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NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

BY

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Recently Bruck initiated the study of the structure of the fixed-point set of a nonexpansive selfmap T of a Banach space, where T satisfies a conditional fixed point property. We generalize many of his results to a Hausdorff locally convex space X. Also, we generalize a result of Holmes and Narayanaswami and use it, along with a procedure of Kiang, to obtain a fixed point theorem for families of asymptotically nonexpansive mappings in X.

Let R^Δ be the product of Δ copies of the real line R. We give R^Δ the product topology. Addition and multiplication in R^Δ are defined pointwise. For f, $g \in R^\Delta$

(1) $f \leq g$ means $f(q) \leq g(q)$ for all $q \in \Delta$;
(2) $f < g$ means $f \leq g$ and there exists $q \in \Delta$ with $f(q) < g(q)$;
(3) $f \ll g$ means $f(q) < g(q)$ for all $q \in \Delta$.

R^Δ is called a *Tychonoff semifield* [1]. A base for the neighborhood system $\mathcal{U}$ of 0 in R^Δ is the collection $\mathcal{B}$ of sets of the form

$$U(\varepsilon; q_1, \ldots, q_n) = \{f : |f(q_i)| < \varepsilon, \qquad \varepsilon > 0, \qquad q_i \in \Delta\}.$$

Throughout this paper X will denote a Hausdorff locally convex space and C a nonempty closed convex subset of X. The topology t of X is generated by the family $\{N_q : q \in \Delta\}$ of continuous seminorms. We define $N: X \to R^\Delta$ by $N(x)(q) = N_q(x)$. The mapping N satisfies the axioms of a norm and is called a *norm over* R^Δ. A natural topology t_N is induced on X by N; a basis of neighborhoods of zero is given by all sets of the form

$$S(0, U) = \{x : N(x) \in U\},$$

where $U = U(\varepsilon; q_1, \ldots, q_n) \in \mathcal{B}$. Since $t_N = t$, X is *normed over* R^Δ by N. A metric ρ for X over R^Δ is obtained by defining $\rho(x, y) = N(x - y)$. It is shown in [1] that the mapping $(x, y) \to \rho(x, y)$ is continuous.

1. Nonexpansive retracts. A mapping T of X into itself is said to be *nonexpansive* if $N(Tx - Ty) \leq N(x - y)$ for all $x, y \in X$. The set $\{x : Tx = x\}$ of fixed points of T is denoted by $F(T)$. Also, for $A \subset C$ we define $M(A) = \{T \mid T : C \to C$ is nonexpansive and $A \subset F(T)\}$.

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LEMMA 1. *Suppose p is a continuous seminorm on X and a net $\{x_d : d \in D\}$ converges weakly to $x \in X$. Then $p(x) \le \underline{\lim}\ p(x_d)$.*

Proof. There exists a continuous linear functional f such that $f(x) = p(x)$ and $|f(y)| \le p(y)$ for all $y \in X$. Thus $p(x) = f(x) = \underline{\lim}|f(x_d)| \le \lim p(x_d)$.

LEMMA 2. *Suppose C is weakly compact and A is a non-empty subset of C. Then $M(A)$ is compact in the topology of weak pointwise convergence.*

Proof. Fix $x_0 \in A$. For each x define $C_x = \{y \in C : N(y - x_0) \le N(x - x_0)\}$. For each $T \in M(A)$ and each $x \in C$ we have $Tx \in C_x$. Thus $M(A) \subset P = \Pi\{C_x : x \in C\}$.

Clearly C_x is convex; and since the mapping $x \to N(x)$ is continuous, C_x is closed. Thus each C_x is a closed convex subset of the weakly compact set C. Therefore C_x is weakly compact; and by Tychonoff's theorem, P is compact in the topology of weak pointwise convergence.

To conclude the proof it suffices to show that $M(A)$ is closed in P. Suppose $\{T_d : d \in D\}$ is a net in $M(A)$ converging to $T \in P$. Clearly $A \subset F(T)$; and since $T \in P$, $T(C) \subset C$. Using lemma 1 we have

$$N_q(Tx - Ty) \le \underline{\lim}\ N_q(T_d x - T_d y) \le N_q(x - y)$$

for all x, $y \in C$ and $q \in \Delta$. Thus T is nonexpansive and $M(A)$ is closed in P.

