

CANONICAL LIFTS OF FAMILIES OF ELLIPTIC CURVES

JAMES BORGER AND LANCE GURNEY

Abstract. We show that the canonical lift construction for ordinary elliptic curves over perfect fields of characteristic $p > 0$ extends uniquely to arbitrary families of ordinary elliptic curves, even over p -adic formal schemes. In particular, the universal ordinary elliptic curve has a canonical lift. The existence statement is largely a formal consequence of the universal property of Witt vectors applied to the moduli space of ordinary elliptic curves, at least with enough level structure. As an application, we show how this point of view allows for more formal proofs of recent results of Finotti and Erdoğan.

§1. Introduction

Fix a prime number p . Let W denote the usual, p -typical Witt vector functor. Let R be a ring in which p is nilpotent, and write $S = \text{Spec } R$. Let $W_n(S)$ denote $\text{Spec } W_n(R)$, and let $W(S)$ denote the direct limit $\text{colim}_n W_n(S)$. We take this limit in the category of sheaves of sets on the category of affine schemes with respect to the étale topology. One could say that $W(S)$ is the correct version of $\text{Spec } W(R)$, a construction which, as we discuss below, does not have good properties.

We say an elliptic curve E over S is ordinary when all fibers of E , necessarily over points of residue characteristic p , are ordinary. For any morphism $f: S' \rightarrow S$, we write $E_{S'}$, or $f^*(E)$, for the base change $S' \times_S E$ regarded as an elliptic curve over S' in the evident way.

The purpose of this paper is to prove the following:

THEOREM. *There is a unique way of lifting ordinary elliptic curves E over affine schemes S on which p is nilpotent to elliptic curves \tilde{E} over $W(S)$ which is compatible with base change in S and has the property that \tilde{E} admits a Frobenius lift $\psi: \tilde{E} \rightarrow F^*(\tilde{E})$, where F is the usual Witt vector Frobenius map $F: W(S) \rightarrow W(S)$.*

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Note that the requirement here that S is affine is only to simplify the exposition. We remove it below and allow S to be any p -adic formal scheme, or even what we call a p -adic sheaf. See Section 4 for the final statement of the theorem and further details.

We call \tilde{E} the canonical lift of E . In the case $S = \text{Spec } k$ where k is a perfect field of characteristic p , our canonical lift agrees with the usual one by the remarks in Section 7.1.

We emphasize that not only does this approach to canonical lifts work in much greater generality than previous ones, but it is also essentially formal. For remarks on theories of canonical lifts in completely general contexts, see the epilogue in Section 7.4.

§2. Background on sheaves

In this section and the next, we define $W(S)$, the infinite-length Witt vector construction when S is a scheme, and even when S is more general. The reason there is something to do is that while $\text{Spec } W_n(R)$ is a well-behaved construction, the naive infinite-length analogue $\text{Spec } W(R)$ is not. For instance, some basic geometric facts like the theorem in Section 3.2 are not true for the naive construction. Thus, to handle Witt vectors of infinite length geometrically, one needs some way of retaining the information of the projective system of the $W_n(R)$ instead of crudely passing to the limit $W(R) = \lim_n W_n(R)$ in the category of rings.

One standard way of doing this is to view $W(R)$ as a topological ring with the inverse-limit topology, each $W_n(R)$ being discrete. In the special case where R is a perfect \mathbb{F}_p -algebra, this topology is an adic topology with ideal of definition generated by p . We can therefore consider its formal spectrum $\text{Spf } W(R)$, and in this way, the theory of formal schemes can accommodate a satisfactory theory of Witt vector constructions of infinite length, as long as R is perfect. For general rings R , the inverse-limit topology on $W(R)$ is not an adic topology, and so the theory of formal schemes cannot even get started.

Another way of proceeding, also standard, is to view $\text{Spec } W_n(R)$ as a sheaf of sets on the category of affine schemes, and to define $W(\text{Spec } R)$ to be the direct limit $\text{colim}_n \text{Spec } W_n(R)$ in this category. Since all limit and colimit constructions are as well behaved in categories of sheaves as they are in the category of sets, there is good reason to be confident in this approach. Indeed it works without problem and is the one we take.

There is a second issue in defining $W(S)$, which is how general we should allow S to be. It will be convenient later if we can allow S to be the moduli space of ordinary elliptic curves (with sufficient level structure) over rings on which p is nilpotent. But this does not exist in the category of schemes. It does exist in the category of formal schemes, where one can construct it as the ordinary locus in the p -adic completion of the moduli space of all elliptic curves. However, it also exists in the category of sheaves mentioned above. Since we will be using this category anyway, it will be simpler to take that approach and forget about formal schemes entirely. It also has the side benefit of working for objects S much more general than p -adic formal schemes.

2.1 Objects representable over sheaves

Let \mathbf{Aff} denote the category of affine schemes, and let \mathbf{Aff}^\sim denote the category of sheaves of sets on \mathbf{Aff} with respect to the étale topology. (See SGA4 [2, exp. VII]) Any scheme S can be viewed as an object of \mathbf{Aff}^\sim via the functor it represents $\mathrm{Spec} R \mapsto \mathrm{Hom}(\mathrm{Spec} R, S)$. This is a fully faithful embedding and we regard the category of schemes as a full subcategory of \mathbf{Aff}^\sim in this way without further comment.

For any $S \in \mathbf{Aff}^\sim$, let \mathbf{Aff}_S^\sim denote the category of sheaves X equipped with a map $X \rightarrow S$, where the morphisms are morphisms over S . If $T \rightarrow S$ is a morphism, and $X \in \mathbf{Aff}_S^\sim$, let X_T denote the sheaf $T \times_S X$ together with the morphism $\mathrm{pr}_T: T \times_S X \rightarrow T$. We can then define familiar scheme-theoretic structures on objects X over S by using affine test schemes. For example, an elliptic curve over S is a sheaf X over S together with a compatible choice of an elliptic curve structure on X_T for every morphism $T \rightarrow S$ from an affine scheme T . (To be clear, an elliptic curve structure on X_T is by definition an isomorphism to X_T from the sheaf represented by a usual elliptic curve over T ; and the compatibility condition is that for any morphism $T' \rightarrow T$ of affine schemes the isomorphism $(X_T)_{T'} \xrightarrow{\sim} X_{T'}$ is a morphism of usual elliptic curves over T' .) Note that it also follows that the sheaf X over S admits a unique group structure (over S) inducing the group structures on the elliptic curves X_T over T .

