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UNIQUENESS OF GENERALIZED SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

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1. Let Ω be an open subset of R and H be a complex Hilbert space; (,) represents scalar product in H. Let also A be a closed linear operator with domain D_A dense in H and A^* with domain D_A^* be its adjoint. Under graph scalar product D_A and D_A^* are also Hilbert spaces. By $\mathscr{D}_\Omega(H)$ we denote the space of all infinitely differentiable functions (H-valued) with compact support defined on Ω. $\mathscr{D}_\Omega(H)$ is equipped with Schwartz topology. Similarly, we define $\mathscr{D}_\Omega(D_A)$, $\mathscr{D}_\Omega(D_A^*)$ and $\mathscr{D}_\Omega(\mathbb{C})$; $\mathbb{C}$ represents the complex plane. By $\mathscr{D}'_\Omega(H)=\mathscr{L}(\mathscr{D}_\Omega(\mathbb{C}); H)$ we mean the space of all continuous linear mappings (H-valued) defined on $\mathscr{D}_\Omega(\mathbb{C})$. In a similar way, we define $\mathscr{D}'_\Omega(D_A)$. For $\Psi \in \mathscr{D}_\Omega(\mathbb{C})$ and $u \in \mathscr{D}'_\Omega(H)$, $\langle u, \Psi\rangle \in H$. It is easy to show that if $\langle u, \Psi\rangle \in D_A$ for all $\Psi \in \mathscr{D}_\Omega(\mathbb{C})$, then $u \in \mathscr{D}'_\Omega(D_A)$. The H-valued distribution space $\mathscr{D}'_\Omega(H)$ is also the dual of $\mathscr{D}_\Omega(H)$. In this case, for $\varphi \in \mathscr{D}_\Omega(H)$ and $u \in \mathscr{D}'_\Omega(H)$, $\langle u, \Psi\rangle \in \mathbb{C}$.

We define Au, for $u \in \mathscr{D}'_R(D_A)$ by the relation

$$\langle Au, \Psi\rangle = A\langle u, \Psi\rangle \tag{1.1}$$

for all $\Psi \in \mathscr{D}_R(\mathbb{C})$; $Au \in \mathscr{D}'_R(H)$.

For convenience, we write $L=(1/i)(d/dt)-A$ and $L^*=(1/i)(d/dt)-A^*$. By $R(\lambda; A)$, we denote the resolvent operator of A, $\lambda \in \mathbb{C}$. In view of imposing condition on A, we need:

Definition. Let $\mathscr{F}$ be a family of parallel lines $\{\text{Im } \lambda=\tau_n, \tau_n\to\infty \text{ as } n\to\infty, \tau_n\to-\infty \text{ as } n\to-\infty\}$ in the complex plane $\mathbb{C}$. Let r be a positive real number and j, m be positive integers. We shall say that the resolvent $R(\lambda; A)$ is of (j, r, m)-growth on $\mathscr{F}$ if $R(\lambda; A)$ exists for λ outside j intervals of length r on every line of $\mathscr{F}$ and for these λ

$$|R(\lambda; A)| \le \text{const. } |\lambda|^m \tag{1.2}$$

Throughout this paper, the 'const.' need not be the same constant.

2. We consider the abstract differential equation

$$\frac{1}{i}\frac{du}{dt} - Au = f \tag{2.1}$$

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The author has recently proved the existence of weak solution of the equation (2.1) imposing condition on the resolvent $R(\lambda; A^*)$ in [2]. In fact, he proved:

THEOREM A. *If $R(\lambda; A^*)$ is of (j, r, m)-growth on $\mathscr{F}$, then for every $f \in \mathscr{D}'_A(H)$ the equation $Lu=f$ has at least one weak solution $u \in \mathscr{D}'_R(H)$, i.e.,*

$$\langle u, L^*\varphi\rangle = \langle f, \varphi\rangle \tag{2.2}$$

for all $\varphi \in \mathscr{D}(D'_R)$.

