NOTE ON t-MINIMAL COMPLETE BIPARTITE GRAPHS

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1. Introduction. The thickness of a graph G is the smallest natural number t such that G is the union of t planar subgraphs. A graph G is t-minimal if its thickness is t and if every proper subgraph of G has thickness < t. (These terms were introduced by Tutte in [3]. In [1, p. 51] Beineke employs the term t-critical instead of t-minimal.) The complete bipartite graph K(m, n) consists of m 'dark' points, n 'light' points, and the mn lines joining points of different types.

In [1, p. 52] Beineke proves that the complete bipartite graph

$$\text{(X)} \qquad K(2s+1,\ 4s^2-2s+1)$$

is (s + 1)-minimal (s = 1, 2, ...). He also states, without proof, that the complete bipartite graph

$$\text{(Y)} \qquad K(4s-1,\ 4s-1)$$

is (s + 1)-minimal if s is odd, and conjectures that it also holds if s is even. The purpose of this note is to prove that for each s the graph (Y) is (s + 1)-minimal. The proof is based upon a technique developed by Beineke, Harary and Moon [2].

Before giving the proof, we motivate the appearance of the families (X) and (Y). The maximal number of lines of a planar subgraph of K(m, n) is 2(m + n - 2) (see for instance [2, p. 1]). It is therefore natural to consider, as candidates for (s + 1)-minimal graphs, those graphs K(m, n) that satisfy:

$$(1) \qquad mn = 2(m+n-2)s + 1 .$$

Since (1) may be written in the form:

$$(m-2s)(n-2s) = (2s-1)^2,$$

we see that to each couple (s, R), where s is any integer $\geqslant 1$ and where R is any (integral) factor of $(2s-1)^2$, there corresponds a

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graph $K(m, n)$ satisfying (1), namely the graph for which

$$(2) \qquad m = 2s + R \quad \text{and} \quad n = 2s + (2s - 1)^2 / R .$$

We may assume $m \leqslant n$, that is, $1 \leqslant R \leqslant 2s - 1$. The correspondence $(s, R) \rightarrow K(m, n)$ defined as above then becomes one-to-one and onto the set of all graphs $K(m, n)$ satisfying (1). The graphs (X) and (Y) appear as the graphs corresponding to the extremal values 1 and $2s - 1$ of R.

2. Proof. In order to prove the result, it is sufficient to show that there exist $s + 1$ planar subgraphs of the given graph (Y) such that their union is (Y) and such that one of the subgraphs contains only one line (so that, by (1), the remaining subgraphs contain the maximum possible number of lines).

We may assume $s \geqslant 2$. From the Main Theorem of [2] (reproduced as Theorem 8 on p. 49 of [1]) the thickness of $K(4s - 1, 4s - 2)$ is found to be s. Construct a $(2s - 1)$ by s array A as follows: The (i, j)th cell of A consists of $N(i, j)$ consecutive integers modulo $4s - 1$, where

$$N(2s - j, j) = N(s - j, j) = N(s, s) = 3 \quad (j = 1, 2, \ldots, s - 1) \quad \text{and}$$

$$N(i, j) = 4 \text{ otherwise } (i = 1, 2, \ldots, 2s - 1;\ j = 1, 2, \ldots, s) ;$$

in the first row of A the integers $1, 2, \ldots, 4s - 1$ appear consecutively; the cells are then filled in, inductively, by letting the penultimate entry of a cell be the first entry of the cell below it. (This array is found from the construction given in the proof of [2, Lemma 2] by setting $m = 4s - 1$, $k = s$ and $r = 4s - 2$.) It can be seen that the total number of distinct entries in the cells of the jth column of A is $4s - 1$ if $j = s$, and $4s - 2$ if $j \neq s$. The integer not appearing among the entries of the first (or jth for $j = 2, 3, \ldots, s-1$; $s > 2$) column is $4s - 1$ (or, respectively, $4(j - 1)$). Construct a subgraph G_j of $K(4s - 1, 4s - 2)$ from the jth column of the array A (as in the proof of [2, Theorem 2]) as follows: the dark points correspond to the integers $1, 2, \ldots, 4s - 1$; the $4s - 2$ light points correspond in pairs to the cells of the column (where we shall assume that the light points i and $(i + 2s - 1)$ correspond to the ith cell); and a dark point and a light point are joined if and only if the integer corresponding to the dark point occurs as an entry of the cell corresponding to the light point. The graphs G_j are planar, and their union is $K(4s - 1, 4s - 2)$ ([2], proof of Theorem 2).

