References

1. Thomas Walker, A Geometric Telescope, *The American Mathematical Monthly* **109**(6) The Mathematical Association of America (2002) p. 524.
2. Roger B. Nelsen, *Proofs without Words III: Further Exercises in Visual Thinking*, The Mathematical Association of America (2015) p. 159.

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107.05 The final solution of a quasi-palindromic

Introduction

We consider the eighth roots of unity. They are obtained by solving the octic $x^8 - 1 = 0$. Also the primitive eighth roots of unity are obtained by solving the eighth cyclotomic $x^4 + 1 = 0$. Let us solve this cyclotomic. From $x^4 = -1$, we have $x^2 = \pm i$. Hence the four roots of the cyclotomic are $x = \pm\sqrt{i}, \pm\sqrt{-i}$. However, the radicals in these roots include the imaginary unit. In general, roots with the imaginary unit included in radicals would be inferior to ones without it. As the latter types of roots of the cyclotomic, $x = \dfrac{1 \pm i}{\sqrt{2}}, \dfrac{-1 \pm i}{\sqrt{2}}$ are well known. Comparing these two types of roots, we notice that the latter types are in the form of $u + vi$ where u, v are real. We thus introduce the concept of *final roots*. These are roots which are in the form of $u + vi$ where u, v are real. The process of finding these final roots is called a *final solution*.

The main result of this Note is the following. The final roots of the quartic

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

where B, C, D, E are real with $D^2 = B^2E$, $B \neq 0$ and $E \neq 0$ (the quartic with $D^2 = B^2E$ and $E \neq 0$ is called a quasi-palindromic) are as follows:

(i) If $B^2 - 4C + \dfrac{8D}{B} \geqslant 0$, then

$$x = \frac{1}{2}\left(-p_k \pm \sqrt{p_k^2 - \frac{4D}{B}}\right) \qquad (k = 1,\ 2)$$

where $p_1 = \dfrac{1}{2}\left(B + \sqrt{B^2 - 4C + \dfrac{8D}{B}}\right)$, $p_2 = \dfrac{1}{2}\left(B - \sqrt{B^2 - 4C + \dfrac{8D}{B}}\right)$.

(ii) If $B^2 - 4C + \frac{8D}{B} < 0$, then

$$x = \frac{-p_k \pm \sqrt{p_k^2 - 4q_k}}{2} \qquad (k = 1, 2)$$

where $p_1 = \dfrac{B + \sqrt{B^2 - 4C + 4t}}{2}$, $p_2 = \dfrac{B - \sqrt{B^2 - 4C + 4t}}{2}$,

$$q_1 = \frac{1}{2}\left(t + \sqrt{t^2 - \frac{4D^2}{B^2}}\right), q_2 = \frac{1}{2}\left(t - \sqrt{t^2 - \frac{4D^2}{B^2}}\right),$$

$$t = \frac{C}{2} - \frac{D}{B} + \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}.$$

All this time, mathematicians have found several solutions of the general quartic equation since Ferrari's discovery in 1540 (e.g. [1]). There are old studies such as Descartes [2], Euler [3] and Lagrange [4], and recent ones such as Christianson [5] and Yacoub–Fraidenraich [6]. Among these solutions, we use a modified Descartes' solution of the quartic as the basis for solving the quartic in the next section. A later section is devoted to a well-known solution of a quasi-palindromic. In the last section, we present the final solution of the quasi-palindromic.

A modified version of Descartes' solution

From the fundamental theorem of algebra, the general monic quartic

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0 \tag{1}$$

with real coefficients can be factorised as

$$\left(x^2 + p_1x + q_1\right)\left(x^2 + p_2x + q_2\right) = 0 \tag{2}$$

where p_k and q_k are real numbers for $k = 1, 2$. We define the following two kinds of subsidiary equations of quartic (1).

Definition 1: We say that the quadratic

$$\left(X - p_1\right)\left(X - p_2\right) = 0$$

is the *first subsidiary equation* of (1) and that the quadratic

$$\left(Y - q_1\right)\left(Y - q_2\right) = 0$$

is the *second subsidiary equation* of (1).

