THE BOOLEAN ALGEBRA OF REGULAR OPEN SETS

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1. Introduction. Let S be a completely regular topological space. Let $C(S)$ denote the set of bounded, real-valued, continuous functions on S. It is well known that $C(S)$ forms a distributive lattice under the ordinary pointwise joins and meets. For any distributive lattice L and any ideal $I \subseteq L$, a quasi-ordering of L can be defined as follows: $f \supseteq g$ if, for all $h \in L, f \cap h \in I$ implies $g \cap h \in I$. If equivalent elements under this quasi-ordering are identified, a homomorphic image of L is obtained. In this paper, the particular case where L is $C(S)$ and I is any principal ideal will be studied. It will be shown that the homomorphic image obtained in this way is isomorphic to a sub-lattice of $\mathfrak{B}(S)$, the Boolean algebra[1] of regular open sets of S. This homomorphic image will be denoted by $\mathfrak{L}(S)$; it does not depend on the choice of the principal ideal I. For the case where S is a normal space, a topological characterization of $\mathfrak{L}(S)$ will be obtained. Finally it will be proved that $\mathfrak{B}(S)$ 'is isomorphic to the normal completion (see [**1**]) of $\mathfrak{L}(S)$. Incidentally these results prove, without using transfinite methods, that for any completely regular space S, $\mathfrak{B}(S)$ is determined to within isomorphism by $C(S)$, a fact which could also be inferred from the theorem of Kaplansky [**3**] that a compact Hausdorff space S is determined to within homeomorphism by $C(S)$.

2. A congruence relation. In everything that follows, S will denote a completely regular topological space. Denote by $\leqslant$, $\cap$, $\cup$ the ordinary pointwise inequality, meet and join operations in $C(S)$.

Definition 2.1. $f \supseteq g$ if, for all $h \in C(S), f \cap h \leqslant 0$ implies[2] $g \cap h \leqslant 0$. Write $f \sim g$ if $f \supseteq g$ and $g \supseteq f$.

LEMMA 2.1. *On $C(S)$, $\supseteq$ is a quasi-ordering and $\sim$ is a congruence relation.*

Proof. If $f \supseteq g$ and $g \supseteq h$, then $f \supseteq h$ is an immediate consequence of the definition. Also $f \supseteq f$ is clear. Thus $\supseteq$ is a quasi-ordering and consequently $\sim$ is an equivalence relation (Birkhoff [**1**]).

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[1]An open subset of a topological space is called regular open if it coincides with the interior of its closure. It is well known (see Birkhoff [1, p. 177]) that the regular open sets form a complete Boolean algebra.

[2]This definition puts the zero function in a distinguished position—a position which the lattice structure of $C(S)$ does not support. It is easy to see however that if 0 is replaced by an arbitrary (fixed) $f_0 \in C(S)$, all the results of the paper go through with only slight modification.

Let $f \supseteq g$ and f_1 be any continuous function; then $f \cap f_1 \supseteq g \cap f_1$ and $f \cup f_1 \supseteq g \cup f_1$. For suppose $(f \cap f_1) \cap h \leqslant 0$. Then $f \cap (f_1 \cap h) \leqslant 0$, so since $f \supseteq g$, $g \cap (f_1 \cap h) \leqslant 0$. Thus $(g \cap f_1) \cap h \leqslant 0$ and, since h was arbitrarily chosen, $f \cap f_1 \supseteq g \cap f_1$. Next suppose $(f \cup f_1) \cap h \leqslant 0$. This means $f_1 \cap h \leqslant 0$, $f \cap h \leqslant 0$ and, because $f \supseteq g$, $g \cap h \leqslant 0$. Hence $(g \cup f_1) \cap h = (g \cap h) \cup (f_1 \cap h) \leqslant 0$ and again by definition $f \cup f_1 \supseteq g \cup f_1$.

