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Co-t-structures, cotilting and cotorsion pairs

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Abstract

Let T be a triangulated category with shift functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$. Suppose (A, B) is a co-t-structure with coheart $\mathsf{S} = \Sigma\mathsf{A} \cap \mathsf{B}$ and extended coheart $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$, which is an extriangulated category. We show that there is a bijection between co-t-structures $(\mathsf{A}', \mathsf{B}')$ in T such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$ and complete cotorsion pairs in the extended coheart C. In the case that T is Hom-finite, $\mathbf{k}$-linear and Krull–Schmidt, we show further that there is a bijection between complete cotorsion pairs in C and functorially finite torsion classes in $\operatorname{mod} \mathsf{S}$.

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Introduction

Happel–Reiten–Smalø (HRS) tilting was introduced in [**13**] as a method to construct new t-structures from torsion pairs in the heart of a given t-structure. Suppose T is a triangulated category with shift functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$, and (U, V) is a t-structure in T with the heart $\mathsf{H} = \mathsf{U} \cap \Sigma\mathsf{V}$. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in H, the HRS tilt of (U, V) at $(\mathcal{T}, \mathcal{F})$ is the t-structure

$$(\mathsf{U}', \mathsf{V}') := (\Sigma\mathsf{U} * \mathcal{T}, \mathcal{F} * \mathsf{V}).$$

In addition to providing a method for constructing new t-structures from old, HRS tilting gives all t-structures that are ‘sufficiently close’ to the initial one; see [**7**, **28**, **32**]. Explicitly, there is a bijection:

$$\{\text{t-structures } (\mathsf{U}', \mathsf{V}') \text{ with } \Sigma\mathsf{U} \subseteq \mathsf{U}' \subseteq \mathsf{U}\} \xleftrightarrow{1-1} \{\text{torsion pairs } (\mathcal{T}, \mathcal{F}) \text{ in } \mathsf{H}\}. \qquad (0{\cdot}1)$$

Such t-structures $(\mathsf{U}', \mathsf{V}')$ are often called *intermediate with respect to* (U, V).

HRS tilting has many applications in representation theory and algebraic geometry. For example, it provides a method for constructing derived equivalences between abelian categories in cases where explicit tilting objects are not available. In this context, HRS tilting was used to study derived equivalences for smooth compact analytic surfaces with no curves [8] or for K3 surfaces [10, 15]. Recently, HRS tilting has been extensively used in the study of Bridgeland stability conditions [10, 27, 29, 32].

A *co-t-structure* in T consists of a pair of full subcategories (A, B) of T, which are closed under direct summands, such that $\Sigma^{-1}\mathsf{A} \subseteq \mathsf{A}$, $\mathsf{T}(\mathsf{A}, \mathsf{B}) = 0$, and $\mathsf{T} = \mathsf{A} * \mathsf{B}$ [9, 26]; note that in [9] co-t-structures are called weight structures. The subcategory $\mathsf{S} = \Sigma\mathsf{A} \cap \mathsf{B}$ is called the *coheart*; it is a *presilting subcategory* of T, see Section 1. Since their introduction, co-t-structures have acquired an important role in representation theory in connection with silting theory and τ-tilting theory [1, 2, 16, 20, 21, 23]; for surveys of recent results see [4, 19]. At first sight, the definitions of t-structure and co-t-structure appear very similar and there are, indeed, a number of parallels between the two theories. However, t-structures and co-t-structures are not dual to each other in a mathematical sense and there are notable differences between them, with the most basic being the failure of abelianness of the coheart.

The main result of this paper is an analogue of bijection (0·1) for co-t-structures. A priori it is not clear what are the co-t-structure counterparts of the HRS tilting procedure and the torsion pair in the heart. The recent introduction of extriangulated categories in [24] provides the right context. If (A, B) is a co-t-structure in T with coheart S, then $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$, which we call the *extended coheart*, is an extriangulated category, in which there is a notion of a *complete cotorsion pair* [14, 24, 30]. A particular example of the extended coheart appears in the context of Amiot cluster categories in the guise of the fundamental domain of the cluster category [3].

THEOREM A (Theorem 2·2). *Suppose* T *is a triangulated category,* (A, B) *is a co-t-structure in* T*, and* C *is its extended coheart. There is a bijection*

$$\{\textit{co-t-structures } (\mathsf{A}', \mathsf{B}') \textit{ with } \mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}\} \xleftrightarrow{1-1} \{\textit{complete cotorsion pairs } (\mathcal{X}, \mathcal{Y}) \textit{ in } \mathsf{C}\}.$$

Let A be a finite-dimensional $\mathbf{k}$-algebra. From [21], there are bijections between the following objects:

(i) silting subcategories, S, in $\mathsf{K}^b(\mathsf{proj}(A))$;

(ii) bounded co-t-structures, $(\mathsf{A}_\mathsf{S}, \mathsf{B}_\mathsf{S})$, in $\mathsf{K}^b(\mathsf{proj}(A))$;

(iii) *algebraic t-structures*, that is, bounded t-structures, $(\mathsf{U}_\mathsf{S}, \mathsf{V}_\mathsf{S})$, in $\mathsf{D}^b(A)$ with length heart.

Fixing a silting subcategory $\mathsf{S} = \mathsf{add}(s)$ for a silting object s, by [16] these bijections restrict to bijections between:

(i) algebraic t-structures intermediate with respect to $(\mathsf{U}_\mathsf{S}, \mathsf{V}_\mathsf{S})$;

(ii) bounded co-t-structures intermediate with respect to $(\mathsf{A}_\mathsf{S}, \mathsf{B}_\mathsf{S})$;

(iii) silting subcategories S' with $\mathsf{S}' \subseteq \mathsf{S} * \Sigma\mathsf{S}$.

Finally, HRS tilting and support τ-tilting theory [1, 16] adds a bijection with:

(iv) functorially finite torsion pairs in $\mathsf{mod}\,\mathrm{End}(s) \simeq \mathsf{mod}\,\mathsf{S}$

into the mix. Theorem A completes the picture with the co-t-structure version of torsion pairs: cotorsion pairs in $\mathsf{S} * \Sigma\mathsf{S}$. Moreover, working with co-t-structures and cotorsion pairs seems to provide a more convenient context for representation theory: one does not have to care about the additional condition on the t-structure that it has a length heart, which may be difficult to check in practice, cf. [11].

Our second result provides a direct and explicit connection between cotorsion pairs in $\mathsf{S} * \Sigma\mathsf{S}$ and torsion pairs in $\mathsf{mod}\,\mathsf{S}$.

THEOREM B (see Theorem 3·7). *Suppose* T *is an essentially small, Hom-finite,* $\mathbf{k}$*-linear, Krull–Schmidt triangulated category. If* $\mathsf{S} = \mathsf{add}(s)$ *is a presilting subcategory of* T *and* $\mathsf{C} = \mathsf{S} * \Sigma\mathsf{S}$*, then the restricted Yoneda functor,* $F\colon \mathsf{C} \to \mathsf{mod}\,\mathsf{S}$*, induces a bijection*

$$\{\textit{complete cotorsion pairs in } \mathsf{C}\} \stackrel{1-1}{\longleftrightarrow} \{\textit{functorially finite torsion classes in } \mathsf{mod}\,\mathsf{S}\}.$$

In particular, F sends cotorsionfree classes to torsion classes. Moreover, if in addition $\mathsf{mod}\,\mathsf{S} \simeq \mathsf{mod}\,A$ *for an artin algebra A, there is a bijection between complete cotorsion pairs in* C *and functorially finite torsion pairs in* $\mathsf{mod}\,\mathsf{S}$.

We note that the connections between torsion and cotorsion pairs have been studied before in a different setting [7]. In particular, Beligiannis and Reiten consider cotorsion pairs in abelian categories and the corresponding cotorsion pairs in pretriangulated categories.

1. *Background*

Let A be an additive category and $\mathsf{B} \subset \mathsf{A}$ a subcategory. For objects a_1, a_2 of A we will write $\mathsf{A}(a_1, a_2) = \mathrm{Hom}_{\mathsf{A}}(a_1, a_2)$. We define the left and right orthogonal categories of B as follows:

$${}^{\perp}\mathsf{B} := \{a \in \mathsf{A} \mid \mathsf{A}(a, b) = 0 \text{ for all } b \in \mathsf{B}\} \text{ and } \mathsf{B}^{\perp} := \{a \in \mathsf{A} \mid \mathsf{A}(b, a) = 0 \text{ for all } b \in \mathsf{B}\}.$$

We will often use the shorthand $\mathsf{A}(a, \mathsf{B}) = 0$ to mean $\mathsf{A}(a, b) = 0$ for all $b \in \mathsf{B}$; similarly for the shorthand $\mathsf{A}(\mathsf{B}, a) = 0$.

Throughout this paper T will be a triangulated category with shift functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$. For two subcategories A, B of T, the full subcategory with objects $\{t \mid$ there exists a triangle $a \to t \to b \to \Sigma a$ with $a \in \mathsf{A}$ and $b \in \mathsf{B}\}$ will be denoted by $\mathsf{A} * \mathsf{B}$. A full additive subcategory C of T is *extension-closed* if $\mathsf{C} * \mathsf{C} = \mathsf{C}$.