Equating (2) with (1), we have

$$p_1 + p_2 = B, \tag{3}$$

$$p_1p_2 + q_1 + q_2 = C, \tag{4}$$

$$p_1q_2 + p_2q_1 = D, \tag{5}$$

$$q_1q_2 = E. \tag{6}$$

From (3) and (5),

$$-(q_1 - q_2)^2 p_1 p_2 = (D - q_1 B)(D - q_2 B). \tag{7}$$

Using (4) and (6),

$$p_1 p_2 = C - (q_1 + q_2) \tag{8}$$

and

$$(q_1 - q_2)^2 = (q_1 + q_2)^2 - 4E, \tag{9}$$

respectively. It follows from (7), (8) and (9) that

$$(q_1 + q_2)^3 - C(q_1 + q_2)^2 + (BD - 4E)(q_1 + q_2) - B^2E + 4CE - D^2 = 0.$$

Let $t = q_1 + q_2$. Then t satisfies

$$t^3 - Ct^2 + (BD - 4E)t - B^2E + 4CE - D^2 = 0. \tag{10}$$

This cubic equation is well known as the resolvent cubic and is the same as that of Ferrari (e.g. [7]). Here it is necessary to remember that p_1, p_2 and q_1, q_2 are all real numbers. Then t is restricted by some conditions. We will mention them later in Remark 1. From (3), (8), and Vièta's formulas, the first subsidiary equation of (1) is

$$X^2 - BX + C - t = 0.$$

Solving this quadratic and using the fact that p_1 and p_2 are exchangeable, we find

$$p_1 = \frac{B + \sqrt{B^2 - 4C + 4t}}{2}, \; p_2 = \frac{B - \sqrt{B^2 - 4C + 4t}}{2}. \tag{11}$$

By $q_1 + q_2 = t$ and (6), the second subsidiary equation is

$$Y^2 - tY + E = 0.$$

Using (5), the roots of this quadratic are

$$q_1 = \frac{t + \sqrt{t^2 - 4E}}{2}, \qquad q_2 = \frac{t - \sqrt{t^2 - 4E}}{2}, \tag{12}$$

and quartic (1) has the roots

$$x = \frac{-p_k \pm \sqrt{p_k^2 - 4q_k}}{2} \qquad (k = 1, 2). \tag{13}$$

Note that the signs of the radicals in (12) are + for q_1 and − for q_2. In fact, (11) and (12) provide (5). However, (11) and $q_1 = \frac{1}{2}\left(t - \sqrt{t^2 - 4E}\right)$, $q_2 = \frac{1}{2}\left(t + \sqrt{t^2 - 4E}\right)$ provide

$$p_1 q_2 + p_2 q_1 = Bt - D.$$

Remark 1: Since p_k and q_k are real for $k = 1, 2$, we find from (11) and (12) that the root $t = q_1 + q_2$ of resolvent (10) satisfies

$$B^2 - 4C + 4t \geqslant 0, \qquad t^2 - 4E \geqslant 0. \tag{14}$$

If $p_k^2 - 4q_k \geqslant 0$, then roots (13) are real. In this case, let $u = \frac{1}{2}\left(-p_k \pm \sqrt{p_k^2 - 4q_k}\right)$, $v = 0$.

If $p_k^2 - 4q_k < 0$, then roots (13) are imaginary. In this case, let $u = -\frac{1}{2}p_k$ and $v = \pm\frac{1}{2}\sqrt{4q_k - p_k^2}$. In both cases, the roots are $x = u + vi$ where u, v are real.

A well-known solution of a quasi-palindromic

We consider the quartic $x^4 + Bx^3 + Cx^2 + Dx + E = 0$ with $D^2 = B^2E$ and $E \neq 0$, which is called a *quasi-palindromic*. If $B \neq 0$, then we have $D \neq 0$.