Descent theory for \mathbf{Aff}^\sim goes through for general sheaf-theoretic reasons. Suppose $S' \rightarrow S$ is an epimorphism in \mathbf{Aff}^\sim ; in other words, every section of S lifts étale locally to one of S' . Then any object of $\mathbf{Aff}_{S'}^\sim$ equipped with descent data to S descends to a unique (up to unique isomorphism) object of \mathbf{Aff}^\sim over S . This remains true if we are interested in objects X with

additional structure, as long as that structure is of an étale-local nature on affine schemes. This is the case for ordinary elliptic curves, as they satisfy effective descent for the étale topology, and is the only example we need.

There is a generalization of this which we also use. Suppose we are given a presentation of an object $S \in \text{Aff}^\sim$ as a colimit: $\text{colim}_i S_i \xrightarrow{\sim} S$. Then the category of objects over $\text{colim}_i S_i$ is equivalent to the category of compatible families of objects over the S_i . Indeed, an object X over S gives rise to a compatible family of objects $X_i = S_i \times_S X$ over the S_i , where compatible means that we are given morphisms $X_i \rightarrow X_j$ lying over each morphism $S_i \rightarrow S_j$ such that the induced maps $X_i \rightarrow S_i \times_{S_j} X_j$ are isomorphisms. Conversely, to such a compatible family X_i we associate the object $X = \text{colim}_i X_i$. As above, the equivalence between objects over a colimit and compatible families also holds for objects with additional structure of an étale-local nature, such as ordinary elliptic curves. For general reasons, any sheaf S admits such a presentation where each S_i is an affine scheme, and so an ordinary elliptic curve (say) over S is equivalent to a compatible family of ordinary elliptic curves over this diagram of affine schemes. More importantly, one often considers objects S which are given as the colimit of some specific family of affine schemes, and then one can describe an elliptic curve over S as a compatible family of elliptic curves over this specific family. So an elliptic curve over an object given in these terms really is an accessible object.

2.2 p -adic sheaves

We say that a sheaf $S \in \text{Aff}^\sim$ is p -adic if it is isomorphic to a colimit $\text{colim}_i S_i$ of affine schemes S_i on which p is nilpotent. (We require nothing of the indexing diagram of the colimit other than that it is small. In other language, this means that it is a set and not a proper class.) For example, a scheme is p -adic if and only if p is locally nilpotent on it. In fact, any p -adic sheaf S has the property that p is nilpotent on any affine scheme admitting a morphism to S (and the converse is true up to issues of set-theoretic size just mentioned).

If T is a scheme, put

$$T_n = \text{Spec } \mathbb{Z}/p^{n+1}\mathbb{Z} \times_{\text{Spec } \mathbb{Z}} T.$$

Then we call $\widehat{T} = \text{colim}_n T_n$ the p -adic completion of T . It is a p -adic sheaf because each T_n is a scheme over $\mathbb{Z}/p^{n+1}\mathbb{Z}$ and hence a colimit of affine schemes over $\mathbb{Z}/p^{n+1}\mathbb{Z}$.

We can do the same with any p -adic formal scheme T , and this defines a fully faithful embedding of the category of p -adic formal schemes into the category of p -adic sheaves.

§3. Background on Witt vectors

3.1 Universal property of Witt vectors for rings

Let A be a p -torsion-free ring with an endomorphism $\psi: A \rightarrow A$ such that $\psi(a) \equiv a^p \pmod{pA}$. Let R be any ring such that $W(R)$ is p -torsion-free. (This does not hold in general, but it does hold at two opposite extremes—when R is a perfect \mathbb{F}_p -algebra and when R is p -torsion-free.) Then any ring map $g: A \rightarrow R$ lifts to a unique ring map \tilde{g}

$$\begin{array}{ccc}
 A & \overset{\tilde{g}}{\dashrightarrow} & W(R) \\
 & \searrow g & \swarrow \\
 & & R
 \end{array}$$

which is Frobenius equivariant, meaning that $F \circ \tilde{g} = \tilde{g} \circ \psi$. In other words, the image of $\tilde{g}(a)$ under the ghost map $W(R) \rightarrow R^{\mathbb{N}}$ is

$$(3.1.1) \quad \langle g(a), g(\psi(a)), g(\psi^2(a)), \dots \rangle.$$

A proof using the traditional definition of Witt vectors can be found in Lazard’s book [14, p. 215]. However, we remark that the traditional definition of Witt vectors will be irrelevant in this paper—it is only the universal property that matters. So a preferable alternative would be to take a definition of Witt vectors making the universal property obvious. For such a development, one can see Joyal [10] for a concise account or [3, Section 1] for a more extensive one.

Observe that $\tilde{g}: A \rightarrow W(R)$ agrees with the composition $W(g) \circ \widetilde{\text{id}}_A: A \rightarrow W(A) \rightarrow W(R)$. We can therefore define a canonical map \tilde{g} without any restrictions on R . Indeed, we simply define $\tilde{g} = W(g) \circ \widetilde{\text{id}}_A$. One might call \tilde{g} the canonical lift of g . We emphasize that while we have dropped all assumptions on R here, we are still assuming A is p -torsion-free. We also emphasize that while \tilde{g} is a canonically defined Frobenius equivariant lift of g , without any assumptions on R it might not be the unique one.

If we now write $Y = \text{Spec } A$, then for any ring R , we have functorial maps $Y(R) \rightarrow Y(W(R))$ given by $g \mapsto \tilde{g}$. In other words, if Y is a p -torsion-free

moduli space parametrizing objects of some given type, a Frobenius lift on Y defines a way of canonically lifting objects defined over R to objects over $W(R)$. We might say that a class of objects has a theory of canonical lifts whenever their moduli space is p -torsion-free and has a Frobenius lift. (For a little more along these lines see Section 7.4.) Indeed, this is the principle we apply below. But we need it in a slightly modified form because in our example, the Frobenius lift ψ exists only on the p -adic completion of the moduli space, which is not a scheme, but the p -adic completion of a scheme, in the sense of Section 2.2. The form of the universal property we need we be given in Section 3.3.