1·1. *Approximations*

Let A be a subcategory of T and let t be an object of T. A morphism $f\colon t \to a$ with $a \in \mathsf{A}$ is called:

(i) a *left* A*-approximation of t* if $\mathsf{T}(f, \mathsf{A})\colon \mathsf{T}(a, \mathsf{A}) \to \mathsf{T}(t, \mathsf{A})$ is surjective;

(ii) *left minimal* if any $g\colon a \to a$ such that $gf = f$ is an automorphism;

(iii) a *minimal left* A*-approximation of t* if it is both left minimal and a left A-approximation of t.

Left A-approximations are sometimes called A-*pre-envelopes*. If every object of T admits a left A-approximation then A is said to be *covariantly finite* in T. There is a dual notion of a *(minimal) right* A*-approximation* (or an A-*precover*); if every object of T admits a right A-approximation, then A is said to be *contravariantly finite* in T. The subcategory A is said to be *functorially finite* if it is both covariantly finite and contravariantly finite.

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Minimal approximations admit the following important property; see, for example, [18] for a triangulated version. We give the statement for left approximations; there is a dual statement for right approximations.

LEMMA 1·1 (Wakamatsu lemma for triangulated categories). *Let* A *be an extension closed subcategory of* T *and suppose* $f: t \to a$ *is a minimal left* A*-approximation of* t*. Then in the triangle*

$$b \longrightarrow t \xrightarrow{f} a \longrightarrow \Sigma b,$$

we have $b \in {}^{\perp}\mathsf{A}$*.*

1·2. *Co-t-structures, silting subcategories and the extended coheart*

We recall the following definitions from [2, 9, 20, 26], respectively.

Definition 1·2. A subcategory S of a triangulated category T is *presilting* if $\mathsf{T}(\mathsf{S}, \Sigma^i \mathsf{S}) = 0$ for all $i > 0$; it is called *silting* if, in addition $\mathsf{thick}\, \mathsf{S} = \mathsf{T}$, where $\mathsf{thick}\, \mathsf{S}$ is the smallest triangulated subcategory of T containing S that is closed under direct summands. An object s of T is a *(pre)silting* object if $\mathsf{add}(s)$ is a (pre)silting subcategory, where $\mathsf{add}(s)$ consists of the direct summands of finite direct sums of copies of s.

Definition 1·3. A *co-t-structure* in T consists of a pair of full subcategories (A, B) of T, which are closed under direct summands, such that $\Sigma^{-1}\mathsf{A} \subseteq \mathsf{A}$, $\mathsf{T}(\mathsf{A}, \mathsf{B}) = 0$, and $\mathsf{T} = \mathsf{A} * \mathsf{B}$. A co-t-structure (A, B) in T is *bounded* if $\bigcup_{i\in\mathbb{Z}} \Sigma^i \mathsf{A} = \mathsf{T} = \bigcup_{i\in\mathbb{Z}} \Sigma^i \mathsf{B}$.

The *coheart* $\mathsf{S} = \Sigma\mathsf{A} \cap \mathsf{B}$ of a co-t-structure (A, B) is always a presilting subcategory. It is silting precisely when the co-t-structure is bounded [23, corollary 5·9].

Definition 1·4. Let (A, B) be a co-t-structure in T. The subcategory $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$ will be called the *extended coheart* of the co-t-structure.

The following lemma shows that the extended coheart of (A, B) consists of precisely the objects of T which are 'two-term' with respect to the coheart S.

LEMMA 1·5 ([16, lemma 2·1]). *Let* (A, B) *be a co-t-structure in* T *with coheart* S*. Then the extended coheart* $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$.

1·3. *Extriangulated categories and complete cotorsion pairs*

We will use the notion of an extriangulated category from [24] without recalling the complete definition.

An *extriangulated category* consists of a triple $(\mathsf{C}, \mathbb{E}, \mathfrak{s})$, where C is an additive category, $\mathbb{E}(-,-): \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{Ab}$ is a biadditive functor and $\mathfrak{s}$ assigns to any element of $\mathbb{E}(c, a)$ an equivalence class of pairs of morphisms $[a \to b \to c]$, called an $\mathbb{E}$*-triangle*. In addition the triple $(\mathsf{C}, \mathbb{E}, \mathfrak{s})$ should satisfy a number of axioms reminiscent of the axioms of a triangulated category (without rotation of triangles).

If C is an additive category, Σ is an equivalence on C and $\mathbb{E} := \mathsf{C}(-, \Sigma-)$, then by [24, proposition 3·22] fixing a triangulated structure on C with the shift functor Σ is equivalent

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to fixing an extriangulated structure on C with the additive bifunctor $\mathbb{E}$, where $\mathfrak{s}$ assigns to an element $\delta \in \mathsf{C}(c, \Sigma a)$ the isomorphism class of distinguished triangles $a \to b \to c \xrightarrow{\delta} \Sigma a$.

All extriangulated categories C used in this paper will be subcategories of some triangulated category T with the induced extriangulated structure, that is, $\mathbb{E}(-,-)$ is the restriction of $\mathsf{T}(-,\Sigma-)$ and $\mathbb{E}$-triangles are distinguished triangles $a \to b \to c \xrightarrow{\delta} \Sigma a$ with a, b, c in the subcategory C. Analogously to triangulated and exact categories, a subcategory $\mathcal{X}$ of an extriangulated category C is called *extension-closed* if for any $\mathbb{E}$-triangle $x' \to x \to x''$ with $x', x'' \in \mathcal{X}$, the object x is also in $\mathcal{X}$.

LEMMA 1·6. *Let* (A, B) *be a co-t-structure in* T. *Then the triangulated structure of* T *induces an extriangulated structure on* $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$.

Proof. Since both A and B are extension-closed subcategories of T, we have that C is an extension-closed subcategory of T. The triangulated structure on T also provides an extriangulated structure on T, see [**24**, example 2·13]. Hence, by [**24**, remark 2·18], the triangulated structure on T restricted to C induces an extriangulated structure on C.

We adopt the following definitions from the exact and abelian settings (see [**14**, **30**]) to the extriangulated setting.

Definition 1·7. Let $(\mathsf{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. A *cotorsion pair* in C consists of a pair of full additive subcategories $(\mathcal{X}, \mathcal{Y})$ closed under direct summands and such that for each $c \in \mathsf{C}$ the following holds:

(i) $c \in \mathcal{X}$ if and only if $\mathbb{E}(c, \mathcal{Y}) = 0$;

(ii) $c \in \mathcal{Y}$ if and only if $\mathbb{E}(\mathcal{X}, c) = 0$.

If $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in an extriangulated category C, we will refer to $\mathcal{X}$ as the *cotorsion class* and to $\mathcal{Y}$ as the *cotorsionfree class*. Since $\mathcal{X}$ and $\mathcal{Y}$ are each realised as orthogonal subcategories, they are closed under extensions. Indeed, by [**24**, proposition 3·3] any $\mathbb{E}$-triangle $a \to b \to c$ gives rise to an exact sequence $\mathsf{C}(-,a) \to \mathsf{C}(-,b) \to \mathsf{C}(-,c) \to \mathbb{E}(-,a) \to \mathbb{E}(-,b) \to \mathbb{E}(-,c)$ and its dual.

Remark 1·8. Let (A, B) be a co-t-structure in T with coheart S and extended coheart $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$, considered with the extriangulated structure induced from T. Let $(\mathcal{X}, \mathcal{Y})$ be an arbitrary cotorsion pair in C. From the definition above one immediately sees that $\mathsf{S} \subseteq \mathcal{X}$ and $\Sigma\mathsf{S} \subseteq \mathcal{Y}$.

Definition 1·9 ([**24**, definition 4·1]) Let $(\mathsf{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. A *complete cotorsion pair* in C consists of a pair of full additive subcategories $(\mathcal{X}, \mathcal{Y})$ closed under direct summands and such that the following hold:

(i) for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we have $\mathbb{E}(x, y) = 0$;

(ii) for each $c \in \mathsf{C}$, there is an $\mathbb{E}$-triangle $c \to y \to x$ with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$;

(iii) for each $c \in \mathsf{C}$, there is an $\mathbb{E}$-triangle $y \to x \to c$ with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

A pair of full subcategories $(\mathcal{X}, \mathcal{Y})$ satisfying only condition (i) will be called an *Ext-orthogonal pair.*

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For each object c of C, the morphism $c \to y$ occurring in the $\mathbb{E}$-triangle above is always a left $\mathcal{Y}$-approximation of c. Similarly, the morphism $x \to c$ in the $\mathbb{E}$-triangle above is a right $\mathcal{X}$-approximation of c.