It follows that if $f \sim g$, then $f \cap f_1 \sim g \cap f_1$ and $f \cup f_1 \sim g \cup f_1$ for any $f_1 \in C(S)$. Thus $\sim$ is a congruence relation and the proof of the lemma is complete.

LEMMA 2.2. *$f \supseteq g$ if and only if*[3] $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$.

Proof. First suppose $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$. Let $h \in C(S)$ be such that $f \cap h \leqslant 0$. It will be shown that this implies $g \cap h \leqslant 0$, so (by definition) $f \supseteq g$.

If $x_0 \in \{x \mid g(x) > 0\}^-$, then by hypothesis, $x_0 \in \{x \mid f(x) > 0\}^-$. This means that if N is a neighbourhood of x_0, there is an $x \in N$ with $f(x) > 0$. Since $f \cap h \leqslant 0$, it follows that $h(x) \leqslant 0$. Thus $x_0 \in \{x \mid h(x) \leqslant 0\}^- = \{x \mid h(x) \leqslant 0\}$ and $h(x_0) \leqslant 0$. Consequently $(g \cap h)(x_0) \leqslant h(x_0) \leqslant 0$. If $x_0 \notin \{x \mid g(x) > 0\}^-$, then $g(x_0) \leqslant 0$ and $(g \cap h)(x_0) \leqslant g(x_0) \leqslant 0$. Hence for all points x_0 of S, $(g \cap h)(x_0) \leqslant 0$; that is, $g \cap h \leqslant 0$.

Now suppose $f \supseteq g$. Let $M = \{x \mid f(x) > 0\}^{-c} \cap \{x \mid g(x) > 0\}$. Clearly, M is open. Also if $y \in M$, then $f(y) \leqslant 0$ and $g(y) > 0$. To show that $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$, it suffices to prove that M is empty, since then $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}$, and taking closures on both sides gives the desired result.

Suppose M is not empty. Let $x_0 \in M$. By complete regularity, $h \in C(S)$ exists so that $h(x_0) = 1$ and $h(y) = 0$ for all $y \notin M$. Then $(g \cap h)(x_0) = \min\{1, g(x_0)\} > 0$, so $g \cap h$ non $\leqslant 0$. On the other hand, $(f \cap h)(y) \leqslant f(y) \leqslant 0$ if $y \in M$, while $(f \cap h)(y) \leqslant h(y) = 0$ if $y \notin M$. Thus $f \cap h \leqslant 0$. This contradicts $f \supseteq g$ and shows that M is empty.

Denote by $\mathfrak{L}(S)$ the set of equivalence classes under $\sim$. Then by Lemma 2.1, $\mathfrak{L}(S)$ is partially ordered under $\supseteq$ and, with this ordering, it is a distributive lattice. If $f \in C(S)$, let A_f denote the congruence class containing f.

LEMMA 2.3. *$\mathfrak{L}(S)$ is isomorphic to the sub-lattice of the Boolean algebra of regular open sets of S which consists of all sets of the form $I(\{x \mid f(x) > 0\}^-)$ where $f \in C(S)$.*

Proof. The mapping $A_f \rightarrow I(\{x \mid f(x) > 0\}^-)$ is one-to-one by Lemma 2.2. Also[4]

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[3]For any set $N \subseteq S$, N^- will denote the closure of N, N^c the complement of N in S, while the interior of N, N^{c-c}, will be represented by $I(N)$.

[4]The lattice operations in the Boolean algebra of regular open sets are given by $P \wedge Q = I([P \cap Q]^-) = P \cap Q$, $P \vee Q = I(P^- \cup Q^-)$ and $P' = P^{-c}$.

$$\begin{aligned} A_{f\cap g} \to I(\{x \mid (f \cap g)(x) > 0\}^-) &= I([\{x \mid f(x) > 0\} \cap \{x \mid g(x) > 0\}]^-) \\ &= I(\{x \mid f(x) > 0\}^-) \wedge I(\{x \mid g(x) > 0\}^-), \\ A_{f\cup g} \to I(\{x \mid (f \cup g)(x) > 0\}^-) &= I(\{x \mid f(x) > 0\}^- \cup \{x \mid g(x) > 0\}^-) \\ &= I(\{x \mid f(x) > 0\}^-) \vee I(\{x \mid g(x) > 0\}^-). \end{aligned}$$

The following lemma gives information on the image of $\mathfrak{L}(S)$ which will be needed later.