LEMMA 3. *Suppose C is weakly compact and A is a nonempty subset of C. Then there exists $S \in M(A)$ such that for each $T \in M(A)$ and $x, y \in C$, $N(TSx - TSy) = N(Sx - Sy)$.*

Proof. Define an order on $M(A)$ by $T' \le T$ if $N(T'x - T'y) \le N(Tx - Ty)$ for all $x, y \in C$. For each $T \in M(A)$ let $I(T) = \{T' \in M(A) : T' \le T\}$. Using the same techniques as used in lemma 2 to show $M(A)$ is closed in P, we obtain $I(T)$ is closed in $M(A)$. Thus $I(T)$ is weakly compact.

If $\mathscr{C}$ is a chain in $M(A)$, then $\{I(T) : T \in \mathscr{C}\}$ is a chain under set inclusion. Since each $I(T)$ is compact there exists $T_0 \in \bigcap \{I(T) : T \in \mathscr{C}\}$. T_0 is a lower bound for $\mathscr{C}$. Zorn's lemma now implies that $M(A)$ has a minimal element S.

For each $T \in M(A)$, $TS \in M(A)$; and since T is nonexpansive, $N(TSx - TSy) \le N(Sx - Sy)$ for all $x, y \in C$. But S is a minimal element in $M(A)$. Thus $N(TSx - TSy) = N(Sx - Sy)$ for all $x, y \in C$ and $T \in M(A)$.

A subset A of C is a *nonexpansive retract* of C if either $A = \emptyset$ or there is a retraction of C onto A which is nonexpansive.

A mapping $T : C \to X$ satisfies the *conditional fixed point property* (CFPP) if either T has no fixed points, or T has a fixed point in every bounded closed convex subset which it leaves invariant.

With lemmas 2 and 3, the proofs of theorem 1 and 2 are the same as for Banach spaces [2] and hence omitted.

THEOREM 1. *Suppose C is weakly compact and A is a nonempty subset of C. Suppose for each $z \in C$ there exists $T \in M(A)$ such that $Tz \in A$. Then A is a nonexpansive retract of C.*

THEOREM 2. *Suppose C is weakly compact and $T: C \to C$ is nonexpansive and satisfies* (CFPP). *Then $F(T)$ is a nonexpansive retract of C.*

The conditional fixed point property does hold for several classes of mappings. Suppose K is a nonempty weakly compact convex subset of X and K has normal structure. Then Tan [8] has shown that a nonexpansive mapping of K into itself has a fixed point. Thus (CFPP) holds for nonexpansive mappings of C into X where C is weakly compact and has normal structure.

A subset B of X is said to have the *fixed point property for nonexpansive mappings* if every nonexpansive mapping of B into itself has a fixed point. A set B is said to have the ***hereditary fixed point property*** (HFPP) if every nonempty bounded closed convex subset of B has the fixed point property for nonexpansive mappings. If $T: C \to C$ is nonexpansive and C satisfies (HFPP), then T satisfies (CFPP). Thus, if C is bounded and compact and $T: C \to C$ is nonexpansive, then T satisfies (CFPP).

A sequence $\{x_n\}$ is said to be *Cauchy* if for each neighborhood U of 0 in R^{Δ}, there exists an integer M such that $N(x_n - x_m) \in U$ for all $n, m \geq M$. We say X is *sequentially complete* if every Cauchy sequence converges.

THEOREM 3. *Suppose X is sequentially complete and $T: C \to C$ is nonexpansive. Then T satisfies* (CFPP) *if any one of the following hold*:

(a) *$(I-T)(K)$ is strongly closed whenever K is a bounded closed convex subset of C*;

(b) *T is compact*;

(c) *C is weakly compact and T is affine.*

Proof. Suppose K is a bounded closed convex subset of C with $T(K) \subset K$.