Remark 1·10. In this paper we revert to the classical distinction between complete cotorsion pairs and cotorsion pairs in [**14**, **30**]. Therefore what is called a cotorsion pair in [**24**] will be called a complete cotorsion pair here. By [**24**, remark 4·4], any complete cotorsion pair is a cotorsion pair, since the 0 element of $\mathbb{E}(c, a)$ is represented, up to equivalence, by a split $\mathbb{E}$-triangle $a \to a \oplus c \to c$.

1·4. *The restricted Yoneda functor*

Assume now that T is essentially small, idempotent complete, Hom-finite, $\mathbf{k}$-linear and Krull-Schmidt, where $\mathbf{k}$ is a commutative noetherian ring. In this situation, Hom-finiteness means that $\mathsf{T}(a, b)$ is a finitely-generated $\mathbf{k}$-module for any $a, b \in \mathsf{T}$. In particular, the endomorphism ring of an object $\mathsf{T}(s, s)$ is a noetherian ring. Suppose S is a presilting subcategory of T and let $\mathsf{C} := \mathsf{S} * \Sigma\mathsf{S}$. We write $\mathsf{Mod}\ \mathsf{S}$ for the category of contravariant additive functors from S to the category $\mathsf{Mod}\ \mathbf{k}$ and $\mathsf{mod}\ \mathsf{S}$ for the full subcategory of finitely presented functors; see [**6**]. Consider the restricted Yoneda functor

$$\begin{aligned} F\colon \mathsf{T} &\longrightarrow \mathsf{Mod}\ \mathsf{S} \\ t &\longmapsto \mathsf{T}(-, t)|_{\mathsf{S}}. \end{aligned}$$

By [**17**, proposition 6·2], [**16**, remark 3·1] the restricted Yoneda functor induces an equivalence of categories,

$$F\colon (\mathsf{S} * \Sigma\mathsf{S})/\Sigma\mathsf{S} \longrightarrow \mathsf{mod}\,\mathsf{S},$$

where $(\mathsf{S} * \Sigma\mathsf{S})/\Sigma\mathsf{S}$ denotes the subfactor category in which morphisms factoring through an object of $\Sigma\mathsf{S}$ are sent to zero. We will usually use this equivalence in case $\mathsf{mod}\,\mathsf{S}$ is an abelian category or, equivalently, when S has weak kernels [**12**]. If $\mathsf{S} = \mathsf{add}(s)$ for some object $s \in \mathsf{T}$ there is an equivalence $\mathsf{mod}\,\mathsf{S} \simeq \mathsf{mod}\,E$, where $E = \mathsf{T}(s, s)$; see [**16**, remark 4·1], and $\mathsf{mod}\,\mathsf{S}$ is automatically abelian.

1·5. *Torsion pairs*

A *torsion pair* in an abelian category H consists of a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of H such that $\mathcal{T}^{\perp} = \mathcal{F}$, ${}^{\perp}\mathcal{F} = \mathcal{T}$, and for each object h of H, there is a short exact sequence

$$0 \longrightarrow t \longrightarrow h \longrightarrow f \longrightarrow 0 \tag{1·1}$$

with $t \in \mathcal{T}$ and $f \in \mathcal{F}$. The subcategory $\mathcal{T}$ is called the *torsion class* and the subcategory $\mathcal{F}$ is called the *torsionfree class*.

By virtue of the short exact sequence (1·1), it follows that $\mathcal{T}$ is contravariantly finite in H and $\mathcal{F}$ is covariantly finite in H. If, in addition, $\mathcal{T}$ is covariantly finite in H, we say that $\mathcal{T}$ is a *functorially finite torsion class*. Note that, if $\mathsf{H} \simeq \mathsf{mod}\,A$ for an artin algebra A and $(\mathcal{T}, \mathcal{F})$ is a torsion pair in H, then $\mathcal{T}$ is covariantly finite in H if and only if $\mathcal{F}$ is contravariantly finite in H [**31**, theorem]. Torsion pairs $(\mathcal{T}, \mathcal{F})$ with functorially finite torsion class $\mathcal{T}$ and functorially finite torsionfree class $\mathcal{F}$ are called *functorially finite*; see e.g. [**1**].

If H is noetherian, for example $\mathsf{H} \simeq \mathsf{mod}\, E$ for a noetherian ring E, then any subcategory closed under extensions and quotients is a torsion class of a torsion pair; see, e.g. [**5**, chapter VI] or [**22**, proposition 3·5]. When H is artinian, the dual statement holds for torsionfree classes.

2. *HRS tilting of co-t-structures at complete cotorsion pairs*

In this section T will be an arbitrary triangulated category. The aim of this section is to prove Theorem A. We will need the following technical lemma.

LEMMA 2·1. *Suppose* T *is a triangulated category,* (A, B) *is a co-t-structure in* T*, and* $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$ *is the extended coheart of* (A, B)*. If* $(\mathcal{X}, \mathcal{Y})$ *is a cotorsion pair in* C *then:*

(i) $\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X} = \mathsf{A} * \Sigma^{-1}\mathcal{X}$*, and*

(ii) $\mathcal{Y} * \Sigma^2\mathsf{B} = \mathcal{Y} * \Sigma\mathsf{B}$.

Proof. We show the first equality holds; the second equality is analogous. The inclusion $\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X} \subseteq \mathsf{A} * \Sigma^{-1}\mathcal{X}$ is immediate because $\Sigma^{-1}\mathsf{A} \subseteq \mathsf{A}$. For the other inclusion, consider a decomposition of $t \in \mathsf{A} * \Sigma^{-1}\mathcal{X}$,

$$a \longrightarrow t \longrightarrow \Sigma^{-1}x \longrightarrow \Sigma a$$

with $a \in \mathsf{A}$ and $x \in \mathcal{X}$. Decompose a with respect to the co-t-structure $(\Sigma^{-1}\mathsf{A}, \Sigma^{-1}\mathsf{B})$ to get a triangle $\Sigma^{-1}a' \to a \to \Sigma^{-1}s \to a'$ with $s \in \mathsf{S} = \Sigma\mathsf{A} \cap \mathsf{B}$. Since $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair, by Remark 1·8, we have $\mathsf{S} \subseteq \mathcal{X}$. Applying the octahedral axiom to the two triangles gives

$$\begin{tikzcd}
 & \Sigma^{-1}a' \arrow[r, equal] \arrow[d] & \Sigma^{-1}a' \arrow[d] & \\
\Sigma^{-2}x \arrow[r] \arrow[d, equal] & a \arrow[r] \arrow[d] & t \arrow[r] \arrow[d] & \Sigma^{-1}x \arrow[d, equal] \\
\Sigma^{-2}x \arrow[r] & \Sigma^{-1}s \arrow[r] \arrow[d] & \Sigma^{-1}x' \arrow[r] \arrow[d] & \Sigma^{-1}x \\
 & a' \arrow[r, equal] & a' &
\end{tikzcd}$$

in which $x' \in \mathcal{X}$, giving a decomposition of $t \in \Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}$. Hence $\mathsf{A} * \Sigma^{-1}\mathcal{X} = \Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}$.

THEOREM 2·2. *Suppose* T *is a triangulated category,* (A, B) *is a co-t-structure in* T*, and* $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$ *is the extended coheart of* (A, B)*. Then there is a bijection*

$$\{\textit{co-t-structures } (\mathsf{A}', \mathsf{B}') \textit{ with } \mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}\} \stackrel{1-1}{\longleftrightarrow} \{\textit{complete cotorsion pairs } (\mathcal{X}, \mathcal{Y}) \textit{ in } \mathsf{C}\}.$$
$$(\mathsf{A}', \mathsf{B}') \longmapsto (\mathsf{B} \cap \Sigma\mathsf{A}', \mathsf{B}' \cap \Sigma^2\mathsf{A})$$
$$(\mathsf{add}(\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}), \mathsf{add}(\mathcal{Y} * \Sigma^2\mathsf{B})) \longleftarrow\!\shortmid (\mathcal{X}, \mathcal{Y}).$$

Remark 2·3. Let (A, B) be a co-t-structure in T. A co-t-structure $(\mathsf{A}', \mathsf{B}')$ such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$ (or, equivalently, $\mathsf{B} \supseteq \mathsf{B}' \supseteq \Sigma\mathsf{B}$) is often said to be *intermediate with respect to* (A, B), cf. [**4**] or [**16**]; in the former case 'intermediate' means with respect to the 'standard co-t-structure' and the interval may be larger.

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Proof. The proof of this theorem consists of three steps: first we construct the map

$$\varphi\colon \{\text{co-t-structures } (\mathsf{A}', \mathsf{B}') \text{ with } \mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}\} \to \{\text{complete cotorsion pairs } (\mathcal{X}, \mathcal{Y}) \text{ in } \mathsf{C}\};$$

then we construct the map ψ in the opposite direction; then we prove that $\varphi\psi = \mathrm{id}$ and $\psi\varphi = \mathrm{id}$.

