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THE WIENER-HOPF INTEGRAL EQUATION FOR FRACTIONAL RIESZ-BESSEL MOTION

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Dedicated to Professor David Elliott on the occasion of his 65th birthday

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Abstract

This paper gives an approximate solution to the Wiener-Hopf integral equation for filtering fractional Riesz-Bessel motion. This is obtained by showing that the corresponding covariance operator of the integral equation is a continuous isomorphism between appropriate fractional Sobolev spaces. The proof relies on properties of the Riesz and Bessel potentials and the theory of fractional Sobolev spaces.

1. Introduction

Let $X(t)$ be a real-valued random field of the form

$$X(t) = S(t) + N(t), \quad t \in \mathbb{R}^n, \tag{1.1}$$

where $S(t)$ is the useful signal and $N(t)$ is noise. We shall assume that $E(S(t)) = 0$, $E(N(t)) = 0$ and the covariance functions

$$R(s, t) = E(X(s)X(t)), \quad g(s, t) = E(X(s)S(t)) \tag{1.2}$$

are known, where E denotes mathematical expectation. Given that $X(t)$ is observed in a bounded domain T of $\mathbb{R}^n$, we want to derive the best linear estimate of $S(t)$ for t in

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the closure of T in the mean-square sense. This is obtained by solving the variational problem

$$E\left|\int_T h(s,t)X(t)\,dt - S(s)\right|^2 = \min, \quad s \in \overline{T}. \tag{1.3}$$

It is known that a necessary condition for (1.3) to hold is

$$\int_T R(s,t)h(z,t)\,dt = g(s,z), \quad s \in \overline{T}$$

(Ramm [12]). Since z appears as a parameter in this equation, we may study the equation

$$\int_T R(s,t)h(t)\,dt = g(s), \quad s \in \overline{T}. \tag{1.4}$$

Equation (1.4) is known as the Wiener-Hopf integral equation for filtering random fields. This paper will study this equation for a class of covariance kernels defined below.

Existing work on (1.4) commonly assumes that the random field $X(t)$ displays a short-range dependence behaviour (for example, the covariance kernel $R(s,t)$ is the Fourier transform of a rational spectrum). On the other hand, recent studies have indicated that self-similarity (SS) / long-range dependence (LRD) and intermittency are the central issues in modelling observed data in a large number of fields including hydrology, geophysics, air pollution, image analysis, economics and finance (see, for example, Beran [2]; Peters [10]; Anh and Lunney [1]). A key example of an SS/LRD process is fractional Brownian motion (fBm) B_H with Hurst index H, $0 < H < 1$ (Mandelbrot and Van Ness [9]). In this paper, we shall consider an extension of the class of fBm, namely, the class of fractional Riesz-Bessel motions (fRBm), which exhibit possible LRD and intermittency. The members of this class have a spectral density of the form

$$f(\lambda) = \frac{1}{(2\pi)^{2n}}\frac{1}{|\lambda|^{2\gamma}}\frac{1}{\left(1+|\lambda|^2\right)^{\alpha}}, \quad 0 < \gamma < n, \quad \alpha \geq 0, \quad \lambda \in \mathbb{R}^n. \tag{1.5}$$

The interpretation of the form (1.5) and the bounds on γ, α are verified in Section 2. It is noted that the exponent γ determines the degree of SS and possible LRD, while the exponent α indicates the extent of intermittency of the random field. When $\alpha = 0$, the spectral density (1.5) reduces to that of fBm.

This paper will obtain an approximate solution to (1.4) for the class of fRBm. This in fact will give an extension to the problem considered in Chapter 1 (Theorem 13 of Subsection 3, pp. 28–30) of [11], where $f(\lambda) \sim A(1+\lambda^2)^{-\beta}$ as $|\lambda| \to \infty$, $A =$

constant > 0, $\lambda \in \mathbb{R}$ and $\beta > 0$ is an integer. When β is assumed to be an integer, an approximation based on the rational form of the spectrum and the theory of Sobolev spaces of integer order can be applied, as detailed in [11]. On the other hand, the exponents γ and α of (1.5) are positive real numbers, hence Ramm's results are not directly applicable. Our approach is based on the theory of Sobolev spaces of fractional order and, in particular, the theory of Riesz and Bessel potentials.