(a) Let $\{t_n\}$ be a sequence from (0, 1) such that $\lim t_n = 1$. By a result of Tan [7, theorem 2.3], the function $t_n T$ has a fixed point $x_n \in K$. Since K is bounded, $(I-T)(x_n) = (1 - 1/t_n)(x_n) \to 0$. Since $(I-T)(K)$ is closed, there exists $x \in K$ with $Tx = x$.

(b) We show $(I-T)(K)$ is closed and the result follows from (a). Suppose $\{x_d - T(x_d) : d \in D\}$ is a net in $(I-T)(K)$ converging to $y \in X$. Since T is compact and $\{x_d\}$ is bounded, $\{T(x_d)\}$ contains a subnet, which we also denote by $T(x_d)$, converging to y_1. Since $T(x_d) \to y_1$ and $x_d - T(x_d) \to y$, $x_d \to y_1 + y = z \in K$. The continuity of T implies $T(z) = y_1$. Thus $y = z - y_1 = z - Tz \in (I-T)(K)$.

(c) We again show that $(I-T)(K)$ is closed. Since T is affine, $(I-T)(K) = z_0 + T'(K)$ where T' is continuous and linear. Thus it suffices to show that $T'(K)$ is closed. Since K is convex and weakly compact, $T'(K)$ is also convex and weakly compact. But X is a Hausdorff space, thus $T'(K)$ is weakly closed and hence closed.

For families of nonexpansive mappings on X the following results of Bruck [2] carry over, with the same proof, to Hausdorff locally convex spaces.

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THEOREM 4. *Suppose C is weakly compact and satisfies* (HFPP). *Suppose F is a finite family of commuting nonexpansive mappings from C into C. Then* $\bigcap\{F(T): T \in F\}$ *is a nonexpansive retract of C.*

THEOREM 5. *Suppose C is compact and* $\mathcal{F}$ *is an arbitrary family of commuting nonexpansive mappings of C into C. Then* $\bigcap\{F(T): T \in \mathcal{F}\}$ *is a nonempty nonexpansive retract of C.*

2. Asymptotically Nonexpansive Mappings. A family $\mathcal{F}$ of mappings from X into X is *asymptotically nonexpansive* [*asymptotically isometric*] if for each $x, y \in X$ there exists a $S \in \mathcal{F}$ such that for all T in $\mathcal{F}$

$$(4) \qquad N(TS(x) - TS(y)) \leq N(x - y)$$

$$[N(TSx - TSy) = N(x - y)]$$

We recall that the family $\mathcal{B}$ of all sets of the form

$$U(\varepsilon; q_1, \ldots, q_n) = \{f : |f(q_i)| < \varepsilon, \varepsilon > 0, q_i \in \Delta\}$$

is a base for $\mathcal{U}$, the neighborhood system of 0 in R^Δ. For $\mathcal{F}$ a commutative semigroup of continuous asymptotically nonexpansive mappings on X and $Y \subset X$ we define,

(5) $Y^{\mathcal{F}} = \{z \in Y$: there exists $x \in Y$ such that for each $T \in \mathcal{F}$ and each $U \in \mathcal{U}$, there exists $S \in \mathcal{F}$ *with* $N(TS(x) - z) \in U\}$.

The following theorem generalizes propositions 1 and 2 of [3].

THEOREM 6. *Suppose* $\mathcal{F}$ *is a commutative semigroup of continuous asymptotically nonexpansive mappings on X.*

(i) *If* $z \in X^{\mathcal{F}}$, *then for every* $T \in \mathcal{F}$ *and* $U \in \mathcal{U}$, *there exists* $S \in \mathcal{F}$ *with* $N(TS(z) - z) \in U$.