3.2 Witt vectors for schemes

Write $W_n: \text{Aff} \rightarrow \text{Aff}$ for the functor defined by $W_n(\text{Spec } R) = \text{Spec } W_n(R)$. The following theorem is fundamental:

THEOREM. *The functor $W_n: \text{Aff} \rightarrow \text{Aff}$ preserves étale maps, étale covering families, and étale base change. In particular, W_n is continuous in the étale topology.*

The first general argument was given in van der Kallen [17, (2.4)], but was written only for the ‘big’ Witt vector functor. For a proof for the p -typical Witt vector functor considered here, one can see [3, Section 9.2]. One can also see the earlier paper by Langer and Zink [13, Appendix A]; note that while the results there are stated only under some finite-type assumptions, the general case can be deduced by a limiting argument. However, for all our applications, it is enough to consider Witt vectors of rings in which p is nilpotent, and in this context the theorems in Langer–Zink do not have any finiteness assumptions and hence apply without modification.

This allows us to extend W_n to Aff^\sim , by SGA4 [1, exp. III]. Indeed, because W_n is continuous, for any sheaf S , the presheaf $U \mapsto S(W_n(U))$, for any $U \in \text{Aff}$, is a sheaf. This defines a functor $W_{n*}: \text{Aff}^\sim \rightarrow \text{Aff}^\sim$, and it has a left adjoint W_n^* . Finally, W_n^* extends W_n from Aff to Aff^\sim in the sense that we have canonical isomorphisms $W_n^*(\text{Spec } R) \xrightarrow{\sim} \text{Spec } W_n(R)$. So from now on, we often abusively write $W_n = W_n^*$.

Since $W_n: \text{Aff}^\sim \rightarrow \text{Aff}^\sim$ is a left adjoint, it preserves colimits. Therefore, any presentation $f: \text{colim}_i \text{Spec } R_i \xrightarrow{\sim} S$ gives rise to a presentation

$$W_n(f): \text{colim}_i \text{Spec } W_n(R_i) \xrightarrow{\sim} W_n(S).$$

Therefore, by the remarks in Section 2.1, an elliptic curve over $W_n(S)$ can be understood as a compatible system of elliptic curves over the $W_n(S_i)$, and similarly for any other kind of object having a local nature.

Observe that if S is a p -adic sheaf, then so is $W_n(S)$. Indeed, it is sufficient (and necessary) to observe that if p is nilpotent in a ring R , then it is nilpotent in $W_n(R)$. One way to show this is to observe that the comonad structure map $W_{m+n}(\mathbb{F}_p) \rightarrow W_n(W_m(\mathbb{F}_p))$ makes $W_n(\mathbb{Z}/p^{m+1}\mathbb{Z})$ into a $\mathbb{Z}/p^{m+n+1}\mathbb{Z}$ -algebra.

In fact, if S is a scheme on which p is locally nilpotent, then so is $W_n(S)$. We will not use this below and mention it only for the reader who is more comfortable with schemes than abstract sheaf theory. As a topological space, $W_n(S)$ agrees with S , and its structure sheaf $\mathcal{O}_{W_n(S)}$ is given by the presheaf $U \mapsto W_n(\mathcal{O}_S(U))$. One way to show this is by using the fact that for any ring R and any element $f \in R$, we have $W_n(R[1/f]) = W_n(R)[1/[f]]$, where $[f]$ denotes the Teichmüller lift of f . Alternatively, open immersions are the same as étale monomorphisms and hence are preserved by W_n , by the theorem above.

Then we define

$$W(S) = \operatorname{colim}_n W_n(S).$$

We emphasize that this colimit is taken in Aff^\sim . Thus W is the left adjoint of $\lim_n W_{n*}$. If $S = \operatorname{colim}_i S_i$, then we have $W(S) = \operatorname{colim}_{n,i} W_n(S_i)$. Therefore, if S is a p -adic sheaf, then so is $W(S)$. But $W(S)$ is typically not a scheme, even if S is. (For instance, if $S = \operatorname{Spec} \mathbb{F}_p$, then $W(S)$ is the colimit of $\operatorname{Spec} \mathbb{Z}/p^{n+1}\mathbb{Z}$, which represents the functor sending $\operatorname{Spec} R$ to the one-point set if p is nilpotent in R and to the empty set otherwise. This is representable by the formal scheme $\operatorname{Spf} \mathbb{Z}_p$ but not by a scheme.) It is, however, still easy to work with. For example, if T is an affine scheme, then the set $\operatorname{Hom}(T, W(S))$ is the filtered colimit $\operatorname{colim}_n \operatorname{Hom}(T, W_n(S))$. (By for example SGA4 [2, exp. VI, Theorem 1.23(ii) on p. 185]) So if S is a scheme, for example, then any map $T \rightarrow W(S)$ factors through the scheme $W_n(S)$ for some n .

Note that W_n and W preserve epimorphisms in Aff^\sim , as they are left adjoints. In particular, if $S' \rightarrow S$ is an étale cover of schemes, then $W(S') \rightarrow W(S)$ is an epimorphism, and hence descent is available.

3.3 The universal property of Witt vectors for p -adic sheaves

Let A be a p -torsion-free ring with a Frobenius lift ψ . Let $Y_m = \operatorname{Spec} A/p^{m+1}A$, $\hat{Y} = \operatorname{colim}_m Y_m$, as in Section 2.2, and $\hat{\psi}: \hat{Y} \rightarrow \hat{Y}$ be the

Frobenius lift induced by ψ . Let S be a p -adic sheaf and let $f: S \rightarrow \widehat{Y}$ be a morphism. Write $S = \text{colim}_i \text{Spec } R_i$, with p nilpotent in each ring R_i . Then for each i , there exists an m_i such that the map $\text{Spec } R_i \rightarrow \widehat{Y}$ factors through the inclusion $Y_{m_i} \rightarrow \widehat{Y}$, thus inducing a map $A/p^{m_i+1}A \rightarrow R_i$. Let g denote the composition $A \rightarrow A/p^{m_i+1}A \rightarrow R_i$, and let \tilde{g} denote the canonical lift as defined in 3.1. Then for each n , the composition

$$A \xrightarrow{\tilde{g}} W(R_i) \longrightarrow W_n(R_i)$$

factors through $A \rightarrow A/p^{N_{i,n}+1}A$ for some $N_{i,n}$ (in fact, for any $N_{i,n} \geq m_i + n$). This defines a compatible family of maps

$$\text{Spec } W_n(R_i) \longrightarrow \text{Spec } A/p^{N_{i,n}+1} = Y_{N_{i,n}} \longrightarrow \widehat{Y},$$

and hence, by the universal property of colimits, a map

$$\tilde{f}: W(S) = \text{colim}_{n,i} W_n(\text{Spec } R_i) \longrightarrow \widehat{Y},$$

which we again call the canonical lift of $f: S \rightarrow \widehat{Y}$. Note that as with the previously defined canonical lift maps, the map \tilde{f} is Frobenius equivariant, which is to say $\tilde{f} \circ \widehat{\psi} = F \circ \tilde{f}$.