Step 1: Let $(\mathsf{A}', \mathsf{B}')$ be a co-t-structure in T such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$ (equivalently $\Sigma\mathsf{B} \subseteq \mathsf{B}' \subseteq \mathsf{B}$) and consider the following subcategories of the extended coheart C:

$$\mathcal{X} := \mathsf{B} \cap \Sigma\mathsf{A}' \subseteq \mathsf{C} \quad \text{and} \quad \mathcal{Y} := \mathsf{B}' \cap \Sigma^2\mathsf{A} \subseteq \mathsf{C}.$$

Note that $\mathcal{X}$ and $\mathcal{Y}$ are closed under summands, since so are $\mathsf{A}, \mathsf{A}', \mathsf{B}$ and B'. We claim that $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair in C. Since $\mathcal{X} \subseteq \Sigma\mathsf{A}'$ and $\Sigma\mathcal{Y} \subseteq \Sigma\mathsf{B}'$, we have $\mathbb{E}(\mathcal{X}, \mathcal{Y}) = \mathsf{T}(\mathcal{X}, \Sigma\mathcal{Y}) = 0$ and condition (i) of Definition 1·9 holds.

To find the $\mathbb{E}$-triangle required for condition (ii), consider the following triangles for $c \in \mathsf{C}$ given by taking approximation triangles coming from the co-t-structures $(\Sigma\mathsf{A}, \Sigma\mathsf{B})$ and $(\mathsf{A}', \mathsf{B}')$, respectively,

$$b \longrightarrow \Sigma a \longrightarrow c \longrightarrow \Sigma b \quad \text{and} \quad a' \longrightarrow \Sigma a \longrightarrow b' \longrightarrow \Sigma a', \tag{2·1}$$

where $a \in \mathsf{A}$, $b \in \mathsf{B}$, $a' \in \mathsf{A}'$ and $b' \in \mathsf{B}'$. Applying the octahedral axiom, we get:

$$\begin{tikzcd}
 & b \arrow[r, equal] \arrow[d] & b \arrow[d] & \\
a' \arrow[r] \arrow[d, equal] & \Sigma a \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & \Sigma a' \arrow[d, equal] \\
a' \arrow[r] & c \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & \Sigma a' \\
 & \Sigma b \arrow[r, equal] & \Sigma b . &
\end{tikzcd}$$

We first observe that $y \in \mathcal{Y}$. In the triangle $b' \to y \to \Sigma b \to \Sigma b'$ the outer terms $b' \in \mathsf{B}'$ and $\Sigma b \in \Sigma\mathsf{B} \subseteq \mathsf{B}'$, so $y \in \mathsf{B}'$. In the triangle $c \to y \to \Sigma a' \to \Sigma c$ the outer terms $c \in \Sigma^2\mathsf{A}$ and $\Sigma a' \in \Sigma\mathsf{A}' \subseteq \Sigma^2\mathsf{A}$, so $y \in \Sigma^2\mathsf{A}$ and thus $y \in \mathcal{Y}$.

Since $\Sigma b \in \mathsf{B}$ and $\Sigma c \in \mathsf{B}$, we get that $\Sigma^2 a \in \mathsf{B}$. In the triangle $b' \to \Sigma a' \to \Sigma^2 a \to \Sigma b'$ the outer terms $b' \in \mathsf{B}' \subseteq \mathsf{B}$ and $\Sigma^2 a \in \mathsf{B}$, so $\Sigma a' \in \Sigma\mathsf{A}' \cap \mathsf{B} = \mathcal{X}$. Thus, the triangle

$$c \longrightarrow y \longrightarrow \Sigma a' \longrightarrow \Sigma c$$

gives the $\mathbb{E}$-triangle required for condition (ii) of Definition 1·9.

To find the $\mathbb{E}$-triangle required for condition (iii), consider $c \in \mathsf{C}$ as above and a triangle

$$a'' \longrightarrow b \longrightarrow b'' \longrightarrow \Sigma a'',$$

where b is the object in the triangle in (2·1), $a'' \in \mathsf{A}'$ and $b'' \in \mathsf{B}'$. Observe that $\Sigma^{-1}c \in \Sigma\mathsf{A}$ and $\Sigma a \in \Sigma\mathsf{A}$, so $b \in \Sigma\mathsf{A} \subseteq \Sigma^2\mathsf{A}$. Since $\Sigma a'' \in \Sigma\mathsf{A}' \subseteq \Sigma^2\mathsf{A}$, we get $b'' \in \Sigma^2\mathsf{A}$. Applying the octahedral axiom again, we get:

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$$\begin{tikzcd}
 & a'' \arrow[r, equal] \arrow[d] & a'' \arrow[d] & \\
\Sigma^{-1}c \arrow[r] \arrow[d, equal] & b \arrow[r] \arrow[d] & \Sigma a \arrow[r] \arrow[d] & c \arrow[d, equal] \\
\Sigma^{-1}c \arrow[r] & b'' \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & c \\
 & \Sigma a'' \arrow[r, equal] & \Sigma a''. &
\end{tikzcd}$$

Clearly $x \in \mathcal{X}$ and $b'' \in \mathcal{Y}$, so the triangle

$$b'' \longrightarrow x \longrightarrow c \longrightarrow \Sigma b''$$

gives the $\mathbb{E}$-triangle required for condition (iii) of Definition 1·9. Thus the assignment $\varphi\colon (\mathsf{A}', \mathsf{B}') \mapsto (\mathcal{X}, \mathcal{Y})$ defines a map from co-t-structures $(\mathsf{A}', \mathsf{B}')$ such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$ to complete cotorsion pairs in C.

Step 2: We now construct the map in the other direction. Let (A, B) be a co-t-structure in T and let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair in the extended coheart $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$. Consider the following pair of subcategories of T:

$$(\mathsf{A}', \mathsf{B}') := (\mathsf{add}(\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}), \mathsf{add}(\mathcal{Y} * \Sigma^2\mathsf{B})).$$

Note that $\mathsf{T}(\Sigma^{-1}\mathcal{X}, \Sigma^2\mathsf{B}) = 0$ since $\Sigma^{-1}\mathcal{X} \subseteq \Sigma\mathsf{A} \subseteq \Sigma^2\mathsf{A}$ and $\mathsf{T}(\Sigma^2\mathsf{A}, \Sigma^2\mathsf{B}) = 0$ because (A, B) is a co-t-structure. Hence, $\mathsf{T}(\mathsf{A}', \mathsf{B}') = 0$, verifying the orthogonality condition of the definition of a co-t-structure.

We claim that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$, from which it follows that $\Sigma^{-1}\mathsf{A}' \subseteq \mathsf{A}'$. By Lemma 2·1(i), we have $\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X} = \mathsf{A} * \Sigma^{-1}\mathcal{X}$, and therefore $\mathsf{A} \subseteq \mathsf{A} * \Sigma^{-1}\mathcal{X} \subseteq \mathsf{A}'$. As $\mathcal{X} \subseteq \Sigma^2\mathsf{A}$, again using Lemma 2·1(i), we have $\mathsf{A}' = \mathsf{add}(\mathsf{A} * \Sigma^{-1}\mathcal{X}) \subseteq \mathsf{add}(\mathsf{A} * \Sigma\mathsf{A}) \subseteq \Sigma\mathsf{A}$ since $\mathsf{A} \subseteq \Sigma\mathsf{A}$ and $\Sigma\mathsf{A}$ is closed under extensions and summands. Similarly, using Lemma 2·1(ii), one can check that $\Sigma\mathsf{B} \subseteq \mathsf{B}' \subseteq \mathsf{B}$, from which it follows that $\Sigma\mathsf{B}' \subseteq \mathsf{B}'$.

It remains for us to construct the approximation triangle from the definition of the co-t-structure. Consider the following triangles for $t \in \mathsf{T}$:

$$a_t \longrightarrow t \longrightarrow b_t \longrightarrow \Sigma a_t \quad \text{and} \quad \Sigma b \longrightarrow \Sigma^2 a \longrightarrow b_t \longrightarrow \Sigma^2 b,$$

where $a_t, a \in \mathsf{A}$ and $b, b_t \in \mathsf{B}$. Since $\Sigma^2 a \in \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{C}$, there is a triangle

$$\Sigma^{-1}x \longrightarrow \Sigma^2 a \longrightarrow y \longrightarrow x$$

with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ coming from the $\mathbb{E}$-triangle occurring in condition (ii) of the definition of complete cotorsion pair. Applying the octahedral axiom twice, we get:

$$\begin{tikzcd}
 & \Sigma b \arrow[r, equal] \arrow[d] & \Sigma b \arrow[d] & \\
\Sigma^{-1}x \arrow[r] \arrow[d, equal] & \Sigma^2 a \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & x \arrow[d, equal] \\
\Sigma^{-1}x \arrow[r] & b_t \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & x \\
 & \Sigma^2 b \arrow[r, equal] & \Sigma^2 b &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Sigma^{-1}x \arrow[r, equal] \arrow[d] & \Sigma^{-1}x \arrow[d] & \\
t \arrow[r] \arrow[d, equal] & b_t \arrow[r] \arrow[d] & \Sigma a_t \arrow[r] \arrow[d] & \Sigma t \arrow[d, equal] \\
t \arrow[r] & b' \arrow[r] \arrow[d] & \Sigma a' \arrow[r] \arrow[d] & \Sigma t \\
 & x \arrow[r, equal] & x, &
\end{tikzcd}$$

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where, in the left-hand diagram, we see $b' \in \mathcal{Y} * \Sigma^2\mathsf{B} \subseteq \mathsf{B}'$, and in the right-hand diagram, we have $a' \in \mathsf{A} * \Sigma^{-1}\mathcal{X} \subseteq \mathsf{A}'$. Thus the triangle

$$a' \longrightarrow t \longrightarrow b' \longrightarrow \Sigma a'$$

is an approximation triangle for t with respect to the co-t-structure $(\mathsf{A}', \mathsf{B}')$. Thus the assignment $\psi\colon (\mathcal{X}, \mathcal{Y}) \mapsto (\mathsf{A}' = \mathsf{add}(\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}), \mathsf{B}' = \mathsf{add}(\mathcal{Y} * \Sigma^2\mathsf{B}))$ defines a map from complete cotorsion pairs in C to co-t-structures $(\mathsf{A}', \mathsf{B}')$ such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$.

