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A NOTE ON SPLIT EXTENSIONS OF FINITE GROUPS

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Let N and K be finite groups with K acting on N, and let G be the semidirect product NK. It was shown by Zassenhaus that if $(|N|, |K|) = 1$ and either N or K is solvable (an assumption later rendered redundant by the Feit-Thompson theorem), then all complements of N in G are conjugate to K. It does not seem to be widely recognized, however, that a simple modification of the averaging argument used in the standard proof of this fact (e.g. Theorem 6.2.1 of [1]) yields the following somewhat stronger result:

THEOREM. *Let G be a finite group such that $G = NK$, where $N \trianglelefteq G$ and $N \cap K = 1$. Suppose $H \leq G$ such that $(|H : H \cap K^x|, |N|) = 1$ for every $x \in N$ (where $K^x = x^{-1}Kx$). If N or H is solvable, then H is contained in some conjugate of K.*

Before proving this, we isolate as a separate lemma the case that the normal subgroup is abelian, not only because this is the critical case but also because in this situation, an apparently weaker hypothesis suffices.

LEMMA. *Suppose $G = AK$, where A is abelian, $A \trianglelefteq G$ and $A \cap K = 1$. If $H \leq G$ such that $(|H : H \cap K|, |A|) = 1$, then H is contained in some conjugate of K.*

Proof. As in the proof of Zassenhaus' theorem, we consider the "crossed homomorphism" $f : H \to A$ defined by

$$xf(x) \in K \quad \text{for all} \quad x \in H.$$

Clearly, $f(x) = f(y)$ if and only if $xy^{-1} \in H \cap K$. Hence, if $T = \{x_i : 1 \leq i \leq m\}$ is a right transversal for $H \cap K$ in H, the elements $f(x_i)$, $1 \leq i \leq m$, represent each element in the image of f exactly once. For any $x \in H$, $Tx = \{x_i x : 1 \leq i \leq m\}$ is again a right transversal for $H \cap K$ in H, so if $b = \prod_{i=1}^m f(x_i)$, it follows that $b = \prod_{i=1}^m f(x_i x)$. But $f(x_i x) = f(x_i)^x f(x)$ for each i, so we obtain $b = b^x f(x)^m$. Now $m = |H : H \cap K|$ is relatively prime to $|A|$ so, choosing an integer n with $mn \equiv 1 \pmod{|A|}$ and letting $a = b^{-n}$, we conclude that $a^{-1} = (a^{-1})^x f(x)$. Therefore, $x = (xf(x))^a \in K^a$ for every x in H, which proves the lemma.

Proof of the Theorem. Suppose first that H is solvable, so H contains a Hall π'-subgroup L (where π denotes the set of prime divisors of $|N|$). Now the

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semidirect product NL acts by right multiplication on the set of right cosets of K in G, with N acting transitively. By a lemma of Glauberman ((13.8) of [2]), L fixes some coset Kn, $n \in N$, implying that $L \leq K^n$. It follows that $|H:H \cap K^n|$ divides $|H:L|$ which is a product of primes in π. But $|H:H \cap K^n|$ is relatively prime to $|N|$ by hypothesis, so we conclude that $H \leq K^n$.

We are reduced to the case that N is solvable, and here we use induction on $|G|$. Let M be a minimal normal subgroup of G contained in N and write $\bar{X} = MX/M$ for any subgroup X of G. If $x \in N$, then $|\bar{H}:\bar{H} \cap \bar{K}^{\bar{x}}| = |MH:MH \cap MK^x|$. Since $MH \cap MK^x = M(H \cap MK^x)$ by Dedekind's lemma, we have $|MH:MH \cap MK^x| = |M(H \cap MK^x)H:M(H \cap MK^x)| = |H:H \cap MK^x|$ which divides $|H:H \cap K^x|$ and so is relatively prime to $|N|$. $\bar{G} = \bar{N}\bar{K}$ therefore satisfies the hypotheses of the theorem so, by induction, $H \leq MH \leq MK^n$ for some $n \in N$. Since M is abelian, the preceding lemma applies to the group MK^n. so $H \leq K^{nm}$ for some $m \in M$. This completes the proof.

This argument remains valid if the assumption that N or H is solvable is replaced by the hypothesis that N is $\pi(H)$-solvable or N is $\pi(N)$-separable, but it does not seem clear whether such a restriction can be eliminated entirely.

As a simple application, we mention a slight variation on the lemma of Glauberman to which we referred earlier. We omit the proof which is a straightforward modification of the argument given in [2], using the above result in place of Zassenhaus' theorem.

Corollary. *Suppose A and G are finite groups with A acting on G, and assume A or G is solvable. Assume A and G both act on a set Ω such that*

(a) $(\alpha \cdot g) \cdot a = (\alpha \cdot a) \cdot g^a$ *for all* $\alpha \in \Omega$, $g \in G$ *and* $a \in A$.
(b) *G is regular on Ω.*
(c) *All orbits of A in Ω have length relatively prime to $|G|$.*

Then A fixes some element of Ω.

This corollary may be used to obtain a slight generalization of some standard facts about coprime action (c.f. [1, Theorem 6.2.2]).

Theorem. *Let A be a group of automorphisms of the finite group G such that A or G is solvable. Assume that every orbit of A in G has length relatively prime to $|G|$. Then the following hold:*

(a) *If H is an A-invariant subgroup of G, any A-invariant coset of H contains an element of $C_G(A)$.*

(b) *If N is an A-invariant normal subgroup of G, $C_{G/N}(A) = C_G(A)N/N$.*

(c) *If G contains an A-invariant Sylow p-subgroup for some prime p, any two such subgroups are conjugate by an element of $C_G(A)$.*

(d) *If two subsets of $C_G(A)$ are conjugate in G, they are conjugate in $C_G(A)$.*

Proof. The first statement follows from the preceding corollary and the fact that H acts regularly by left (right) multiplication on each of its right (left) cosets. The second statement is an immediate consequence of the first.

To prove (c) and (d), observe that if X is any A-invariant subset of G (and in particular, an A-invariant Sylow subgroup or a subset of $C_G(A)$) and if X^g is also A-invariant for some $g \in G$, then

$$X^g = (X^g)^a = X^{g^a}$$

so $g^a g^{-1} \in N_G(X)$ for every $a \in A$. Hence, the coset $N_G(X)g$ is A-invariant and so, by (a), $ng \in C_G(A)$ for some $n \in N_G(X)$. Since $X^{ng} = X^g$, statements (c) and (d) now follow.

References

1. Daniel Gorenstein, "Finite Groups," Harper and Row, New York, 1968.
2. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.

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