§4. Statement of the theorem

We return to the notation of the introduction, where R is a ring in which p is nilpotent and $S = \text{Spec } R$. Then the usual Witt vector Frobenius map $F: W_{n+1}(R) \rightarrow W_n(R)$ induces functorial maps $F: W_n(S) \rightarrow W_{n+1}(S)$, for any $S \in \text{Aff}^\sim$, and upon taking colimits, maps $F: W(S) \rightarrow W(S)$. It also satisfies the relation $F(x) \equiv r(x)^p \pmod{pW_n(R)}$, where $r: W_{n+1}(R) \rightarrow W_n(R)$ denotes the usual projection. So the maps $F: W(S) \rightarrow W(S)$ agree with the usual p th power Frobenius map on the locus $\text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} W(S)$.

Let us then say that a *Frobenius lift* on an elliptic curve E over $W(S)$ is a morphism $E \rightarrow F^*(E)$ of elliptic curves over $W(S)$ restricting to the usual Frobenius map on $\text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} W(S)$.

THEOREM. There is a unique way of lifting ordinary elliptic curves E over p -adic sheaves S to elliptic curves \tilde{E} over $W(S)$ such that the construction $E \mapsto \tilde{E}$ is compatible with base change in S and such that each \tilde{E} admits a Frobenius lift.

We make some remarks to clarify the statement. First, the base-change condition, which can be written $(E_{S'})^\sim = \tilde{E}_{W(S')}$ for any map $S' \rightarrow S$, is more properly expressed as a coherent family of isomorphisms. Second, the uniqueness statement is to be understood as follows: if $E \mapsto \check{E}$ is any other such construction, then there is a unique family of isomorphisms $\tilde{E} \rightarrow \check{E}$, where E runs over all ordinary elliptic curves E over all p -adic base sheaves S , which are Frobenius equivariant and compatible with restriction of the base S . We emphasize that such a uniqueness statement does not apply to lifts of a single elliptic curve or even all elliptic curves over a given base S but only to the family of all elliptic curves over all bases. However, see Theorem 6.1 for a result in this direction.

§5. Existence

In this section, we construct the canonical lift functor. In the presence of enough level structure, it is nothing more than the universal property of Witt vectors applied to the moduli space of ordinary elliptic curves with its canonical Frobenius lift. In general, we use a descent argument to pass from the case with level structure to the general setting.

5.1 Existence with level structure

Let $Y(N)$ denote the moduli space of elliptic curves E/S with full level- N structure $\xi: (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N](S)$. We assume $p \nmid N$ and that N is large enough to make the moduli problem representable. (So $N \geq 3$ is enough.) In this case, $Y(N)$ is a smooth affine scheme of relative dimension one over $\text{Spec } \mathbb{Z}[1/N]$. Let T denote the open subscheme of $Y(N)$ which is the complement of the supersingular locus on the fiber over p , and let $Y(N)^\circ$ denote its p -adic completion \widehat{T} , in the sense of Section 2.2. Then $Y(N)^\circ$ has the form $\text{colim}_n \text{Spec } A_N/p^{n+1}A_N$, where A_N is p -adically complete and p -torsion-free. Indeed, if we write $Y(N) = \text{Spec } R$, then A_N is the p -adic completion of $R[Q^{-1}]$, for any subset $Q \subset R$ such that $\text{Spec } R[Q^{-1}]/(p)$ is the ordinary locus of the fiber of $Y(N)$ over p . It follows that A_N is p -torsion-free because R smooth over $\mathbb{Z}[1/N]$ and hence p -torsion-free. Therefore, it has the form needed to apply the universal property of Section 3.3.

Recall the standard Frobenius lift ψ on $Y(N)^\circ$. Let E be an ordinary elliptic curve over an affine scheme S on which p is nilpotent, and let E^{can} denote its canonical subgroup, the connected component of its p -torsion subgroup-scheme [11, Chapter 3]. It is a finite flat closed subgroup-scheme

of E . Let E/E^{can} denote the usual quotient (so not the quotient object in Aff^\sim , but the quotient with respect to the fppf topology). Thus E/E^{can} is also a family of ordinary elliptic curves over S , and the quotient map $E \rightarrow E/E^{\text{can}}$ is faithfully flat. Further, if ξ is a level- N structure on E , then its image $\bar{\xi}$ in E/E^{can} is still a level- N structure. We then let ψ denote the map $Y(N)^\circ \rightarrow Y(N)^\circ$ that, for any S , sends an S -valued point (E, ξ) to $(E/E^{\text{can}}, \bar{\xi})$. It is a Frobenius lift because for ordinary elliptic curves over \mathbb{F}_p -algebras, the connected component of the p -torsion subgroup-scheme agrees with the kernel of Frobenius.

Now let (E, ξ) be an ordinary elliptic curve with level- N structure over a p -adic sheaf S . This is the pull-back of the universal object (\mathcal{E}, ξ^u) through a unique map $c: S \rightarrow Y(N)^\circ$. Since the ring A_N is p -torsion-free, we can apply the universal property of Witt vectors as given in Section 3.3 and write

$$\tilde{c}: W(S) \rightarrow Y(N)^\circ$$

for the morphism induced by c . We then let $(E, \xi)^\sim$ denote the canonical lift of (E, ξ) , which is to say the pull-back $\tilde{c}^*((\mathcal{E}, \xi^u))$ of the universal elliptic curve with level- N structure. The assignment

$$(E, \xi) \mapsto (E, \xi)^\sim$$

now defines our theory of canonical lifts for elliptic curves with full level- N structure. We could write $(E, \xi)^\sim = (\tilde{E}, \tilde{\xi})$, but it will not be until the next section that we know \tilde{E} is canonically independent of the choice of ξ .