Step 3: We now show that the maps φ and ψ defined in Steps 1 and 2 are mutually inverse. Let (A, B) be a co-t-structure in T, let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair in the extended coheart $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B}$ and let $(\mathsf{A}', \mathsf{B}') = \psi\big((\mathcal{X}, \mathcal{Y})\big)$ be the co-t-structure constructed in Step 2. Let $(\mathcal{X}', \mathcal{Y}') = \varphi\big((\mathsf{A}', \mathsf{B}')\big)$ be the complete cotorsion pair constructed from $(\mathsf{A}', \mathsf{B}')$ in Step 1. That is,

$$\mathcal{X}' := \mathsf{B} \cap \Sigma\mathsf{A}' \subseteq \mathsf{C} \quad \text{and} \quad \mathcal{Y}' := \mathsf{B}' \cap \Sigma^2\mathsf{A} \subseteq \mathsf{C}.$$

Since $\Sigma\mathcal{Y}' \subseteq \mathsf{add}(\Sigma\mathcal{Y} * \Sigma^3\mathsf{B})$ and $\mathcal{X} \subseteq \Sigma^2\mathsf{A}$, we get that $\mathsf{T}(\mathcal{X}, \Sigma\mathcal{Y}') = 0$. For an object $y' \in \mathcal{Y}'$, we can consider the triangle $y' \to y \to x \to \Sigma y'$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, coming from the complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$. Since the map $x \to \Sigma y'$ is zero, the triangle splits and y' is a summand of y. Since $\mathcal{Y}$ is closed under summands, we get $\mathcal{Y}' \subseteq \mathcal{Y}$. Similarly, $\mathsf{T}(\mathcal{X}', \Sigma\mathcal{Y}) = 0$, and so the splitting of the triangle $y \to y' \to x' \to \Sigma y$, where $x' \in \mathcal{X}'$ and $y' \in \mathcal{Y}'$, coming from the complete cotorsion pair $(\mathcal{X}', \mathcal{Y}')$ gives that $\mathcal{Y} \subseteq \mathcal{Y}'$. Since $\mathcal{X} = \mathsf{C} \cap ({}^{\perp}\Sigma\mathcal{Y}) = \mathsf{C} \cap ({}^{\perp}\Sigma\mathcal{Y}') = \mathcal{X}'$, the cotorsion pairs coincide and $\varphi\psi = \mathrm{id}$.

Let $(\mathsf{A}', \mathsf{B}')$ be a co-t-structure such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$ (equivalently, $\Sigma\mathsf{B} \subseteq \mathsf{B}' \subseteq \mathsf{B}$). Consider the co-t-structure

$$(\mathsf{A}'', \mathsf{B}'') := \psi\varphi\big((\mathsf{A}', \mathsf{B}')\big) = (\mathsf{add}(\Sigma^{-1}\mathsf{A} * (\Sigma^{-1}\mathsf{B} \cap \mathsf{A}')), \mathsf{add}((\mathsf{B}' \cap \Sigma^2 A) * \Sigma^2\mathsf{B})).$$

Clearly, $\mathsf{A}'' \subseteq \mathsf{A}'$ and $\mathsf{B}'' \subseteq \mathsf{B}'$ and, since both pairs of subcategories are co-t-structures, we get $(\mathsf{A}', \mathsf{B}') = (\mathsf{A}'', \mathsf{B}'')$ and $\psi\varphi = \mathrm{id}$. Thus we get the desired bijection.

Remark 2·4. Let (A, B) be a co-t-structure in T with coheart S and extended coheart $\mathsf{C} = \Sigma^2\mathsf{A} \cap \mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$. Suppose $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair in C.

(i) In the definition of the intermediate co-t-structure $(\mathsf{A}', \mathsf{B}')$ obtained from $(\mathcal{X}, \mathcal{Y})$ in Theorem 2·2 it is not obvious that $\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}$ and $\mathcal{Y} * \Sigma^2\mathsf{B}$ are closed under summands, hence we are required to take the additive closure. However, one can check that $\mathsf{T}(\mathsf{A}, \mathcal{X}) = 0$ and $\mathsf{T}(\mathcal{Y}, \Sigma^2\mathsf{B}) = 0$, so that in the case that T is Krull-Schmidt, applying [**17**, proposition 2·1], we see that $\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X} = \mathsf{A} * \Sigma^{-1}\mathcal{X}$ and $\mathcal{Y} * \Sigma^2\mathsf{B}$ are closed under direct summands.

(ii) We have $\Sigma^{-1}\mathcal{X} * \mathcal{Y} = \Sigma^{-1}\mathsf{S} * \mathsf{S} * \Sigma\mathsf{S}$. The inclusion $\Sigma^{-1}\mathcal{X} * \mathcal{Y} \subseteq \Sigma^{-1}\mathsf{S} * \mathsf{S} * \Sigma\mathsf{S}$ is clear since $\mathcal{X}, \mathcal{Y} \subseteq \mathsf{S} * \Sigma\mathsf{S}$. The inclusion $\Sigma^{-1}\mathcal{X} * \mathcal{Y} \supseteq \Sigma^{-1}\mathsf{S} * \mathsf{S} * \Sigma\mathsf{S}$ follows from the inclusions $\mathsf{S} * \Sigma\mathsf{S} \subseteq \Sigma^{-1}\mathcal{X} * \mathcal{Y}$ and $\Sigma^{-1}\mathsf{S} * \mathsf{S} \subseteq \Sigma^{-1}\mathcal{X} * \mathcal{Y}$ coming from the two $\mathbb{E}$-triangles in Definition 1·9, and the fact that $\Sigma^{-1}\mathcal{X} * \mathcal{Y}$ is extension closed by [**25**, lemma 8] since $\mathsf{T}(\Sigma^{-1}\mathcal{X}, \Sigma\mathcal{Y}) = 0$.

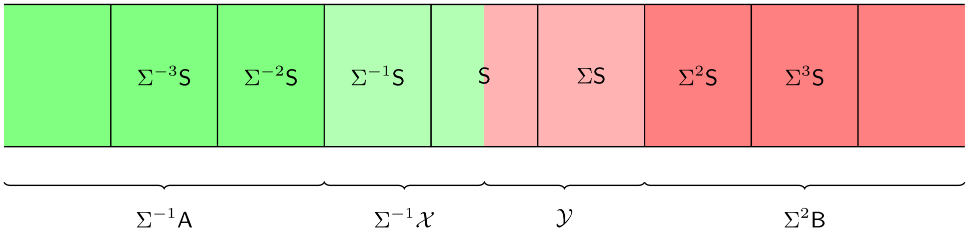

Fig. 1. Schematic showing the construction of the intermediate co-t-structure $(\mathsf{A}', \mathsf{B}')$ in Theorem 2·2 from a complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in the extended coheart $\mathsf{C} = \mathsf{S} * \Sigma\mathsf{S}$ of the co-t-structure (A, B).

(iii) In light of Lemma 2·1, there are two descriptions of A' and B'. Namely,

$$\mathsf{A}' = \mathsf{add}(\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}) = \mathsf{add}(\mathsf{A} * \Sigma^{-1}\mathcal{X})$$
$$\mathsf{B}' = \mathsf{add}(\mathcal{Y} * \Sigma^{2}\mathsf{B}) = \mathsf{add}(\mathcal{Y} * \Sigma\mathsf{B}).$$

The description we have chosen to present in the statement of Theorem 2·2 has two advantages: it is the closest parallel to classic HRS tilting for t-structures using torsion pairs in that it is written in terms of non-overlapping classes, and it makes the equality $\Sigma^{-1}\mathcal{X} * \mathcal{Y} = \Sigma^{-1}\mathsf{S} * \mathsf{S} * \Sigma\mathsf{S}$ in (ii) above intuitive; see Figure 1 for a schematic.

The corollary below shows that Theorem 2·2 recovers the bijection between co-t-structures intermediate with respect to (A, B) and silting subcategories $\mathsf{S}' \subseteq \mathsf{S} * \Sigma\mathsf{S}$ in [**16**, theorem 2·3].