(ii) *If* $z \in X^{\mathcal{F}}$, *then* $\mathcal{F} \mid \mathcal{F}(z)$ *is a family of asymptotic isometries.*

(iii) *If* $A \subset X$ *and* $\mathcal{F} \mid A$ *is a family of asymptotic isometries, then* $\mathcal{F} \mid A$ *is a family of isometries.*

Proof of (i). Suppose $z \in X^{\mathcal{F}}$, $T \in \mathcal{F}$ and $U \in \mathcal{U}$. There are a finite number of subbase elements $U(\varepsilon, q_i)$ such that $\bigcap_i U(\varepsilon, q_i) \subset U$. Let $V = \bigcap_i U(\varepsilon/2, q_i)$, then from (5) there exists $T_1 \in \mathcal{F}$ with $N(T_1T(x) - z) \in V$. Since $\mathcal{F}$ is commutative and asymptotically nonexpansive, there exists $T_2 \in \mathcal{F}$ such that $N(ST_2T_1T(x) - ST_2(z)) \leq N(T_1T(x) - z)$ for all $S \in \mathcal{F}$. Also, from (5), there exists $T_3 \in \mathcal{F}$ with $N(TT_2T_1TT_3(x) - z) \in V$. Let $S_0 = T_2T_3$. Then, for each i, we have

$$N_{q_i}(TS_0(z) - z) \leq N_{q_i}(TS_0(z) - TT_2T_1TT_3(x)) + N_{q_i}(TT_2T_1TT_3(x) - z)$$
$$\leq N_{1_i}(z - T_1T(x)) + \varepsilon/2 < \varepsilon.$$

Therefore, $N(TS_0(z) - z) \varepsilon U$.

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Proof of (ii). Suppose it is not true. Then there exist $u = T_1(z)$ and $v = T_2(z)$ such that for each $T \in \mathcal{F}$ there exists $S \in \mathcal{F}$ and $q \in \Delta$ with $N_q(TSu - TSv)) \neq N_q(u - v)$. Since $\mathcal{F}$ is asymptotically nonexpansive, there exists $T_3 \in \mathcal{F}$ such that $N(ST_3(u) - ST_3(v)) \leq N(u - v)$ for all $S \in \mathcal{F}$. Thus there exists $S_0 \in \mathcal{F}$ and $q \in \Delta$ such $N_q(S_0T_3(u) - S_0T_3(v)) < N_q(u - v)$. Let $\varepsilon = -N_q(S_0T_3(u) + S_0T_3(v)) + N_q(u - v)$ and $V = U(\varepsilon/2, q)$. By (4), there exists $T_4 \in \mathcal{F}$ such that

$$N(ST_4S_0T_3(u) - ST_4S_0T_3(v)) \leq N(S_0T_3(u) - S_0T_3(v)) \tag{6}$$

for all $S \in \mathcal{F}$.

Since T_1 and T_2 are continuous at z, there exists a $U \in \mathcal{U}$ such that $N(y - z) \in U$ implies $N(T_1(y) - T_1(z))$ and $N(T_2(y) - T_2(z))$ are in V. Applying (i) to $T_4S_0T_3 = S_1$ and U, there exists $T_5 \in \mathcal{F}$ such that $N(T_5S_1(z) - z) \in U$. Since $u = T_1(z)$, $v = T_2(z)$ and $N(T_5S_1(z) - z) \in U$, we have $N(T_5S_1(u) - u) \in V$ and $N(T_5S_1(v) - v) \in V$. Finally using (6) we obtain

$$\begin{aligned} N_q(u - v) &\leq N_q(u - T_5S_1(u)) + N_q(T_5S_1(u) - T_5S_1(v)) \\ &\quad + N_q(T_5S_1(v) - v) \\ &< \varepsilon/2 + N_q(S_0T_3(u) - S_0T_3(v) + \varepsilon/2 = N_q(u - v). \end{aligned}$$

This contradiction implies $\mathcal{F} \mid \mathcal{F}(z)$ is a family of asymptotic isometries.

Proof of (iii). Suppose $x, y \in A$ and $T \in \mathcal{F}$. Then there exist T_1, $T_2 \in \mathcal{F}$ such that $N(ST_1(x) - ST_1(y)) = N(x - y)$ and $N(ST_2T(x) - ST_2T(y)) = N(T_x - Ty)$ for all $S \in \mathcal{F}$. Since T_1 and T_2T are in $\mathcal{F}$, we have

$$\begin{aligned} N(x - y) &= N(T_2TT_1(x) - T_2TT_1(y)) \\ &= N(T_1T_2Tx - T_1T_2Ty) = N(Tx - Ty). \end{aligned}$$

Remark. The proof of theorem 6 does not use the norm property that $N(ax) = aN(x)$. Thus if X is a Hausdorff uniform space, hence metrizable over R^Δ, the theorem is valid with the same proof.