5.2 Existence in general

We use a standard descent argument. So let $Y(N, N)$ denote the moduli space of elliptic curves with a pair of full level- N structures (ξ_1, ξ_2) . Forgetting one or the other defines projections $Y(N, N) \rightrightarrows Y(N)$, both which are finite étale. Now we proceed as we did with $Y(N)^\circ$. Let $Y(N, N)^\circ$ denote the p -adic completion of the complement of the supersingular locus on the fiber over p . Then $Y(N, N)^\circ$ is of the form $\text{colim}_n \text{Spec } A_{N,N}/p^{n+1}A_{N,N}$, where $A_{N,N}$ is a p -adically complete and p -torsion-free ring, and it has a Frobenius lift, also denoted ψ , defined by sending an elliptic curve to its quotient by the canonical subgroup, with the image level structures. This defines a theory of canonical lifts for elliptic curves with pairs of full level- N structure. It is compatible with the projections $Y(N, N) \rightrightarrows Y(N)$ in the sense that taking the canonical lift commutes with forgetting each of the

level structures. This is simply because the Frobenius lift ψ commutes with the projections.

We can now define the canonical lift of an arbitrary ordinary elliptic curve E over a p -adic sheaf S . It will in fact be enough to do this for affine schemes S , as the construction will be compatible with base change along morphisms of affine schemes, so that if S is an arbitrary p -adic sheaf and E is an ordinary elliptic curve over S , writing $S = \text{colim}_i S_i$ as a colimit of affine schemes, we may define \tilde{E} over $W(S)$ to be $\text{colim}_i (E_{S_i})^\sim$ over $W(S) = \text{colim}_i W(S_i)$, as in Section 2.1.

So let E be an ordinary elliptic curve over an affine p -adic scheme S . Let S' be the universal cover of S over which E admits a level- N structure ξ . The covering morphism $S' \rightarrow S$ is then finite and étale, and we have a diagram

$$\begin{array}{ccc} S' \times_S S' & \xrightarrow{c_2} & Y(N, N)^\circ \\ \Downarrow & & \Downarrow \\ S' & \xrightarrow{c_1} & Y(N)^\circ. \end{array}$$

Each column has the structure of a groupoid object in the category Aff^\sim in which the displayed projections are the source and target structure maps. Indeed, $S' \times_S S'$ is an equivalence-relation object on S' over S , and hence it admits a unique groupoid structure over S . On the right column, the groupoid structure $Y(N, N)^\circ$ over $Y(N)^\circ$ is the one with composition given by

$$(E, \xi_1, \xi_2) \circ (E', \xi'_1, \xi'_2) = (E, \xi_1, \beta^*(\xi'_2)),$$

whenever there is an isomorphism $\beta: (E, \xi_2) \xrightarrow{\sim} (E', \xi'_1)$, which is unique when it exists. With respect to these groupoid structures, this diagram becomes a morphism of groupoid objects. Because the right column has Frobenius lifts which are compatible with the projections, the universal property of Witt vectors then gives us a diagram

$$\begin{array}{ccc} W(S' \times_S S') & \xrightarrow{\tilde{c}_2} & Y(N, N)^\circ \\ \Downarrow & & \Downarrow \\ W(S') & \xrightarrow{\tilde{c}_1} & Y(N)^\circ. \end{array}$$

By the theorem in Section 3.2, we have

$$\begin{aligned} W(S' \times_S S') &= \operatorname{colim}_n W_n(S' \times_S S') = \operatorname{colim}_n W_n(S') \times_{W_n(S)} W_n(S') \\ &= W(S') \times_{W(S)} W(S'), \end{aligned}$$

and so the diagram above can be identified with

$$\begin{array}{ccc} W(S') \times_{W(S)} W(S') & \longrightarrow & Y(N, N)^\circ \\ \Downarrow & & \Downarrow \\ W(S') & \longrightarrow & Y(N)^\circ, \end{array}$$

which is easily seen to be a morphism of groupoid objects again. In other words, there is a family over $W(S')$ equipped with descent data to $W(S)$. Since W preserves epimorphisms, as explained above, the map $W(S') \rightarrow W(S)$ is an effective descent morphism for elliptic curves. So we can define \tilde{E} to be the descended object over $W(S)$. It is well defined up to unique isomorphism, in the usual sense. As remarked earlier, it is at this point clear that the construction $E/S \mapsto \tilde{E}/W(S)$ for affine schemes S is compatible with base change and therefore can be extended to any ordinary elliptic curve over any p -adic sheaf S and we do so without further comment.

Our construction of the canonical lift \tilde{E} appears to depend on the auxiliary choice of the level N . One could easily show at this point that it does not, up to canonical isomorphism, but this is a consequence of the uniqueness statement proved in Section 6, and there is no need to establish it earlier.

5.3 Remark: a stack-theoretic interpretation

The language of stacks is well-suited for expressing the descent argument above. For simplicity, we explain it for Witt vectors of finite length.

Given a stack X , let $W_{n*}(X)$ denote the fibered category sending any affine scheme S to the category $X(W_n(S))$. This is often called the arithmetic jet space or the Greenberg transform of X . It is straightforward to show that W_{n*} sends affine schemes to affine schemes, that it is a right adjoint and hence sends groupoid objects to groupoid objects, and that it preserves étale morphisms. (See [4].) Finally, by the theorem in 3.2 the functor W_n is continuous in the étale topology. It then follows for general reasons that W_{n*} takes stacks to stacks. (See [16, Tag 04WC].)

The compatible Frobenius lifts on $Y(N)^\circ$ and $Y(N, N)^\circ$ can then be packaged as a morphism of étale groupoid objects

$$\begin{array}{ccc} Y(N, N)^\circ & \longrightarrow & W_{n*}(Y(N, N)^\circ) \\ \Downarrow & & \Downarrow \\ Y(N)^\circ & \longrightarrow & W_{n*}(Y(N)^\circ). \end{array}$$

It therefore induces a morphism (unique up to unique isomorphism) of the quotient stacks

$$\begin{array}{ccc} Y(N, N)^\circ & \longrightarrow & W_{n*}(Y(N, N)^\circ) \\ \Downarrow & & \Downarrow \\ Y(N)^\circ & \longrightarrow & W_{n*}(Y(N)^\circ) \\ \downarrow & & \downarrow \\ Y(1)^\circ & \dashrightarrow & W_{n*}(Y(1)^\circ). \end{array}$$

This morphism of stacks is then nothing more than the family of canonical lift functors, truncated at length n . The image of an elliptic curve E over S is the elliptic curve $\tilde{E}_{W_n(S)}$ over $W_n(S)$.