COROLLARY 2·5. *Suppose* (A, B) *is a co-t-structure in* T*, and* $\mathsf{C} = \Sigma^{2}\mathsf{A} \cap \mathsf{B}$ *is the extended coheart of* (A, B)*. If* $(\mathcal{X}, \mathcal{Y})$ *is a complete cotorsion pair in* C*, then its core* $\mathcal{W} = \mathcal{X} \cap \mathcal{Y}$ *is the coheart of the corresponding intermediate co-t-structure* $(\mathsf{A}', \mathsf{B}') = (\mathsf{add}(\Sigma^{-1}\mathsf{A} * \Sigma^{-1}\mathcal{X}), \mathsf{add}(\mathcal{Y} * \Sigma^{2}\mathsf{B}))$.

Proof. Let $\mathsf{S}' = \Sigma\mathsf{A}' \cap \mathsf{B}'$ be the coheart of the co-t-structure $(\mathsf{A}', \mathsf{B}')$. By Theorem 2·2, $\mathcal{X} = \mathsf{B} \cap \Sigma\mathsf{A}'$ and $\mathcal{Y} = \mathsf{B}' \cap \Sigma^{2}\mathsf{A}$. Hence $\mathcal{W} = \mathcal{X} \cap \mathcal{Y} = \mathsf{S}' \cap \mathsf{C}$. But since $\mathsf{S}' = \Sigma\mathsf{A}' \cap \mathsf{B}' \subseteq \Sigma^{2}\mathsf{A} \cap \mathsf{B} = \mathsf{C}$, we have $\mathcal{W} = \mathsf{S}'$.

We finish this section with a straightforward example illustrating Theorem 2·2.

Example 2·6. Let $\mathbf{k}$ be a field and let A_3 be the equi-oriented Dynkin diagram of type A_3. In the diagram below we show the indecomposable objects in the Auslander–Reiten quiver of $\mathsf{T} = \mathsf{D}^{b}(\mathbf{k}A_3)$; note that we suppress the arrows in the AR quiver. The diagram depicts a co-t-structure (A, B), its extended coheart $\mathsf{C} = \Sigma^{2}\mathsf{A} \cap \mathsf{B}$, an intermediate co-t-structure $(\mathsf{A}', \mathsf{B}')$ such that $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$, and a complete cotorsion pair $(\mathcal{X}, \mathcal{Y}) = (\mathsf{B} \cap \Sigma\mathsf{A}', \mathsf{B}' \cap \Sigma^{2}\mathsf{A})$. We highlight these objects in the diagram as follows:

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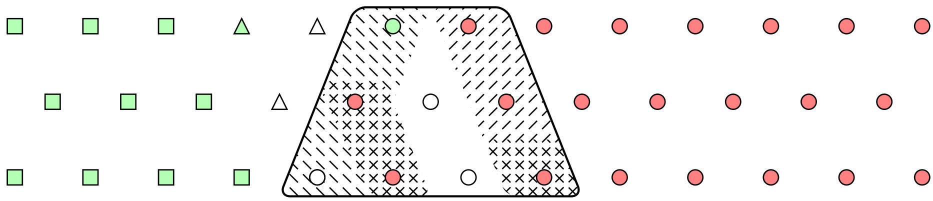
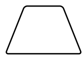


symbols	*indecomposable objects of*
□	*the aisle* A *of the co-t-structure* (A,B)
○	*the co-aisle* B *of the co-t-structure* (A,B)
△	*neither the aisle* A *nor the co-aisle* B *of the co-t-structure* (A,B)
□, △ *or* ○	*the aisle* A' *of the co-t-structure* $(\mathsf{A}',\mathsf{B}')$
○	*the co-aisle* B' *of the co-t-structure* $(\mathsf{A}',\mathsf{B}')$
	*the cotorsion class* $\mathcal{X}$ *of the complete cotorsion pair* $(\mathcal{X},\mathcal{Y})$
	*the cotorsionfree class* $\mathcal{Y}$ *of the complete cotorsion pair* $(\mathcal{X},\mathcal{Y})$

The extended coheart C *is indicated by the outlined region:* 

3. *Cotorsion pairs versus torsion pairs*

The aim of this section is to provide a direct proof of Theorem B. We restrict to the following setup so that we can apply the setup of Section 1·4.

Setup 3·1. From now on we will assume that T is essentially small, Hom-finite, $\mathbf{k}$-linear and Krull–Schmidt, where $\mathbf{k}$ is a commutative noetherian ring. Note that in that case T is automatically idempotent complete. Let S be a presilting subcategory of T such that $\mathsf{mod}\,\mathsf{S}$ is abelian and noetherian and set $\mathsf{C}=\mathsf{S}*\Sigma\mathsf{S}$. Note that, since S is silting in $\mathsf{thick}\,\mathsf{S}$, the subcategory $\mathsf{S}*\Sigma\mathsf{S}$ is closed under summands and extensions.

Under the assumptions of Setup 3·1, if $\mathsf{S}=\mathsf{add}(s)$, then $\mathsf{mod}\,\mathsf{S}\simeq\mathsf{mod}\,E$, where $E=\mathsf{T}(s,s)$ is a noetherian ring, making $\mathsf{mod}\,\mathsf{S}$ abelian and noetherian.

PROPOSITION 3·2. *Suppose that the hypotheses of Setup 3·1 hold. Then the equivalence* $F\colon \mathsf{C}/\Sigma\mathsf{S}\to\mathsf{mod}\,\mathsf{S}$ *induces a well-defined map*

$$\Phi\colon \{\textit{cotorsion pairs in } \mathsf{C}\}\longrightarrow\{\textit{torsion pairs in } \mathsf{mod}\,\mathsf{S}\},$$
$$(\mathcal{X},\mathcal{Y})\longmapsto(\mathcal{T}=F\mathcal{Y},\mathcal{F}=\mathcal{T}^{\perp})$$

which restricts to a well-defined map

$$\Phi\colon\{\textit{complete cotorsion pairs in } \mathsf{C}\}\longrightarrow\left\{\begin{array}{c}\textit{torsion pairs in } \mathsf{mod}\,\mathsf{S} \textit{ whose torsion class is}\\ \textit{functorially finite}\end{array}\right\}.$$

Proof. Let $(\mathcal{X},\mathcal{Y})$ be a cotorsion pair in C. We claim that the essential image $\mathcal{T}=F\mathcal{Y}$ is a torsion class in $\mathsf{mod}\,\mathsf{S}$. Since $\mathsf{mod}\,\mathsf{S}$ is noetherian, by [22, proposition 3·5] it is enough to show that $\mathcal{T}$ is closed under quotients and extensions.

We start by showing that $\mathcal{T}=F\mathcal{Y}$ is closed under quotients. Consider an exact sequence $t\xrightarrow{\varphi}u\to 0$ in $\mathsf{mod}\,\mathsf{S}$ with $t\in\mathcal{T}$. Lifting this to C via F, there are objects $y\in\mathcal{Y}$ and $v\in\mathsf{C}$ and a morphism $f\colon y\to v$ such that $Fy\simeq t$, $Fv\simeq u$ and $Ff\simeq\varphi$. Completing the morphism f

to a distinguished triangle in T gives

$$c \longrightarrow y \xrightarrow{f} v \xrightarrow{g} \Sigma c.$$

Applying F to this triangle, we get the exact sequence

$$\mathsf{T}(-,y)|_{\mathsf{S}} \xrightarrow{\mathsf{T}(-,f)|_{\mathsf{S}}} \mathsf{T}(-,v)|_{\mathsf{S}} \xrightarrow{\mathsf{T}(-,g)|_{\mathsf{S}}} \mathsf{T}(-,\Sigma c)|_{\mathsf{S}} \longrightarrow \mathsf{T}(-,\Sigma y)|_{\mathsf{S}}.$$

Since $\mathsf{T}(-,f)|_{\mathsf{S}} \simeq \varphi$ is an epimorphism, we have $\mathsf{T}(-,g)|_{\mathsf{S}} = 0$. Moreover, $\Sigma y \in \Sigma\mathsf{S} * \Sigma^2\mathsf{S}$ so that S presilting implies that $\mathsf{T}(-,\Sigma y)|_{\mathsf{S}} = 0$. Hence, $\mathsf{T}(-,\Sigma c)|_{\mathsf{S}} = 0$. In particular, it follows that $\Sigma c \in (\mathsf{S} * \Sigma\mathsf{S} * \Sigma^2\mathsf{S}) \cap \mathsf{S}^{\perp}$, in which case we get that $c \in \mathsf{S} * \Sigma\mathsf{S}$. Now applying $\mathsf{T}(\mathcal{X},-)$ to the triangle above gives $\mathsf{T}(\mathcal{X},\Sigma v) = 0$, which means $v \in \mathcal{Y}$ because $(\mathcal{X},\mathcal{Y})$ is a cotorsion pair. Hence, $u \simeq Fv \in \mathcal{T}$ and $\mathcal{T}$ is closed under quotients.