Proposition 1: The resolvent of the quasi-palindromic

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

with $B \neq 0$ has three roots

$$t_1 = \frac{2D}{B}, \qquad t_2, t_3 = \frac{C}{2} - \frac{D}{B} \pm \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}.$$

Proof: Since $E = \dfrac{D^2}{B^2}$, the resolvent of the quasi-palindromic is

$$t^3 - Ct^2 + \left(BD - \frac{4D^2}{B^2}\right)t - 2D^2 + \frac{4CD^2}{B^2} = 0. \tag{15}$$

This can be factorised as

$$\left(t - \frac{2D}{B}\right)\left(t^2 + \left(\frac{2D}{B} - C\right)t + BD - \frac{2CD}{B}\right) = 0.$$

Finding zeros of the above quadratic polynomial, we obtain the conclusion.

For the sake of simplicity, we choose $t_1 = \dfrac{2D}{B}$. Then the quasi-palindromic has roots

$$x = \frac{1}{2}\left(-p_k \pm \sqrt{p_k^2 - \frac{4D}{B}}\right) \qquad (k = 1,\ 2) \tag{16}$$

where

$$p_1 = \frac{1}{2}\left(B + \sqrt{B^2 - 4C + \frac{8D}{B}}\right),\ p_2 = \frac{1}{2}\left(B - \sqrt{B^2 - 4C + \frac{8D}{B}}\right).$$

The final solution of a quasi-palindromic

We consider the three roots of resolvent (10). Let t be a root of the resolvent. Then it is not always true that t satisfies (14). If t satisfies (14), roots (13) of quartic (1) are in the form of $u + vi$ where u, v are real.

Definition 2: If the roots of an equation are in the form of $u + vi$ where u, v are real, we say that they are the *final roots* of the equation. The process of finding these final roots is called the *final solution* of the equation.

Roots (16) in the previous section are well known. However, they are not necessarily final. In fact, if $B^2 - 4C + \frac{8D}{B} \geqslant 0$, they are final because $\left(\frac{2D}{B}\right)^2 - 4E = 0$. If $B^2 - 4C + \frac{8D}{B} < 0$, they are not final.

We consider the case $B^2 - 4C + \frac{8D}{B} < 0$. In this case, the roots of resolvent (15) are all real. If t_2 and t_3 are imaginary, each root of the resolvent does not satisfy (14). This fact contradicts Remark 1. Also $t_2 > 0$ because $\frac{C}{2} - \frac{D}{B} > 0$. Now, let us find a root t of resolvent (15) satisfying (14). In the following, we examine $t_2 = \frac{C}{2} - \frac{D}{B} + \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}$. First,

$$B^2 - 4C + 4t_2 = B^2 - 2C - \frac{4D}{B} + 4\sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}.$$

When $B^2 - 2C - \frac{4D}{B} > 0$, we obtain $B^2 - 4C + 4t_2 > 0$. When $B^2 - 2C - \frac{4D}{B} \leqslant 0$, we find

$$16\left(\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD\right) - \left(-B^2 + 2C + \frac{4D}{B}\right)^2$$
$$= B^2\left(-B^2 + 4C - \frac{8D}{B}\right) > 0.$$

Therefore, $B^2 - 4C + 4t_2 > 0$ holds. In addition,

$$t_2^2 - 4E = t_2^2 - \frac{4D^2}{B^2} = \left(t_2 + \frac{2D}{B}\right)\left(t_2 - \frac{2D}{B}\right).$$

When $BD > 0$, we obtain $t_2 + \frac{2D}{B} > 0$ from $t_2 > 0$. When $BD < 0$, we find

$$t_2 + \frac{2D}{B} = \frac{C}{2} + \frac{D}{B} + \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD} > \frac{C}{2} + \frac{D}{B} + \left|\frac{C}{2} + \frac{D}{B}\right| \geqslant 0.$$

Also,

$$t_2 - \frac{2D}{B} = \frac{C}{2} - \frac{3D}{B} + \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}.$$

When $\frac{C}{2} - \frac{3D}{B} > 0$, we obtain $t_2 - \frac{2D}{B} > 0$. When $\frac{C}{2} - \frac{3D}{B} \leqslant 0$ and $BD > 0$, we find

$$\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD - \left(-\frac{C}{2} + \frac{3D}{B}\right)^2 = \frac{D}{B}\left(-B^2 + 4C - \frac{8D}{B}\right) > 0.$$

Therefore, $t_2 - \frac{2D}{B} > 0$ holds. When $BD < 0$, we have $t_2 - \frac{2D}{B} > 0$ from $t_2 > 0$. From the above, in any case t_2 satisfies (14). Thus we obtain the final solution.