It is worth noting that Buium’s work on differential modular forms [6] also touches on the connection between canonical lifts, moduli spaces, and arithmetic jet spaces.

5.4 Frobenius lifts

In this section, we define isomorphisms

$$\eta_E: \tilde{E}/\tilde{E}^{\text{can}} \xrightarrow{\sim} F^*(\tilde{E}),$$

where F denotes the Frobenius map $W(S) \rightarrow W(S)$, such that the composition

$$\tilde{E} \longrightarrow \tilde{E}/\tilde{E}^{\text{can}} \xrightarrow{\sim} F^*(\tilde{E})$$

with the quotient map is a Frobenius lift, in the sense of Section 4. First observe that such an isomorphism η_E , if it exists, is necessarily unique.

Indeed, to show two isomorphisms with the property above agree, it is enough to show they agree after base change to any $W_n(S_i)$, where S_i is any affine scheme mapping to S on which p is nilpotent; but then the difference of two such maps would be a map of elliptic curves which is zero over $\text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} W_n(S_i)$, and hence zero over all of $W_n(S_i)$ by the rigidity theorem [12, Theorem 2.4.2, p. 76].

Therefore, local existence on the base will imply global existence. In particular, we may assume that E admits a level- N structure and then, by base change, that $S = Y(N)^\circ$ and that E is the universal curve \mathcal{E} .

The classifying morphism $c = \tilde{\text{id}}: W(Y(N)^\circ) \rightarrow Y(N)^\circ$ for $\tilde{\mathcal{E}}$ is Frobenius equivariant, by construction. In other words, there is a unique identification $c^*\psi^*(\mathcal{E}) = \psi^*F^*(\mathcal{E})$ compatible with the level structure. We also have $\psi^*(\mathcal{E}) = \mathcal{E}/\mathcal{E}^{\text{can}}$, by the definition of ψ . Thus we have identifications

$$\tilde{\mathcal{E}}/\tilde{\mathcal{E}}^{\text{can}} = c^*(\mathcal{E}/\mathcal{E}^{\text{can}}) = c^*\psi^*(\mathcal{E}) = F^*c^*(\mathcal{E}) = F^*(\tilde{\mathcal{E}})$$

which are compatible with the level structure (and hence unique). Finally, the composition $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}/\tilde{\mathcal{E}}^{\text{can}} = F^*(\tilde{\mathcal{E}})$ reduces to the relative p th power Frobenius map modulo p because, writing $\tilde{\mathcal{E}}_0 = \text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} \tilde{\mathcal{E}}$, the actual Frobenius map $\tilde{\mathcal{E}}_0 \rightarrow \text{Fr}^*(\tilde{\mathcal{E}}_0)$ has kernel $\tilde{\mathcal{E}}_0^{\text{can}}$ and is compatible with the level structure.

From the stack-theoretic point of view, the map η_E can be viewed as an invertible natural transformation $\psi \circ c \rightarrow c \circ F$ and hence as providing a Frobenius equivariant structure on the morphism $c: W(S) \rightarrow Y(1)^\circ$.

5.5 Remark: avoiding abstract sheaf theory

It is possible to avoid abstract sheaf theory by working with compatible systems of elliptic curves. Indeed, as explained in Sections 2.1 and 3.2, an elliptic curve over $W(S)$, where S is a scheme on which p is nilpotent, is equivalent to a compatible family of elliptic curves over the $W_n(S)$, which are also schemes on which p is nilpotent (and even of finite type over \mathbb{Z}_p if S is). Thus the canonical lift \tilde{E} can be viewed as a compatible system of elliptic curves \tilde{E}_n over the schemes $W_n(S)$, which are not much harder to understand than S itself.

Of course one could construct this compatible system directly, without going through Witt vectors of infinite length. For each n , the truncated canonical lift \tilde{E}_n is constructed exactly as we did above with Witt vectors of infinite length but using W_n everywhere instead of W . One then shows

directly that the \tilde{E}_n form a compatible system, and we never have to leave the category of schemes. The Frobenius morphism then becomes a family of morphisms

$$\tilde{E}_n \longrightarrow \tilde{E}_n/\tilde{E}_n^{\text{can}} \xrightarrow{\sim} F^*(\tilde{E}_{n+1})$$

of elliptic curves over the $W_n(S)$, where F now denotes the truncated Witt vector Frobenius map $W_n(S) \rightarrow W_{n+1}(S)$.

§6. Uniqueness

To prove the uniqueness part of the theorem, we need the following result in the particular case of the universal elliptic curve. But since the result is just as easy to prove in a more general form, we do that.

THEOREM 6.1. *Let R be a p -adically complete ring such that $W(R)$ is p -torsion-free. Let $S = \text{colim}_n \text{Spec } R/p^{n+1}R$, and let E be an ordinary elliptic curve over S . Suppose X_1 and X_2 are lifts of E to $W(S)$ with Frobenius lifts ψ_1 and ψ_2 . Then there is a unique Frobenius equivariant isomorphism $\beta: X_1 \rightarrow X_2$ restricting to the identity on E .*

We recall again that $W(R)$ is p -torsion-free if R is either p -torsion-free or a perfect \mathbb{F}_p -algebra. Also note that by formal GAGA, elliptic curves over S are equivalent to elliptic curves over $\text{Spec } R$.

Proof. The Frobenius lifts are morphisms $X_i \rightarrow F^*(X_i)$ of elliptic curves over $W(S)$ which reduce to the Frobenius map modulo p , and hence are (representable and) flat of degree p , by rigidity [12, Theorem 2.4.2, p. 76]. Therefore, the kernel is a (representable) finite flat lift of the kernel of the Frobenius map and hence equals the canonical subgroup. Thus the Frobenius lifts ψ_i induce isomorphisms $\bar{\psi}_i: X_i/X_i^{\text{can}} \xrightarrow{\sim} F^*(X_i)$.