Next we show that $\mathcal{T}$ is closed under extensions. Consider a short exact sequence

$$0 \longrightarrow t' \xrightarrow{\varphi} t \longrightarrow t'' \longrightarrow 0$$

in $\operatorname{mod}\mathsf{S}$ with $t', t'' \in \mathcal{T}$. Lift the morphism $\varphi\colon t' \to t$ to C to obtain a morphism $f\colon y' \to y$ such that $y' \in \mathcal{Y}$, $Ff \simeq \varphi$, $Fy' \simeq t'$ and $Fy \simeq t$. Extend f to a distinguished triangle to get

$$y' \xrightarrow{f} y \longrightarrow c \longrightarrow \Sigma y'.$$

Applying the restricted Yoneda functor to this triangle and noting that $\mathsf{T}(-,\Sigma y')|_{\mathsf{S}} = 0$ gives a commutative diagram,

$$\begin{tikzcd}
Fy' \arrow[r, "Ff"] \arrow[d, "\sim"] & Fy \arrow[r] \arrow[d, "\sim"] & Fc \arrow[r] & 0 \\
t' \arrow[r, "\varphi"'] & t \arrow[r] & t'' \arrow[r] & 0,
\end{tikzcd}$$

whence $Fc \simeq t'' \simeq Fy''$ for some $y'' \in \mathcal{Y}$.

If $c \in \mathcal{Y} \subseteq \mathsf{C}$, then we are done. However, we do not know this to be the case. Reading off from the triangle above, $c \in \mathsf{S} * \Sigma\mathsf{S} * \Sigma^2\mathsf{S}$, so we can consider a decomposition $m \to c \to n \to \Sigma m$ in which $m \in \mathsf{S} * \Sigma\mathsf{S}$ and $n \in \Sigma^2\mathsf{S}$. Consider the octahedral diagram obtained from the two triangles below.

$$\begin{tikzcd}
& m \arrow[r, equal] \arrow[d] & m \arrow[d] & \\
y \arrow[r] \arrow[d, equal] & c \arrow[r] \arrow[d] & \Sigma y' \arrow[r] \arrow[d] & \Sigma y \arrow[d, equal] \\
y \arrow[r] & n \arrow[r] \arrow[d] & \Sigma e \arrow[r] \arrow[d] & \Sigma y \\
& \Sigma m \arrow[r, equal] & \Sigma m, &
\end{tikzcd}$$

Applying the restricted Yoneda functor to a rotation of the lower horizontal triangle gives an isomorphism $Fe \xrightarrow{\sim} Fy$. Rotating the right-hand vertical triangle gives

$$y' \to e \to m \to \Sigma y',$$

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showing that $e \in \mathsf{C}$. Consider the octahedral diagram obtained using this triangle together with the triangle $e \to y \to n \to \Sigma e$.

$$\begin{tikzcd}
 & \Sigma^{-1}n \arrow[r, equal] \arrow[d] & \Sigma^{-1}n \arrow[d] & \\
y' \arrow[r] \arrow[d, equal] & e \arrow[r] \arrow[d] & m \arrow[r] \arrow[d] & \Sigma y' \arrow[d, equal] \\
y' \arrow[r] & y \arrow[r] \arrow[d] & c' \arrow[r] \arrow[d] & \Sigma y' \\
 & n \arrow[r, equal] & n, &
\end{tikzcd}$$

Note that from the previous diagram c' is isomorphic to the cone of the map $y' \xrightarrow{f} y$, so $c' \simeq c$ and $Fc' \simeq Fc$. Applying the restricted Yoneda functor to the two horizontal triangles gives,

$$\begin{tikzcd}
Fy' \arrow[r] \arrow[d, equal] & Fe \arrow[r] \arrow[d, "\sim"] & Fm \arrow[r] \arrow[d, "\sim"] & 0 \\
Fy' \arrow[r] & Fy \arrow[r] & Fc' \arrow[r] & 0.
\end{tikzcd}$$

In particular, there is an isomorphism $Fy'' \xrightarrow{\sim} Fm$, so by the argument showing $\mathcal{T}$ is closed under quotients, we see that m lies in $\mathcal{Y}$. Since $\mathcal{Y}$ is closed under extensions, it follows that $e \in \mathcal{Y}$. Hence $Fe \simeq Fy \simeq t$, showing that $t \in \mathcal{T}$, as required.

Finally, we check that the map induced by the restricted Yoneda functor restricts as claimed. We need to show that $\mathcal{T} = F\mathcal{Y}$ is covariantly finite when $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair. Let $m \in \operatorname{mod} \mathsf{S}$. Suppose $c \in \mathsf{C}$ is such that $Fc \simeq m$. Without loss of generality we can assume $Fc = m$. Consider a decomposition triangle of c with respect to the complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$,

$$c \xrightarrow{f} y \longrightarrow x \longrightarrow \Sigma c.$$

Note that $f\colon c \to y$ is a left $\mathcal{Y}$-approximation of c in C. We claim that $Ff\colon m \to Fy$ is a left $\mathcal{T}$-approximation of m in $\operatorname{mod} \mathsf{S}$. Suppose $\varphi\colon m \to t$ is a morphism in $\operatorname{mod} \mathsf{S}$ with $t \in \mathcal{T}$. Then there exist $y' \in \mathcal{Y}$ together with an isomorphism $e\colon Fy' \xrightarrow{\sim} t$ and $g\colon c \to y'$ such that $\varphi = eFg$. Since f is a left $\mathcal{Y}$-approximation of c in C, there exists $h\colon y \to y'$ such that $g = hf$. Applying F to this composition gives $\varphi = eFg = eFhFf$, that is, Ff is a left $\mathcal{T}$-approximation, as required.

Remark 3·3. If we relax the assumption of $\operatorname{mod} \mathsf{S}$ being noetherian and require simply that $\operatorname{mod} \mathsf{S}$ is abelian, then each instance of the term 'torsion class' appearing in Proposition 3·2 can be replaced by 'nullity class' in the sense of [**22**].

The next lemma provides a useful criterion to detect when a cotorsion pair is complete.

LEMMA 3·4. *Let* S *be a presilting subcategory of a triangulated category* T *and* $\mathsf{C} = \mathsf{S} * \Sigma \mathsf{S}$. *Suppose* $(\mathcal{X}, \mathcal{Y})$ *is a cotorsion pair in* C, *then* $(\mathcal{X}, \mathcal{Y})$ *is a complete cotorsion pair in* C *if and only if for each object* s *of* S *there exists a triangle* $s \to y \to x \to \Sigma s$ *with* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$.

Proof. If $(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair in C then by definition such a triangle exists for $s\in\mathsf{S}$ because such a triangle exists for each $c\in\mathsf{C}$.

Conversely, suppose $(\mathcal{X},\mathcal{Y})$ is a cotorsion pair in C and that for each $s\in\mathsf{S}$, there is a triangle $s\to y\to x\to\Sigma s$ with $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. Let $c\in\mathsf{C}$ and take a decomposition triangle, $s_2\to s_1\to c\to\Sigma s_2$, and the triangle $s_2\to y_2\to x_2\to\Sigma s_2$ given by the assumption. Applying the octahedral axiom to these triangles gives the following diagram.

$$\begin{tikzcd}
 & \Sigma^{-1}x_2 \arrow[r, equal] \arrow[d] & \Sigma^{-1}x_2 \arrow[d] & \\
\Sigma^{-1}c \arrow[r] \arrow[d, equal] & s_2 \arrow[r] \arrow[d] & s_1 \arrow[r] \arrow[d] & c \arrow[d, equal] \\
\Sigma^{-1}c \arrow[r] & y_2 \arrow[r] \arrow[d] & e \arrow[r] \arrow[d] & c \\
 & x_2 \arrow[r, equal] & x_2. &
\end{tikzcd}$$

We have $e\in\mathcal{X}$ since $\mathsf{S}\subseteq\mathcal{X}$, because $(\mathcal{X},\mathcal{Y})$ is a cotorsion pair. Thus, the triangle $y_2\to e\to c\to\Sigma y_2$ provides the second triangle required for completeness in Definition 1·9.

To obtain the first triangle in Definition 1·9, we use the octahedral axiom again together with the triangle $s_1\to y_1\to x_1\to\Sigma s_1$ given by the assumption:

$$\begin{tikzcd}
 & \Sigma^{-1}x_1 \arrow[r, equal] \arrow[d] & \Sigma^{-1}x_1 \arrow[d] & \\
s_2 \arrow[r] \arrow[d, equal] & s_1 \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \Sigma s_2 \arrow[d, equal] \\
s_2 \arrow[r] & y_1 \arrow[r] \arrow[d] & d \arrow[r] \arrow[d] & \Sigma s_2 \\
 & x_1 \arrow[r, equal] & x_1. &
\end{tikzcd}$$

Analogously, we have $d\in\mathcal{Y}$ because $\Sigma\mathsf{S}\subseteq\mathcal{Y}$, making $c\to d\to x_1\to\Sigma c$ the required triangle.

Remark 3·5. We make two observations regarding Lemma 3·4.