In this paper, we show that the solution u in Theorem A is not unique and actually $u \in \mathscr{D}'(D_A)$ yielding a solution of (2.1). We also study the uniqueness of the solution u (of Lu=0) vanishing in a neighbourhood of $x \in R$.

3. We prove:

THEOREM 1. *If $R(\lambda; A^*)$ is of (j, r, m)-growth on $\mathscr{F}$, the space of weak solutions of $Lu=0$ consists of more than one element.*

In the proof of Theorem 1, we need the following:

DEFINITION. We define V_Ω as the set of all $u \in \mathscr{D}'_\Omega(H)$ such that

$$\langle u, L^*\varphi\rangle = 0 \tag{3.1}$$

for all $\varphi \in \mathscr{D}_\Omega(D^*_A)$.

LEMMA 1. *Let the hypothesis of Theorem 1 be satisfied and Ω_1, Ω_2 be two open subsets of R with $\Omega_1 \subset \Omega_2$. Then V_R is dense in V_{Ω_2} under the topology of $\mathscr{D}'_{\Omega_1}(H)$, i.e., for $\varphi \in \mathscr{D}_{\Omega_1}(H)$ if $\langle \chi, \varphi\rangle=0$ for all $\chi \in V_R$ then $\langle \mu, \varphi\rangle=0$ for all $\mu \in V_{\Omega_2}$.*

S. Zaidman [4] has proved a similar result for $L^2_{\text{loc}}(H)$, the space of locally square integrable H-valued functions and Lemma 1 can be proved along the same line.

From Lemma 1, we immediately have:

LEMMA 2. *Under the hypothesis of Theorem 1, if $V_R=\{0\}$ then for any $\Omega \subset R$, $V_\Omega=\{0\}$.*

Proof of Theorem 1. Suppose on contrary, that $V_R=\{0\}$. In such a case, we shall show that the weak solution of $Lu=\delta\otimes x$ does not exist i.e., there exists no $u \in \mathscr{D}'_R(H)$ satisfying

$$\langle u, L^*\varphi\rangle = \langle \delta\otimes x, \varphi\rangle \tag{3.2}$$

for all $\varphi \in \mathscr{D}(D^*_A)$. As it contradicts Theorem A the proof will be complete.

Now suppose there exists $u \in \mathscr{D}'_R(H)$ satisfying (3.2). For $\varphi \in \mathscr{D}(D^*_A)$ with supp $\varphi \subset (0, \infty)$ we obviously have $\varphi(0)=0$ and so

$$\langle u, L^*\varphi\rangle = 0 \tag{3.3}$$

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since $V_R=\{0\}$, from Lemma 2, $u=0$ on $(0, \infty)$; let $\Omega=(0, \infty)$. Similarly, $u=0$ on $(-\infty, 0)$. So the supp u is concentrated at the origin and u may therefore be expressed as a finite linear combination of Dirac distribution and its derivative, hence:

$$u = \sum_0^n a_k \otimes \delta^{(k)} \tag{3.4}$$

$a_k \in H$. Substituting (3.4) in (3.2) and using (1.1) after transposing the derivative, we have

$$\sum_0^n (a_k, (-i)^{k+1}\varphi^{(k+1)}(0) - A^*\varphi^{(k)}(0)) = (x, \varphi(0)) \tag{3.5}$$

for all $\varphi \in \mathscr{D}_R(D_A^*)$ and $\chi \in H$. A choice of φ in (3.5) such that $\varphi^{(k)}(0)=0$ for $k=0, 1, 2, \ldots, n$ whereas $\varphi^{(n+1)}(0) \neq 0$ implies the leading coefficient $a_n=0$. Thus $u \equiv 0$. It contradicts (3.2). This completes the proof.