We say a subset K of X if *strictly convex* if for $x, y \in K$, $N(x + y) = N(x) + N(y)$ implies $\{x, y\}$ is linearly dependent.

Suppose $\mathcal{F}$ is a commutative semigroup of continuous asymptotically nonexpansive mappings on X and $z \in X^{\mathcal{F}}$. Then by theorem 6(i), for each $T \in \mathcal{F}$ and each $U \in \mathcal{U}$, there exists $\varphi_{T,U} \in \mathcal{F}$ such that $N(T\varphi_{T,U}(z) - z) \in U$. Let $B = \{x \in X : N(x - \Phi_{T,V}T(z)) \in V$ for all $T \in \mathcal{F}$ and for all $V \in \mathcal{B}\}$ where $\mathcal{B}$ is the base for $\mathcal{U}$ define in the introduction. B is not empty since $z \in B$. The following theorem generalizes theorem 5 of Kiang [5].

Theorem 7. *Suppose C is a weakly compact subset of a strictly convex space X and $F : C \to C$ is a cummutative semigroup of continuous asymptotically nonex-*

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pansive mappings on C. Suppose $z \in X^{\mathcal{F}} \neq \emptyset$. *Then* $\mathcal{F}$ *has a common fixed point in the closed convex hull of* $\{z\} \cup \mathcal{F}(z)$.

The proof of theorem 7 is based on the following lemmas concerning the set $\mathcal{B}$ defined in the preceeding paragraph.

LEMMA 4. $T(B) \subset B$ *for all* $T \in \mathcal{F}$.

Proof. Let $T \in \mathcal{F}$. If $x \in B$, then $x \in X^{\mathcal{F}}$. Thus by theorem 6, $\mathcal{F} \mid \mathcal{F}(x)$ is a family of isometries. Therefore, for $T \in \mathcal{F}$, $N(T(x) - \Phi_{S,V}ST(x)) = N(x - \Phi_{S,V}S(x)) \in V$ for all $V \in \mathcal{B}$ and all $S \in \mathcal{F}$. Thus $Tx \in B$.

LEMMA 5. $\mathcal{F} \mid \bar{B}$ *is a family of isometries.*

The proof of lemma 5 is the same as Kiang's proof of lemma 2 [5, p. 68] and hence is omitted.

LEMMA 6. *Suppose X is strictly convex, then for every* $T \in \mathcal{F}$, $x, y \in B$ and $\lambda \in [0, 1]$; $T(\lambda x_1 + (1-\lambda)x_2) = \lambda T(x_1) + (1-\lambda)T(x_2)$.

Proof. Partially order $\mathcal{B}$ by $V_1 \geq V_2$ if $V_1 \subset V_2$. Then, for $x \in B$ and $T \in \mathcal{F}$, the net $\{\Phi_{T,V}T(x) : V \in B\}$ converges to x. Using this, the Banach space proof of Kiang [5] carries over to locally convex spaces and hence the details are omitted.

LEMMA 7. *Suppose X is strictly convex. Then* $\bar{B}$ *is convex.*

Proof. We show that B is convex. Let x_1, $x_2 \in B$, $\lambda \in [0, 1]$ and $x = \lambda x_1 + (1-\lambda)x_2$. Then by definition of B, for $i = 1, 2$, $N(x_i - \Phi_{T,V}Tx_i) \in V$ for all $T \in \mathcal{F}$ and all $V \in \mathcal{B}$. Let $T \in \mathcal{F}$ and $V \in \mathcal{B}$. Since $V \supset \bigcap_{j=1}^{n} U(\varepsilon, q_j)$, using lemma δ, we have

$$\begin{aligned} N_{q_j}(x - \Phi_{T,V}Tx) &= N_{q_j}[\lambda(x_1 - \Phi_{T,V}Tx_1) + (1-\lambda)(x_2 - \Phi_{T,V}Tx_2)] \\ &\leq \lambda N_{q_j}(x_1 - \Phi_{T,V}Tx_1) + (1-\lambda)N_{q_j}(x_2 - \Phi_{T,V}Tx_2) \\ &< \varepsilon \text{ for each } j. \end{aligned}$$

Thus $N(x - \Phi_{T,V}Tx) \in V$. Therefore, B and hence $\bar{B}$ are convex.