(i) Lemma 3·4, in fact, holds in the case that $(\mathcal{X},\mathcal{Y})$ is an Ext-orthogonal pair of subcategories of C closed under extensions and direct summands and such that $\mathsf{S}\subseteq\mathcal{X}$ and $\Sigma\mathsf{S}\subseteq\mathcal{Y}$.

(ii) Let S, C and $(\mathcal{X},\mathcal{Y})$ be as in the statement of Lemma 3·4. In the triangle $s\to y\to x\to\Sigma s$ we observe that since $\mathsf{S}\subseteq\mathcal{X}$ and $\Sigma\mathsf{S}\subseteq\mathcal{Y}$, we have $x,y\in\mathcal{X}\cap\mathcal{Y}$. In the context of Section 2 this provides a decomposition of S in $\Sigma^{-1}\mathsf{S}'*\mathsf{S}'$, where $\mathsf{S}'=\mathcal{X}\cap\mathcal{Y}$ is the coheart of the co-t-structure $(\mathsf{A}',\mathsf{B}')$.

We now define an inverse to the restricted map in Proposition 3·2. For a subcategory $\mathcal{T}$ of $\operatorname{mod}\mathsf{S}$, denote by $F^{-1}(\mathcal{T})$ the full subcategory of C whose objects are $\{c\in\mathsf{C}\mid Fc\in\mathcal{T}\}$.

PROPOSITION 3·6. *Let* S *be a presilting subcategory of* T *and* $\mathsf{C}=\mathsf{S}*\Sigma\mathsf{S}$. *There is a well-defined map*

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$$\Theta\colon \left\{ \begin{array}{c} \textit{torsion pairs in } \mathsf{mod}\,\mathsf{S} \textit{ whose torsion class is} \\ \textit{functorially finite} \end{array} \right\} \longrightarrow \{\textit{complete cotorsion pairs in } \mathsf{C}\},$$
$$(\mathcal{T},\mathcal{F}) \longmapsto (\mathcal{X} = {}^{\perp}(\Sigma\mathcal{Y}) \cap \mathsf{C}, \mathcal{Y} = F^{-1}(\mathcal{T})).$$

Proof. Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathsf{mod}\,\mathsf{S}$ whose torsion class is functorially finite, $\mathcal{Y} = \{c \in \mathsf{C} \mid Fc \in \mathcal{T}\}$ and $\mathcal{X} = {}^{\perp}(\Sigma\mathcal{Y}) \cap \mathsf{C}$. The subcategories $\mathcal{Y}$ and $\mathcal{X}$ are closed under direct summands, since $\mathcal{T}$ is closed under direct summands and $\mathcal{X}$ is defined as an orthogonal subcategory. We set $\Theta\big((\mathcal{T},\mathcal{F})\big) = (\mathcal{X},\mathcal{Y})$. We claim that $(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair. We start by showing that $\mathcal{Y}$ is closed under extensions (in C).

Let $y' \xrightarrow{f} y \longrightarrow y'' \longrightarrow \Sigma y'$ be an extension with $y', y'' \in \mathcal{Y}$. Applying F to this $\mathbb{E}$-triangle in C gives an exact sequence $Fy' \xrightarrow{Ff} Fy \longrightarrow Fy'' \longrightarrow 0$ in $\mathsf{mod}\,\mathsf{S}$ since $\Sigma y' \in \Sigma\mathsf{C} = \Sigma\mathsf{S} * \Sigma^2\mathsf{S}$. We thus obtain a short exact sequence $0 \to \operatorname{im} Ff \to Fy \to Fy'' \to 0$. Since $\mathcal{T}$ is a torsion class, it follows that $\operatorname{im} Ff \in \mathcal{T}$ and $Fy \in \mathcal{T}$. Hence, we conclude that $y \in \mathcal{Y}$.

For $s \in \mathsf{S}$, we will construct a triangle $s \to y \to x \to \Sigma s$ in which $y \in \mathcal{Y}$ and $x \in \mathcal{X}$, which will allow us to apply Lemma 3·4 to conclude that $(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair in C. First, we need to check that Lemma 3·4 applies. Since $\mathcal{Y} \subseteq \mathsf{C}$, we have that $\mathsf{T}(\mathsf{S}, \Sigma\mathcal{Y}) = 0$ so that $\mathsf{S} \subseteq \mathcal{X}$. Furthermore, if $s \in \mathsf{S}$ then $F(\Sigma s) = 0 \in \mathcal{T}$ so that $\Sigma s \in \mathcal{Y}$. By Remark 3·5(i), Lemma 3·4 applies.

Now let $s \in \mathsf{S}$. Consider a left $\mathcal{T}$-approximation $\varphi\colon Fs \to t$ of Fs in $\mathsf{mod}\,\mathsf{S}$. Replacing t by an isomorphic object we can assume without loss of generality that $t = Fy$ for some $y \in \mathcal{Y}$. Let $f\colon s \to y$ be such that $Ff = \varphi$. We claim that $f\colon s \to y$ is a left $\mathcal{Y}$-approximation of s. Consider a morphism $g\colon s \to y'$ with $y' \in \mathcal{Y}$. Applying F to f and g gives a diagram,

$$\begin{tikzcd} Fs \arrow[r, "Ff=\varphi"] \arrow[dr, "Fg=\theta"'] & Fy = t, \arrow[d, dashed, "\psi"] \\ & Fy', \end{tikzcd}$$

where the morphism $\psi\colon t \to Fy'$ exists because φ is a left $\mathcal{T}$-approximation. Since the functor F is full, there exists $h\colon y \to y'$ such that $Fh = \psi$. We claim that $g = hf$.

Applying F to $g - hf$ shows that $g - hf = 0$ in $\mathsf{C}/\Sigma\mathsf{S}$. Hence, $g - hf$ factors through $\Sigma\mathsf{S}$.

$$\begin{tikzcd} s \arrow[rr, "g-hf"] \arrow[dr] & & y' \\ & \Sigma s' \arrow[ur] & \end{tikzcd}$$

Hence, $g - hf = 0$ in C since S is presilting. It follows that $f\colon s \to y$ is a left $\mathcal{Y}$-approximation of S. As T is Krull–Schmidt, without loss of generality, we may assume that $f\colon s \to y$ is a minimal left $\mathcal{Y}$-approximation and extend it to a distinguished triangle, $s \xrightarrow{f} y \longrightarrow x \longrightarrow \Sigma s$. By the Wakamatsu lemma for triangulated categories, Lemma 1·1, we see that $x \in {}^{\perp}(\Sigma\mathcal{Y})$. Since y and Σs are in C, we get that $x \in \mathsf{C}$, and so $x \in \mathcal{X}$.

THEOREM 3·7. *Suppose the hypotheses of Setup 3·1 hold. Then, there is a bijection*

$$\{\textit{complete cotorsion pairs in } \mathsf{C}\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{torsion pairs in } \mathsf{mod}\,\mathsf{S} \textit{ whose torsion class is} \\ \textit{functorially finite} \end{array} \right\}.$$

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Moreover, if in addition $\mathsf{mod}\,\mathsf{S} \simeq \mathsf{mod}\,A$ *for an artin algebra A, there is a bijection between complete cotorsion pairs in* C *and functorially finite torsion pairs in* $\mathsf{mod}\,\mathsf{S}$.

Proof. We show that the maps Φ and Θ defined in Propositions 3·2 and 3·6 are mutually inverse. Let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair in C. By Proposition 3·2 we have $\Phi\big((\mathcal{X}, \mathcal{Y})\big) = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = F\mathcal{Y}$ and $\mathcal{F} = \mathcal{T}^{\perp}$ in $\mathsf{mod}\,\mathsf{S}$. Applying Θ to $(\mathcal{T}, \mathcal{F})$ produces a complete cotorsion pair $(\mathcal{X}', \mathcal{Y}')$ in which $\mathcal{Y}' = \{c \in \mathsf{C} \mid Fc \in \mathcal{T}\}$. Clearly, $\mathcal{Y} \subseteq \mathcal{Y}'$. To see that $\mathcal{Y}' \subseteq \mathcal{Y}$ take $y' \in \mathcal{Y}'$ and observe that there is an isomorphism $\varphi\colon Fy \to Fy'$ in $\mathsf{mod}\,\mathsf{S}$ for some $y \in \mathcal{Y}$ by the definition of $\mathcal{T}$. Now, applying the same argument used to show that $F\mathcal{Y}$ is closed under quotients in the proof of Proposition 3·2 shows that $\mathsf{T}(\mathcal{X}, \Sigma y') = 0$, whence by completeness of the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ and the fact that any complete cotorsion pair is a cotorsion pair, we get $y' \in \mathcal{Y}$. As a cotorsionfree class determines a cotorsion pair uniquely, we get $(\mathcal{X}', \mathcal{Y}') = (\mathcal{X}, \mathcal{Y})$. The equality $\Phi\Theta = 1$ follows from $F(F^{-1}(\mathcal{T})) = \mathcal{T}$. The last statement follows from [**31**, theorem].

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