THEOREM 2. *Let $R(\lambda; A^*)$ be of (j, r, m)-growth on $\mathscr{F}$. Then for any $f \in \mathscr{D}'_R(H)$, the abstract differential equation $Lu=f$ has more than one solution $u \in \mathscr{D}'_R(D_A)$.*

Proof of Theorem 2. From Theorem 1, there exists more than one $u \in \mathscr{D}'_R(H)$ such that

$$\langle u, L^*\varphi\rangle = \langle f, \varphi\rangle. \tag{3.6}$$

We shall show that $u \in \mathscr{D}'_R(D_A)$. Putting $\varphi=\Psi\otimes x$, $\Psi \in \mathscr{D}_R(\mathbb{C})$ and $\chi \in D_A^*$ in (3.6) we have

$$\left\langle u, \frac{1}{i}\frac{d}{dt}\Psi\otimes x - A^*\Psi\otimes x\right\rangle = \langle f, \Psi\otimes x\rangle \tag{3.7}$$

from where

$$\left(\left\langle \frac{1}{i}\frac{du}{dt} - f, \Psi\right\rangle, x\right) = (\langle u, \Psi\rangle, A^*x) \tag{3.8}$$

for all $\chi \in D_A^*$. This implies that $\langle u, \Psi\rangle \in D_A^{**} = D_A$ as A is a closed linear operator with domain D_A dense in H; (see [3], pages 196–197). Consequently, $u \in \mathscr{D}'(D_A)$ and satisfies the equation

$$\frac{1}{i}\frac{du}{dt} - Au = f.$$

THEOREM 3. *Let $u \in \mathscr{D}'_R(D_A)$ be a solution of*

$$\frac{1}{i}\frac{du}{dt} - Au = 0 \tag{3.9}$$

and the resolvent $R(\lambda; A)$ is of (j, r, m)-growth on $\mathscr{F}$. If for some $\chi \in R$ and $\varepsilon > 0$, u vanishes on $(x-\varepsilon, x+\varepsilon)$, then $u \equiv 0$.

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LEMMA 3. [2] *Let $R(\lambda; A)$ be of (j, r, m)-growth on $\mathscr{F}$ and $\xi \in C^\infty(D_A)$ be a solution of $L\varphi=0$ on $a \le t \le b$ with $\xi(c)=0$, $a<c<b$. Then $\xi \equiv 0$ on $[a, b]$.*

Proof of Theorem 3. Consider a sequence $\{\alpha_n; \alpha_n \in \mathscr{D}_R(\mathbb{C}), \operatorname{supp} \alpha_n \subset [-1/n, 1/n]\}$ such that $\alpha_n \to \delta$, the Dirac distribution. Let $u \in \mathscr{D}'_R(D_A)$ be a solution of (3.9). Consider the convolution $u * \alpha_n$. It is clear that $u * \alpha_n \in C^\infty(D_A)$, $L(u * \alpha)=0$ and for sufficiently large n,

$$\operatorname{supp}(u * \alpha_n) \cap \left(x-\varepsilon+\frac{1}{n}, x+\varepsilon-\frac{1}{n}\right) = \phi$$

so $(u * \alpha_n)(x)=0$. In view of Lemma 3, $u * \alpha_n=0$ on any interval $a \le t \le b$ containing x and so $(u * \alpha_n)(t)=0$ for all $t \in R$. Consequently $u \equiv 0$.

4. Finally, we present the following version of an example of S. Agmon and L. Nirenberg [1] where the conclusion like of Lemma 2 is not true.

EXAMPLE. In the space of all continuous complex functions defined on R, consider a closed linear operator $A=i(d/dx)$ with domain D_A consisting of all C^1 functions vanishing at $-\infty$. Consider the homogeneous equation

$$\frac{1}{i}\frac{\partial}{\partial t}u - i\frac{\partial}{\partial x}u = 0 \tag{4.1}$$

on $t_1 \le t \le t_2$, $u(t, \cdot) \in D_A$. The operator $-iL=(1/i)(\partial/\partial t)-i(\partial/\partial x)$ is a directional derivative in the (t, x)-plane. Any solution u of (4.1) is constant on the line with direction (1.1) lying in the strip $[t_1, t_2] \times R$ and need not be zero. However, if $t_1=-\infty$ and $u(t, x)$ is a solution of (4.1), then $u \equiv 0$; in fact, u is constant on the line $x=t+c$ and vanishes at $x=-\infty$.

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