černikov group. Then*

(a) *G is a finite direct product of finite cyclic and quasicyclic (Prüfer) groups,*

(b) *G is a so-called FO-group, i.e. for every possible element order n, there exist only finitely many elements of order n in G.*

Proof. See [**6**, § 4.2.11, pp. 104, 446]. □

We recall that a group G is an $\mathfrak{F}C$*-group* if $|G : \boldsymbol{C}_G(x)| < \infty$ for all elements $x \in G$, i.e. all elements have only finitely many conjugates in G. Clearly, the finiteness of the quotient $G/\zeta(G)$ is sufficient for G to be an $\mathfrak{F}C$-group. Moreover, a sufficient condition for being an $\mathfrak{F}C$-group is the finiteness of G', since for fixed $x \in G$, the map $x^g \to [x, g] (g \in G)$ is injective. It is also clear that

$$G/\zeta(G) \text{ is finite} \iff G \text{ is an } \mathfrak{F}C\text{-group and } G/\zeta(G) \text{ is Černikov.} \tag{2.1}$$

Now we can prove the following extension of a result of Baer [**1**] (see [**5**, Theorem 3.14, p. 69]), which is essential for the proof of our Theorem 1.3.

Proposition 2.7. *Let G be a nilpotent group such that $G/\zeta(G)$ is a Černikov group. Then $G/\zeta(G)$ is even finite.*

Remark 2.8. Baer proves the finiteness of $G/\zeta(G)$ under the hypothesis that the whole group G satisfies the minimum condition.

Proof. By (2.1) we only have to show that G is an $\mathfrak{F}C$-group.

We prove the assertion by induction on the nilpotency class c of G. The hypothesis is clearly inherited by every subgroup X of G and every quotient G/N ($N \trianglelefteq G$), since $X/\zeta(X)$ and $(G/N)/\zeta(G/N)$ are isomorphic to sections of $G/\zeta(G)$. For $c \leqslant 1$ there is nothing to prove.

Let $c = 2$, i.e. $G' \leqslant \zeta(G)$. Since $G/\zeta(G)$ is a Černikov group, we see by Lemma 2.5 that G' is also Černikov. Since G' is abelian, it is an FO-group by Lemma 2.6. Let $x \in G$ have order $o(x) \leqslant \infty$. We have $\langle x\rangle \cap \zeta(G) \trianglelefteq G$. Since $G/\zeta(G)$ is a torsion group, certainly $\langle x\rangle/\langle x\rangle \cap \zeta(G) \cong \langle x\zeta(G)\rangle$ is finite. We set $N = 1$ if $o(x) < \infty$ and $N = \langle x\rangle \cap \zeta(G)$ if $o(x) = \infty$. Since G' is a torsion group, $N \cap G' = 1$. Let $m = |\langle x\rangle/N|$, so that $x^m \in N$. For $g \in G$ we have $x^g = xc$ with $c = [x,g] \in G' \leqslant \zeta(G)$. It follows that $(x^g)^m = x^m c^m \in N$ and therefore $c^m \in N \cap G' = 1$. So $o(c)$ divides m. Since G' is an FO-group, there are only finitely many possible choices for c. It follows that G is an $\mathfrak{F}C$-group, i.e. $G/\zeta(G)$ is finite by (2.1).

Now let $c \geqslant 3$. By induction, the quotient by the centre of the group $G/\zeta(G)$ is already finite, i.e. $|G/\zeta_2(G)| < \infty$, where $\zeta_2(G)/\zeta(G) = \zeta(G/\zeta(G))$. For any $x \in G$ the subgroup $X = \zeta_2(G)\langle x\rangle$ is of finite index in G and of class less than or equal to 2. By the first part, $X/\zeta(X)$ is finite. So x is an $\mathfrak{F}C$-element of G, i.e. G is an $\mathfrak{F}C$-group. Again, by (2.1) we conclude that $G/\zeta(G)$ is even finite. □

Corollary 2.9. *Let $G/\zeta(G)$ be a Černikov group and let $G_0/\zeta(G)$ be its finite residual. Then G/G_0 is finite and G_0 is abelian.*

Proof. Clearly, G/G_0 is finite and $G_0/\zeta(G)$ is an abelian divisible group. Since $\zeta(G) \leqslant \zeta(G_0)$, we see that G_0 is nilpotent and $G_0/\zeta(G_0)$ is an abelian divisible Černikov group. By Proposition 2.7, $G_0 = \zeta(G_0)$ is abelian. □

We now have the following, which is analogous to Lemma 2.1.