We first consider the case where E admits a level- N structure ξ . Let ξ_i denote the unique lift of ξ to X_i , and let $\alpha_i: W(S) \rightarrow Y(N)^\circ$ denote the classifying map for (X_i, ξ_i) . Now observe that the isomorphisms $\bar{\psi}_i$ must preserve the level structure, where X_i/X_i^{can} is given the image level structure and $F^*(X_i)$ is given the pull-back level structure. Indeed, consider the elliptic curves $\bar{X}_i = \text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} X_i$ over $\text{Spec } \mathbb{F}_p \times_{\text{Spec } \mathbb{Z}} W(S)$. Because $p \nmid N$, the N -torsion is finite étale. Therefore, the Frobenius map $\bar{X}_i \rightarrow \text{Fr}_S^*(\bar{X}_i)$ preserves the level structure and, again because the N -torsion is finite étale, so does any lift of the Frobenius map. In particular, the maps $\bar{\psi}_i$ preserve it.

It follows that the elliptic curves X_i/X_i^{can} and $F^*(X_i)$ have the same classifying map $W(S) \rightarrow Y(N)^\circ$. The classifying map for X_i/X_i^{can} is $\psi \circ \alpha_i$, by the definition of ψ , and that for $F^*(X_i)$ is $\alpha_i \circ F$. Therefore, we have $\psi \circ \alpha_i = \alpha_i \circ F$, which is to say that the maps $\alpha_i : W(S) \rightarrow Y(N)^\circ$ commute with the Frobenius lifts.

But at the same time, since both (X_i, ξ_i) lift (E, ξ) , the two compositions

$$S \longrightarrow W(S) \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} Y(N)^\circ$$

agree. Therefore, writing $Y(N)^\circ = \text{colim}_n \text{Spec } A/p^{n+1}A$ with A p -adically complete, we see that the two compositions

$$A \begin{array}{c} \xrightarrow{\alpha_1^*} \\ \xrightarrow{\alpha_2^*} \end{array} W(R) \longrightarrow R$$

agree. Further, the maps α_1^* and α_2^* are Frobenius equivariant because α_1 and α_2 are. Since A and $W(R)$ are p -torsion-free, the universal property of Witt vectors implies $\alpha_1^* = \alpha_2^*$ and hence $\alpha_1 = \alpha_2$.

When E is arbitrary, the existence and uniqueness of β follow formally because étale locally β exists and is unique. We write out the details. Let S' denote the universal étale cover of S over which E admits a level- N structure. Write $E' = S' \times_S E$ and $X'_i = W(S') \times_{W(S)} E_i$. Then each X'_i is a lift of E' with a Frobenius lift $\psi'_i = F \times \psi_i$. Since each X'_i also admits a level- N structure, the construction above gives a canonical morphism $\beta' : X'_1 \rightarrow X'_2$ (and hence in fact a unique one, because N is large) commuting with the Frobenius lifts ψ'_i . Further, β is equivariant with respect to descent data. Indeed, write $S'' = S' \times_S S'$ and $X''_i = W(S'') \times_{W(S)} X$. Then each X''_i is a lift of $S'' \times_S E$ with Frobenius lift $\psi''_i = F \times \psi_i$. Therefore, there is a unique Frobenius equivariant morphism $\beta'' : X''_1 \rightarrow X''_2$. In particular, the two restrictions of β' to $W(S'')$ agree, which is to say that β' is equivariant with respect to descent data.

6.2 Uniqueness of the canonical lift functor

Suppose we have an assignment $E \mapsto \check{E}$, sending elliptic curves E/S to elliptic curves $\check{E}/W(S)$ with a Frobenius lift which is compatible with change of the base S . To give an isomorphism $\check{E} \rightarrow \tilde{E}$, it is enough give isomorphisms locally on S which are compatible with change of S . Therefore,

we may assume E admits a level- N structure. Let $d: S \rightarrow Y(N)^\circ$ denote the classifying morphism of E , and let $W(d): W(S) \rightarrow W(Y(N)^\circ)$ denote the induced morphism. Now since we have $Y(N)^\circ = \operatorname{colim}_n \operatorname{Spec} A_N/p^{n+1}A_N$, where A_N is p -adically complete and p -torsion-free, we can apply the theorem above to the universal ordinary elliptic curve \mathcal{E} over $Y(N)^\circ$. Therefore, there is a unique Frobenius equivariant isomorphism $\check{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ restricting to the identity on \mathcal{E} . Since the constructions $E \mapsto \check{E}$ and $E \mapsto \tilde{E}$ are compatible with restriction of S (in the first case by assumption and in the second case by Section 5), the family of morphisms

$$\check{E} = W(d)^*(\check{\mathcal{E}}) \xrightarrow{\sim} W(d)^*(\tilde{\mathcal{E}}) = \tilde{E},$$

is the unique family of Frobenius equivariant isomorphisms $\check{E} \rightarrow \tilde{E}$ which is compatible with restriction of S .

In particular, the functor $E \mapsto \tilde{E}$ constructed in Section 5 is independent of the choice of the level N , up to unique isomorphism.

§7. Remarks and applications

7.1 Comparison with the classical canonical lifts

In the classical context where $S = \operatorname{Spec} k$ with k a perfect field of characteristic p , it is known that the Serre–Tate canonical lift has a Frobenius lift and is characterized, up to unique isomorphism, by this property. (See Messing’s book [15, p. 177, cor. (1.2) and p. 174, cor. (3.4)].) The theorem in Theorem 6.1 then implies that our canonical lifts are isomorphic to the Serre–Tate canonical lifts by a unique Frobenius equivariant morphism.

Also observe that in this context, we only need a weak case of the theorem in Section 3.2 in the descent argument. This is because level structure exists over some finite extension k'/k , and then it is well known that $W(k')/W(k)$ is finite étale and we have $W(k' \otimes_k k') = W(k') \otimes_{W(k)} W(k')$. Thus in the classical context, our argument really is little more than an application of the universal property of Witt vectors.

While our approach has the benefit of allowing families in mixed characteristic, it also requires them, even if one is only interested in canonical lifts of elliptic curves in characteristic p ; and this could be viewed as a drawback. The reason is simply that the special fiber $Y(N)_{\mathbb{F}_p}^\circ$ does not admit a map from $W(Y(N)_{\mathbb{F}_p}^\circ)$. Indeed, no nonempty Witt vector scheme maps to $\operatorname{Spec} \mathbb{F}_p$.