LEMMA 2.4. *Let P be a regular open set of S.*

(a) *If $P \neq 0$ (the null set), $f \in C(S)$ exists so that* $0 \neq I(\{x \mid f(x) > 0\}^-) \subseteq P$.

(b) *If $P \neq S$ (the unit set), $f \in C(S)$ exists so that* $S \neq I(\{x \mid f(x) > 0\}^-) \supseteq P$.

Proof. (a) Let $x_0 \in P$. By complete regularity, choose $f \in C(S)$ so that $f(x_0) = 1$ and $f(y) = 0$ if $y \notin P$. Then $x_0 \in \{x \mid f(x) > 0\} \subseteq P$ so $x_0 \in I(\{x \mid f(x) > 0\}^-) \subseteq I(P^-) = P$.

(b) $P^- \neq S$ since otherwise $P = I(P^-) = I(S) = S$. Let $x_0 \in P^{-c}$. Choose $f \in C(S)$ so that $f(x_0) = -1$ and $f(y) = 1$ for all $y \in P^-$. Then

$$P \subseteq \{x \mid f(x) > 0\} \subseteq I(\{x \mid f(x) > 0\}^-) \subseteq \{x \mid f(x) \geqslant 0\} \neq S.$$

It may happen that the image of $\mathfrak{L}(S)$ under the mapping of Lemma 2.3 coincides with the whole Boolean algebra of regular open sets. For this to be the case, it is clearly necessary and sufficient that every regular open set P of S be of the form $P = I(\{x \mid f(x) > 0\}^-)$ for some $f \in C(S)$. This condition prevails in two cases of special importance. The first is that in which S is a metric space. For then, if P is a regular open set and $f \in C(S)$ is defined by $f(x) = \rho(x, P^c)$, ρ being the metric of S, $P = \{x \mid f(x) > 0\} = I(\{x \mid f(x) > 0\}^-)$ and the criterion is fulfilled. The second important case is that of an extremal space, that is, a space in which the closure of every open set is open. In such a space, the regular open sets are evidently just the open and closed sets. Hence the characteristic function ϕ_P of a regular open set P is continuous and $P = \{x \mid \phi_P(x) > 0\} = I(\{x \mid \phi_P(x) > 0\}^-)$. An important consequence is:

THEOREM 2.1. *The Boolean algebra of regular open sets of a completely regular topological space S is isomorphic to the complete Boolean algebra obtained by reducing the normal completion of $C(S)$ modulo the congruence of Definition* 2.1.

Proof. By the theorem of Dilworth [**2**], the normal completion of $C(S)$ is isomorphic to $C(\mathfrak{S})$ where $\mathfrak{S}$ is the Boolean space associated with $\mathfrak{B}(S)$, the Boolean algebra of regular open sets of S. Then (see Stone [**4**; **5**]) $\mathfrak{S}$ is extremal and its Boolean algebra of open and closed sets is isomorphic to $\mathfrak{B}(S)$. This means that S and $\mathfrak{S}$ have the same Boolean algebra of regular open sets. The theorem then follows from the above remark.

In spite of the above results, it is not true that, for all completely regular spaces S, the image of $\mathfrak{L}(S)$ is the whole Boolean algebra of regular open sets. This is shown by the following example.