Lemma 2.10. *Let A be an abelian normal subgroup of a group G such that G/A is a Černikov group. If $X/\zeta(X)$ is Černikov for all metabelian subgroups X of G, then $G/\zeta(G)$ is a Černikov group.*

Proof. Let B/A be the finite residual of the Černikov group G/A. Then B/A is abelian and B is metabelian. Therefore, $B/\zeta(B)$ is Černikov, by hypothesis. Moreover, B is of finite index in G. Let $B_0/\zeta(B)$ be the finite residual of $B/\zeta(B)$. Then B_0 has finite index in G. By Corollary 2.9 applied to B, we see that B_0 is abelian. Let $r = |G : B_0|$ and let $\{g_1, g_2, \dots, g_r\}$ be a transversal for B_0 in G. The metabelian subgroups $X_i = B_0\langle g_i\rangle$ have Černikov quotient by their centre $X_i/\zeta(X_i)$, $i = 1, 2, \dots, r$. For $D = \bigcap_{i=1}^{r} \zeta(X_i)$ it follows that $D \leqslant \zeta(G)$ and that G/D is a Černikov group. This proves the assertion. □

Lemma 2.10 allows us to prove the Consequence 1.6 from Theorem 1.3.

Proof of Consequence 1.6. Let G be soluble-by-finite satisfying the hypothesis that the metabelian subgroups of G have Černikov quotients by their centres. Let $S \trianglelefteq G$ be such that S is soluble and G/S is finite. By Theorem 1.3, $S/\zeta(S)$ is Černikov. So $A = \zeta(S)$ is an abelian normal subgroup of G with Černikov quotient $G/\zeta(S)$. By Lemma 2.10, $G/\zeta(G)$ is Černikov. □

For the proof of Theorem 1.3 we still need the following.

Lemma 2.11. *Let G be an $\mathfrak{F}C$-group and let M be a maximal abelian subgroup of G. If $M/\zeta(G)$ satisfies the minimal condition, then $G/\zeta(G)$ is finite.*

Proof. For every subset $X \subseteq G$ with $|X| < \infty$ we have $|G : \boldsymbol{C}_G(X)| < \infty$ and $\zeta(G) \leqslant \boldsymbol{C}_M(X) \leqslant M$. By hypothesis, the family

$$\mathfrak{M} = \{\boldsymbol{C}_M(X) \mid X \subseteq G,\ |X| < \infty\}$$

contains a minimal element. Therefore, there exists a finite subset $X_0 \subseteq G$ such that $\boldsymbol{C}_M(X_0)$ is minimal in $\mathfrak{M}$. We claim that $\boldsymbol{C}_M(X_0) = \zeta(G)$. If $\zeta(G) < \boldsymbol{C}_M(X_0)$, let $c \in \boldsymbol{C}_M(X_0) \setminus \zeta(G)$. There exists a $t \in G$ such that $tc \neq ct$. We have $\boldsymbol{C}_M(X_0) > \boldsymbol{C}_M(X_0 \cup \{t\}) \in \mathfrak{M}$: a contradiction. Thus, $\boldsymbol{C}_M(X_0) = \zeta(G)$. Now

$$|M : \zeta(G)| = |M : M \cap \boldsymbol{C}_G(X_0)| \leqslant |\langle \boldsymbol{C}_G(X_0), M\rangle : \boldsymbol{C}_G(X_0)| \leqslant |G : \boldsymbol{C}_G(X_0)| < \infty$$

and therefore $M/\zeta(G)$ is finite. Let $M/\zeta(G) = \zeta(G)m_1 \cup \zeta(G)m_2 \cup \cdots \cup \zeta(G)m_s$ with $m_1, m_2, \ldots, m_s \in M$. Then

$$M = \boldsymbol{C}_G(M) = \boldsymbol{C}_G(m_1, m_2, \ldots, m_s) = \bigcap_{i=1}^{s} \boldsymbol{C}_G(m_i).$$

Since G is an $\mathfrak{F}C$-group, we have

$$\left| G : \bigcap_{i=1}^{s} \boldsymbol{C}_G(m_i) \right| < \infty$$

and so $G/\zeta(G)$ is finite. □

Proof of Theorem 1.3. Again, we prove the assertion by induction on the derived length of the soluble group G. Since every subgroup of G inherits the hypothesis, the quotient by its centre $G'/\zeta(G')$ of the derived group G' is Černikov by induction. To prove that $G/\zeta(G)$ is Černikov, it suffices, by Lemma 2.10, to exhibit in G a normal abelian subgroup A with Černikov quotient G/A.