We now represent each graph G_j as follows: consider rectangular coordinate axes in the plane. Let the dark points correspond, in the order in which they occur in the jth column (starting with the first entry of the first cell and excluding repetitions), to the points on the horizontal axis with abscissas: 0, -1, 1, -2, 2, -3, ... Let the light points i and $(i + 2s - 1)$ $(i = 1, 2, \ldots, 2s - 1)$ correspond to the points on the vertical axis with ordinates i and $-i$, respectively. The lines of G_j may now be drawn (for instance as straight lines) without causing intersections.

From the graphs $G_1, \ldots, G_s$, represented as above, we form graphs $H_1, \ldots, H_s$, respectively, as follows:

(i) If $s \geq 4$ we interchange in G_j $(j = 1, 2, \ldots, s - 3)$ the positions of the dark points $(4j + 1)$ and $(4j + 2)$, and in G_s the positions of the dark points 1 and 2. If $s = 3$ we interchange in G_3 the positions of the dark points 8 and 9, and the positions of the dark points 10 and 11. (In each case the dark points that are interchanged, are joined to the same set of light points.)

(ii) The dark point $(4s - 1)$ is inserted in G_1 and joined to the light points 1 and 2, while the dark point $(4j - 4)$ is inserted in G_j $(j = 2, 3, \ldots, s - 1; s > 2)$ and joined to the light points $2s$ and $(2s + 1)$ if j is even, and to the light points 1 and 2 if j is odd.

(iii) The pairs of lines inserted as above in $G_1, G_2, \ldots, G_{s-1}$ appear in $G_s, G_1, G_2, \ldots, G_{s-2}$ respectively, from which they are now removed.

It may be checked that each graph H_j $(j \neq s - 1)$ contains an octagonal region. For $j = 1, 2, \ldots, s - 2$, the boundary of the octagonal region contains the dark points $(4j - 2)$, $(4j - 1)$, $4j$, $(4j + 1)$; and for $j = s$, the dark points $(4s - 3)$, $(4s - 2)$, $(4s - 1)$, 1.

We now postulate a new light point P and form graphs I_k $(k = 1, 2, \ldots, s + 1)$ as follows: In each H_j $(j \neq s - 1)$ we insert P in the octagonal region of H_j and join it to the four dark points occurring on the boundary of the octagonal region to form the graph I_j. In H_{s-1} we insert P in the region between the dark points $(4s - 6)$ and $(4s - 4)$, and join it to these two dark points to form I_{s-1}. The graph I_{s+1} consists of the single line joining P and the dark point $(4s - 5)$.

Clearly the graphs I_k $(k = 1, 2, \ldots, s + 1)$ are planar and their union is the graph (Y).

3. Conclusion. Under the lexicographic ordering of the couples (s, R) the graphs $K(m, n)$ for which (2), or equivalently (1), is

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satisfied, become linearly ordered. With respect to this order the smallest graph which is not of the form (X) or (Y), is the graph K(13, 37) (s = 5, R = 3). We have found a construction showing that this graph is 6-minimal. The next larger such graphs are K(19, 91) (s = 8, R = 3), K(21, 61) (s = 8, R = 5) and K(25, 41) (s = 8, R = 9). We do not know if these graphs are 9-minimal. It would be of interest to know if all the graphs K(m, n) satisfying (1) are (s + 1)-minimal and if all (s + 1)-minimal graphs K(m, n) satisfy (1).

Added in proof: The same result (with a different construction used) has appeared in (A. M. Hobbs and J. W. Grossman, A Class of Thickness-Minimal Graphs, J. Res. Nat. Bur. Standards. 72(B) (1968) 145-153.)

REFERENCES

1. L. W. Beineke, Complete bipartite graphs: decomposition into planar subgraphs, Lecture 7 in F. Harary, A seminar on graph theory (Holt, Rinehart and Winston, New York, 1967).

2. L. W. Beineke, F. Harary, and J. W. Moon, On the thickness of the complete bipartite graph, Proc. Cambridge Philos. Soc. 60 (1964) 1-5.

3. W. T. Tutte, The thickness of a graph, Nederl. Akad. Wetensch. Proc. Ser. A, 66 = Indag. Math. 25 (1963) 567-577.

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