LEMMA 8. *Suppose B is a closed convex subset of a* T_2 *locally convex space X and* $T : B \to B$ *is continuous and affine. Then T is weakly continuous.*

Proof. Suppose T is not weakly continuous. Then there exists a net $\{x_d\}$ in B that converges weakly to x in B and a continuous linear functional f such that $f(Tx_d) \not\to f(Tx)$. Hence there exists an $\varepsilon > 0$ and a subnet $\{x_\alpha\}$ of $\{x_d\}$ such that for each α, $|f(Tx_\alpha) - f(Tx)| \geq \varepsilon$. Without loss of generality we may assume that $f(Tx_\alpha) - f(Tx) \geq \varepsilon$ for all α. Since $\{x_\alpha\}$ converges weakly to x, there exists a net $\{y_\beta\}$ that converges strongly to x where each y_β is a convex combination of the

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x_α's. Now since $\sum_{i=1}^{n} \alpha_i = 1$, f is linear and T is affine, we have

$$\begin{aligned} f(Ty_\beta)-f(Tx) &= f\left[\left(\sum_{i=1}^{n} \alpha_i x_{\alpha_i}\right)\right]-f(Tx) \\ &= f\left(\sum_{i=1}^{n} \alpha_i Tx_{\alpha_i}\right)-f(Tx) \\ &= \sum_{i=1}^{n} \alpha_i f(Tx_{\alpha_i})-f(Tx) \\ &= \sum_{i=1}^{n} \alpha_i [f(Tx_{\alpha_i})-f(Tx)] \\ &\geq \sum_{i=1}^{n} \alpha_i \varepsilon = \varepsilon \end{aligned}$$

However $y_\beta \to x$ and f and T are continuous gives the contradiction that $f(Ty_\beta)-f(Tx)\to 0$.

Proof of Theorem 7. Lemmas 4–8 imply that $\mathscr{F}\,|\,\bar{B}$ is a commutative semigroup of weakly continuous affine isometries and the remainder of the proof is the same as for Banach spaces [5] and hence is omitted.

Examples of a semigroup of continuous asymptotically non-expansive mappings which are not nonexpansive and a non-normable locally convex space which is strictly convex can be constructed. For these and other examples see [10].

References

1. M. Y. Antonovskii, V. G. Boltyanskii and T. A. Sarymsakov, *An outline of the theory of topological semi-fields*, Russian Math. Surveys, **21** (1966), 163–192.

2. R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc., **179** (1973), 251–262.

3. R. D. Holmes and P. P. Narayanaswami, *On asymptotically nonexpansive semigroups of mappings*, Can. Math. Bull., **13** (1970), 209–214.

4. J. Horvath, *Topological Vector Spaces and Distributions*, Addison–Wesley, Reading, Mass., 1966.

5. M. T. Kiang, *Fixed point theorems for certain classes of semigroups of mappings*, Trans. Amer. Math. Soc., **189** (1974), 63–76.

6. K. K. Tan, *Common fixed point theorems for almost weakly periodic nonexpansive mappings*, Proc. Amer. Math. Soc., **33** (1972), 355–360.

7. ——, *Fixed point theorems for nonexpansive mappings*, Pacific Math. J., **41** (1972), 829–842.

8. ——, *Some fixed point theorems for nonexpansive mappings in Hausdorff locally convex spaces*, Ph.D. Thesis, University of British Columbia, 1970.

9. A. E. Taylor, *Introduction to Functional Analysis*, Wiley, New York, 1958.

10. John D. Kubicek, *Applications of Metrics over Tikonov semifields and fixed points in Hilbert spaces*, Ph.D. Thesis, University of Missouri–Rolla, 1975.

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