(a) *There is an abelian subgroup $G_0 \trianglelefteq G$ such that $G_0 \leqslant G'$ and G'/G_0 is finite.*

Let $G_0/\zeta(G')$ be the finite residual of the Černikov quotient $G'/\zeta(G')$. Then G_0 has the asserted properties by Corollary 2.9 applied to G'.

https://doi.org/10.1017/S0013091508000692 Published online by Cambridge University Press

(b) *There is a (normal) subgroup V of finite index in G, an abelian $N \trianglelefteq V$ with $\zeta(V') \leqslant N \leqslant V'$ and $[V', V] \leqslant N$ such that $|V' : N| < \infty$.*

Let $V = \boldsymbol{C}_G(G'/G_0)$. Then G/V is finite since G'/G_0 is finite. By (a), $G_0 \cap V'$ is abelian, so that $N = \zeta(V')(G_0 \cap V')$ is also abelian. So $N \trianglelefteq V$ and $\zeta(V') \leqslant N \leqslant V'$. Also $|V' : N| < \infty$, as G'/G_0 is finite and $V'/V' \cap G_0 \cong V'G_0/G_0 \leqslant G'/G_0$. Moreover, $[V', V] \leqslant [G', V] \cap V' \leqslant G_0 \cap V' \leqslant N$.

(c) *V/N is a nilpotent $\mathfrak{F}C$-group of class less than or equal to 2.*

Since $(V/N)' = V'/N$ is finite by (b), we see that V/N is an $\mathfrak{F}C$-group. Also $[V', V] \leqslant N$. This means exactly that $V'/N \leqslant \zeta(V/N)$. So V/N is nilpotent of class less than or equal to 2.

We choose a maximal abelian subgroup M/N of V/N. Clearly, $\zeta(V/N) \leqslant M/N$ and $M \trianglelefteq V$.

(d) *$(M/N)/\zeta(V/N)$ has the minimal condition.*

Let $Z/N = \zeta(V/N)$. Since N is abelian, M is a metabelian subgroup of V. Therefore, $M/\zeta(M)$ is a Černikov group, by hypothesis. Now, since $V' \leqslant M$, we see that $\zeta(M) \leqslant \boldsymbol{C}_V(V')$. Moreover, $\zeta(M) \trianglelefteq V$. Therefore, $[V, \zeta(M)] \leqslant V' \cap \zeta(M) \leqslant \zeta(V') \leqslant N$ and so $\zeta(M)N/N \leqslant \zeta(V/N) = Z/N$, i.e. $\zeta(M) \leqslant Z$.

It follows that $M/Z \cong (M/N)/\zeta(V/N)$ is Černikov.

(e) *Conclusion of the proof.*

Since $(M/N)/\zeta(V/N)$ has the minimal condition by (d), and since V/N is an $\mathfrak{F}C$-group, by (c), we see that $(V/N)/\zeta(V/N)$ is finite by Lemma 2.11 applied to V/N. In particular, $|V/N : M/N| < \infty$. Thus, $|V : M|$ is finite. Since also $|G : V| < \infty$, we see that $|G : M| < \infty$ and G/M_G also is finite, where $M_G = \bigcap_{g \in G} M^g$ is the core of M in G. Since M_G is metabelian, $M_G/\zeta(M_G)$ is Černikov by the assumption of the theorem. So $G/\zeta(M_G)$ is Černikov and Lemma 2.10 shows with $A = \zeta(M_G)$ that $G/\zeta(G)$ must be a Černikov group.

□

Acknowledgements. R.M.B. was supported by CNPq and by CAPES (Brazil).

References

1. R. Baer, Nilgruppen, *Math. Z.* **62** (1955), 402–437.
2. R. Baer, Abzählbar erkennbare gruppentheoretische Eigenschaften, *Math. Z.* **89** (1962), 344–363.
3. R. M. Behs, Testando propriedades de grupos em subgrupos pequenos, Tese de Doutorado, Universidade de Brasilia (2008).
4. Ya. D. Polovickii, Locally extremal and layer-extremal groups, *Mat. Sb.* **58** (1962), 685–694.
5. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1 (Springer, 1972).
6. D. J. S. Robinson, *A course in the theory of groups* (Springer, 1996).