Let S be the set W, consisting of all ordinal numbers $\leqslant \Omega$ (the least ordinal of the third class), together with the real interval $I = [0, 1]$, where Ω is identified with 0. Simply order S by defining $\alpha \leqslant x$ for all $\alpha \in W$ and $x \in I$, and by retaining the usual order in W and I. Impose the interval topology (see [**1**]) on S. It is easily seen that S is completely regular. Now the interval $P = \{a \in S \mid a < \Omega\}$ is a regular open set of S. Suppose $f \in C(S)$ exists with $P = I(\{a \mid f(a) > 0\}^-)$. Then $f(a) \leqslant 0$ holds for all $a \geqslant 0 = \Omega$. Let $\alpha_n = \sup \{a \mid f(a) \geqslant 1/n\}$. For all n, $\alpha_n < \Omega$ since $\{a \in S \mid f(a) < 1/n\}$ is open and contains I. Thus α_n is an ordinal of the second class. $\alpha = \sup \alpha_n$, as the limit of a sequence of ordinals of the second class, is itself an ordinal of the second class. Hence $\alpha < \Omega$. Because

$$\{a \mid f(a) > 0\} = \bigcup_n \{a \mid f(a) \geqslant 1/n\} \subseteq \bigcup_n \{a \mid a \leqslant \alpha_n\} \subseteq \{a \mid a \leqslant \alpha\},$$

it follows that $I(\{a \mid f(a) > 0\}^-) \subseteq \{a \mid a < \alpha\} \neq P$. This is a contradiction, showing that no such $f \in C(S)$ can exist.

The preceding example indicates the need for a topological characterization of $\mathfrak{L}(S)$. For the important case where S is a normal space, the following theorem answers this need.

THEOREM 2.2. *Let S be a normal topological space. In the mapping $M \to I(M^-)$ of the lattice of open sets onto the Boolean algebra $\mathfrak{B}(S)$, $\mathfrak{L}(S)$ is the image of the sub-lattice of all open sets which are the countable unions of closed sets (that is, open F_σ sets).*

Proof. $\mathfrak{L}(S)$ consists of all sets of the form $P = I(\{x \mid f(x) > 0\}^-)$ where $f \in C(S)$. But

$$\{x \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geqslant 1/n\}$$

is an open F_σ set. Thus every $P \in \mathfrak{L}(S)$ is the image of an open F_σ set.

On the other hand, suppose $P = I(M^-)$ where M is open and

$$M = \bigcup_{n=1}^{\infty} F_n$$

is the countable union of the closed sets F_n. By Urysohn's lemma, for each n, it is possible to select $f_n \in C(S)$ so that $f_n(x) = 1$ for all $x \in F_n$, $f_n(x) = 0$ for all $x \in M^c$ and $0 \leqslant f_n \leqslant 1$. Define

$$f = \sum_{n=1}^{\infty} (2^{-n}) f_n.$$

Then $f \in C(S)$ and $f(x) = 0$ if $x \in M^c$. If $x \in M$, then $x \in F_n$ for some n, whence

$$f(x) \geqslant 2^{-n} f_n(x) = 2^{-n} > 0.$$

Consequently $M = \{x \mid f(x) > 0\}$ and $P = I(M^-) = I(\{x \mid f(x) > 0\}^-) \in \mathfrak{L}(S)$ Thus the image of any open F_σ is in $\mathfrak{L}(S)$ and the proof is complete.

3. Completion. Having constructed the homomorphic image of $C(S)$, it will now be shown that the normal completion of $\mathfrak{L}(S)$ is isomorphic to the Boolean algebra of regular open sets.

Let it be recalled that the normal completion of a lattice $\mathfrak{L}$ consists of the normal subsets (or closed ideals) of $\mathfrak{L}$ ordered by inclusion, that is, those subsets which contain all lower bounds to the set of their upper bounds. The mapping $A \to \{B \in \mathfrak{L} \mid B \subseteq A\}$ imbeds $\mathfrak{L}$ as a sub-lattice of its normal completion (for proofs, see Birkhoff [**1**]).