7.2 j -invariants

We can describe the j -invariant of the canonical lift of the universal family. Recall that $Y(N)^\circ = \text{Spf } A_N$ and $Y(N, N)^\circ = \text{Spf } A_{N,N}$, where A_N and $A_{N,N}$ are p -adically complete and p -torsion-free, and that forgetting the level structures induces a pair of Frobenius equivariant maps $A_N \rightrightarrows A_{N,N}$. Thus, writing A_1 for the equalizer of these two maps, the map $(\text{id}_{A_N})^\sim: A_N \rightarrow W(A_N)$ restricts to a map $s: A_1 \rightarrow W(A_1)$, and so the Frobenius lift on A_N restricts to a Frobenius lift on A_1 . Now write $A_1 = \mathbb{Z}_p[j, 1/f(j)]^\wedge$, where j is an indeterminate identified with the j -function, $f(j)$ is a monic polynomial whose roots lift the supersingular j -invariants, and $(\cdot)^\wedge$ denotes p -adic completion. The image of j under the map $s: A_1 \rightarrow W(A_1)$ has ghost components $\langle j, \psi(j), \psi^{\circ 2}(j), \dots \rangle \in A_1^\infty$. It follows that if E is any family of ordinary elliptic curves, the ghost components of $j(\tilde{E})$ are obtained by evaluating the universal expressions $\psi^{\circ n}(j) \in \mathbb{Z}_p[j, 1/f(j)]^\wedge$ at $j = j(E)$.

One can similarly consider the usual Witt components (j_0, j_1, \dots) of $s(j)$. Then the Witt components of $j(\tilde{E})$ are also obtained by evaluating the universal expressions $j_n \in \mathbb{Z}_p[j, 1/f(j)]^\wedge$ at $j = j(E)$. For example, by the definition of the ghost map, we have $\psi^{\circ n}(j) = \sum_{i=0}^n p^i j_i^{p^{n-i}}$, and so the Witt components of the j -invariant of the canonical lift of an elliptic curve with j -invariant j are given by

$$\begin{aligned} j_0 &= j, & j_1 &= (\psi(j) - j^p)/p, \\ j_2 &= (\psi(\psi(j)) - j^{p^2})/p^2 - (\psi(j) - j^p)^p/p^{p+1}, \end{aligned}$$

and so on. For elliptic curves over perfect fields of characteristic p , such a result was found by Erdoğ an [7] (strengthening earlier results of Finotti [8]) by more traditional means, but where the universal expressions are of course the reductions of ours modulo p . We note that Erdoğ an also constructs canonical lifts for families over perfect \mathbb{F}_p -algebras.

We can do the same for elliptic curves over a general base. Let E be a family over $\text{Spf } R$ for a p -adically complete ring R . Then the j -invariant $j(\tilde{E})$ is an element of $W(R)$, and as in (3.1.1) its ghost components are

$$\langle j(E_0), j(E_1), j(E_2), \dots \rangle = \langle j(E), \psi(j(E)), \psi^{\circ 2}(j(E)), \dots \rangle \in R^\infty,$$

where $E_0 = E$ and $E_{n+1} = E_n/E_n^{\text{can}}$. This already determines $j(\tilde{E})$ if R is p -torsion-free, since in that case the ghost map $W(R) \rightarrow R^\mathbb{N}$ is injective. In general, one can lift E arbitrarily to a family E' over a p -torsion-free

extension R' of R , and then $j(\tilde{E})$ is the image under $W(R') \rightarrow W(R)$ of $j((E')^\sim)$, which has the simple description above since R' is p -torsion-free.

This can be expressed in an especially pleasant way in the particular case where R is a perfect \mathbb{F}_p -algebra k , and R' is $W(k)$. Then one can show that the projection $W(\text{pr}_0): W(W(k)) \rightarrow W(k)$ sends a Witt vector with ghost components $\langle a_0, a_1, \dots \rangle \in W(k)^\infty$ to $\lim_{n \rightarrow \infty} F^{-n}(a_n) \in W(k)$, and so we have

$$j(\tilde{E}) = \lim_{n \rightarrow \infty} F^{-n}(j(E'_n)) = \lim_{n \rightarrow \infty} F^{-n}(\psi^{\circ n}(j(E'))),$$

where E' is an arbitrary lift to $W(k)$ of E and the E'_n are as above. If k is a field with p^r elements, then $F^r = \text{id}_{W(k)}$ and so we simply have $j(\tilde{E}) = \lim_{n \rightarrow \infty} j(E'_{rn})$.

7.3 Canonical lifts of canonical lifts

Amusingly, one can take canonical lifts of canonical lifts and so on, any number of times. Indeed, if E is an ordinary elliptic curve over a p -adic sheaf S , then \tilde{E} is also an ordinary elliptic curve over the p -adic sheaf $W(S)$. However, one can show that the double canonical lift $\tilde{\tilde{E}}$ is canonically isomorphic to the pull-back of \tilde{E} through the composition, or plethysm, map $W(W(S)) \rightarrow W(S)$ of [4, (10.6.13)], and similarly for the higher iterates. So these higher canonical lifts are all determined by the first one in a way that has nothing to do with elliptic curves and only the general theory of Witt vectors.

7.4 Epilogue: on a general theory of canonical lifts

We proposed above that a theory of canonical lifts for a class of objects parametrized by a moduli space Y should be defined to be a Frobenius lift $\psi: Y \rightarrow Y$, as long as Y is p -torsion-free. In the absence of this assumption, one should define a theory of canonical lifts to be a slightly stronger structure. Namely, the functor $W_* = \lim_n W_{n*}$ has a natural comonad structure (coming from the maps $W_{m+n*} \rightarrow W_{m*} \circ W_{n*}$ of [4, (10.6.14)]) and a theory of canonical lifts for the objects parametrized by Y would then be a coaction of the comonad W_* on Y . This is called a δ -structure in Joyal [10] and Buium [5] and a p -typical Λ -structure in [3, 4].

This can also be done for other varieties of Witt vectors in the sense of [4], such as the big Witt vectors. One would then define a theory of canonical lifts, relative to the given variety of Witt vectors, to be an action of the

comonad W_* on Y . In other words, a theory of canonical lifts for a class of objects should be defined to be a Λ -structure on their moduli space. This was carried out by the second author [9] in the case of elliptic curves with complex multiplication and the variety of big Witt vectors associated with an imaginary quadratic field.

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James Borger
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
Australia
james.borger@anu.edu.au

Lance Gurney
Korteweg-de Vries Instituut
Universiteit van Amsterdam
1090 GE Amsterdam
The Netherlands
l.r.gurney@uva.nl