Theorem 1: The final roots of the quasi-palindromic

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

with $B \neq 0$ are as follows:

(i) If $B^2 - 4C + \frac{8D}{B} \geqslant 0$, then

$$x = \frac{1}{2}\left(-p_k \pm \sqrt{p_k^2 - \frac{4D}{B}}\right) \qquad (k = 1, 2)$$

where $p_1 = \frac{1}{2}\left(B + \sqrt{B^2 - 4C + \frac{8D}{B}}\right)$, $p_2 = \frac{1}{2}\left(B - \sqrt{B^2 - 4C + \frac{8D}{B}}\right)$.

(ii) If $B^2 - 4C + \frac{8D}{B} < 0$, then

$$x = \frac{-p_k \pm \sqrt{p_k^2 - 4q_k}}{2} \qquad (k = 1, 2)$$

where $p_1 = \frac{1}{2}\left(B + \sqrt{B^2 - 4C + 4t}\right)$, $p_2 = \frac{1}{2}\left(B - \sqrt{B^2 - 4C + 4t}\right)$,

$$q_1 = \frac{1}{2}\left(t + \sqrt{t^2 - \frac{4D^2}{B^2}}\right), q_2 = \frac{1}{2}\left(t - \sqrt{t^2 - \frac{4D^2}{B^2}}\right),$$

$$t = \frac{C}{2} - \frac{D}{B} + \sqrt{\left(\frac{C}{2} + \frac{D}{B}\right)^2 - BD}.$$

At the end of this Note, we make the following remark.

Remark 2: When all the roots of the quasi-palindromic in Theorem 1 are real, the three roots t_k $(k = 1, 2, 3)$ of resolvent (15) satisfy

$$B^2 - 4C + 4t_k \geqslant 0, \qquad t_k^2 - 4E \geqslant 0.$$

Hence, in that case it does not matter which of the roots of the resolvent we choose because each one provides the final roots of the quasi-palindromic.

References

1. R. B. King, *Beyond the quartic equation*, Birkhaüser, Boston (1996).
2. R. Descartes, *The Geometry of René Descartes*, with a facsimile of the 1st edition, Dover, New York (1954).

3. L. Euler, *Elements of Algebra*, reprint of the 5th edition, Springer-Verlag, New York (1984).
4. J. L. Lagrange, Réflexions sur la résolution algébrique des équations, in J. A. Serret (ed.), *Œuvres de Lagrange*, III, Gauthier-Villars (1869) pp. 205-421.
5. B. Christianson, Solving quartics using palindromes, *Math. Gaz.* **75** (October 1991) pp. 327-328.
6. M. D. Yacoub and G. Fraidenraich, A solution to the quartic equation, *Math. Gaz.* **96** (July 2012) pp. 271-275.
7. J. E. Hacke, Jr., A simple solution of the general quartic, *Amer. Math. Monthly* **48** (1941) pp. 327-328.

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107.06 Proving inequalities via definite integration: a visual approach

Fascination with inequalities has encouraged numerous visual proofs. It is quite interesting to see and feel the beauty. There are several techniques to do these proofs logically. Definite integration of one variable is seemed to be a greater tool in this case. Geometrically, definite integration means area under a given curve. So, basically it will assign a number. If we use different curves in the same region then it will give us different numerical expressions and we can compare between them. We can use this tool in such a way that it will give us the required expression for an inequality. Also, it will give us a clear visual representation in order to prove our claims. In this Note, we provide another area argument on the general inequality (see [1])

$$e \leqslant A < B \Rightarrow A^B > B^A$$

and also two visual proofs of two different inequalities using area under the curves.

Inequality 1

The constants e and π have encouraged numerous visual proofs of the inequality $\pi^e < e^\pi$ (see [2]). In [3], Gallant provided the most general proof for which this inequality is a consequence, showing that when $e \leqslant A < B$, we have $A^B > B^A$; he used slopes of secant lines connecting the origin to points on the curve $y = \ln(x)$. We provide an alternate visual proof for this general inequality.