LEMMA 3.1.[5] *Let $\mathfrak{L}$ be a sub-lattice of the complete Boolean algebra $\mathfrak{B}$. Suppose, moreover, that the following dual conditions are satisfied: If $X \in \mathfrak{B}$ and $X \neq Z$ (zero of $\mathfrak{B}$), then $A \in \mathfrak{L}$ exists with $Z \neq A \subseteq X$. If $X \in \mathfrak{B}$ and $X \neq I$ (unit of $\mathfrak{B}$), then $A \in \mathfrak{L}$ exists with $I \neq A \supseteq X$. Then the normal completion of $\mathfrak{L}$ is isomorphic to $\mathfrak{B}$.*

Proof. For $X \in \mathfrak{B}$, let $\mathfrak{N}_X = \{A \in \mathfrak{L} \mid A \subseteq X\}$. The first task is to show that $\mathfrak{N}_X$ is a normal subset of $\mathfrak{L}$.

First note that, if $A \supseteq B$ holds for some $A \in \mathfrak{L}$ and all $B \in \mathfrak{N}_X$, then $A \supseteq X$. For otherwise $A' \cap X \neq Z$ and, by hypothesis, $B \in \mathfrak{L}$ exists with $A' \cap X \supseteq B \neq Z$. This means A non $\supseteq B$ while $B \subseteq X$ which contradicts the assumption on A. Thus the upper bounds in $\mathfrak{L}$ of all $B \in \mathfrak{N}_X$ are precisely those A satisfying $A \supseteq X$. As a result, the preceding argument can be dualized to show that a lower bound, A, of the set of all upper bounds of $\mathfrak{N}_X$ satisfies $A \subseteq X$. Consequently $\mathfrak{N}_X$ is normal.

Next, every normal subset of $\mathfrak{L}$ is of the form $\mathfrak{N}_X$ for some $X \in \mathfrak{B}$. In fact, if $\mathfrak{M}$ is normal, let $X = \bigcup \mathfrak{M}$ (sup in $\mathfrak{B}$). Then if $A \in \mathfrak{M}$, $A \subseteq X$, so $\mathfrak{M} \subseteq \mathfrak{N}_X$. On the other hand, by the definition of X, if $C \supseteq B$ holds for all $B \in \mathfrak{M}$, $C \supseteq X$. Thus $A \in \mathfrak{N}_X$ (that is, $A \subseteq X$) implies $A \subseteq C$ for all upper bounds of $\mathfrak{M}$. Since $\mathfrak{M}$ is normal, this means $A \in \mathfrak{M}$, so $\mathfrak{N}_X \in \mathfrak{M}$.

The above argument shows $X \to \mathfrak{N}_X$ is a mapping from $\mathfrak{B}$ onto the normal completion of $\mathfrak{L}$. The mapping obviously preserves order: $X_1 \subseteq X_2$ implies $\mathfrak{N}_{X_1} \subseteq \mathfrak{N}_{X_2}$. To complete the proof, it is sufficient to show that this mapping is one-to-one.

Suppose X_1 non $\subseteq X_2$. Then $Z \neq X_1 \cap X_2'$, so $A \in \mathfrak{L}$ exists with $Z \neq A \subseteq X_1 \cap X_2'$. It follows that $A \in \mathfrak{N}_{X_1}$ and $A \notin \mathfrak{N}_{X_2}$, that is, $\mathfrak{N}_{X_1}$ non $\subseteq \mathfrak{N}_{X_2}$.

COROLLARY. *If S_1 and S_2 are two completely regular topological spaces such that $\mathfrak{L}(S_1)$ is isomorphic to $\mathfrak{L}(S_2)$, then S_1 and S_2 have isomorphic Boolean algebras of regular open sets.*

According to Lemma 2.4 the above principle is applicable to the image of $\mathfrak{L}(S)$ in the complete Boolean algebra of regular open sets. The result of this application is:

[5]This lemma, in essence, was presented by Professor Dilworth during his 1950–51 seminar at C.I.T.

THEOREM 3.1. *The Boolean algebra of regular open sets of a completely regular topological space S is isomorphic to the normal completion of the lattice obtained by reducing $C(S)$ modulo the congruence of Definition* 2.1.

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