The necessary results will be developed in Section 2. In particular, we shall establish that the covariance operator corresponding to (1.5) is a continuous isomorphism between the fractional Sobolev spaces $H^{-(\alpha+\gamma)}(T)$ and $\overline{H}^{\alpha+\gamma}(T)$ (defined in Section 2). This key result provides a solution method for problem (1.4). Its proof is given in Section 3. Section 4 will outline an approximate solution to (1.4) by a least squares method. Some comments on its implementation will then be given.

2. The covariance operator

In this section, we obtain the covariance operator corresponding to (1.5) on appropriate Sobolev spaces. These spaces are constructed from the spaces of C^∞-functions with compact support in $\mathbb{R}^n$, $\mathscr{D}(\mathbb{R}^n)$ and the space of C^∞-functions with rapid decay at infinity, $\mathscr{S}(\mathbb{R}^n)$. The duals of these spaces are respectively the space of distributions, $\mathscr{D}'(\mathbb{R}^n)$, and the space of tempered distributions, $\mathscr{S}'(\mathbb{R}^n)$.

DEFINITION 2.1. For $s \in \mathbb{R}$, $H_p^s(\mathbb{R}^n)$ is the space of tempered distributions u such that

$$(1 + |\xi|^2)^{s/2}\hat{u} \in L^p(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

In this space, we use the inner product

$$(u, v)_s = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi,$$

with associated norm

$$\|u\|_s = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2\, d\xi\right)^{1/2}.$$

We shall consider the case $p = 2$, that is, $H_2^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

The definition of fractional Sobolev spaces can also be given in terms of Bessel potentials and their inverse operators (see, for example, Stein [14]). This approach plays a key role in our work. Let I denotes the identity operator and Δ the Laplacian operator. The integral operator

$$\mathscr{J}_\alpha = (I - \Delta)^{-\alpha/2}$$

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for $\alpha \in \mathbb{R}_+$ is called the Bessel potential of order α, whose kernel I_α is given by

$$I_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/s} e^{-s/4\pi} s^{(-n+\alpha)/2} \frac{ds}{s}.$$

The following proposition gives some fundamental properties of Bessel potentials.

PROPOSITION 2.1. *For each $\alpha \in \mathbb{R}_+$, $I_\alpha(x) \in L^1(\mathbb{R}^n)$ and its Fourier transform is*

$$\hat{I}_\alpha(\lambda) = (2\pi)^{-n/2} \left(1 + |\lambda|^2\right)^{-\alpha/2}, \quad \lambda \in \mathbb{R}^n. \tag{2.1}$$

For $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$,

$$\mathscr{I}_\alpha(f) = I_\alpha * f$$

(the convolution of I_α and f) and

$$I_\alpha * I_\beta = I_{(\alpha+\beta)}.$$

Therefore

$$\mathscr{I}_\alpha \cdot \mathscr{I}_\beta = \mathscr{I}_{(\alpha+\beta)}, \quad \alpha \ge 0, \beta \ge 0.$$

On the other hand, the inverse of $\mathscr{I}_\alpha$ is the operator $\mathscr{I}_{-\alpha} = (I - \Delta)^{\alpha/2}$, for $\alpha \ge 0$.

PROOF. See [14, pp. 130–135].

The potential spaces are defined in terms of Bessel potentials as

$$\mathscr{L}_p^\alpha = \mathscr{I}_\alpha(L^p(\mathbb{R}^n)), \quad \alpha \in \mathbb{R}_+$$

(Stein [14]). These spaces coincide with the spaces $H_p^s(\mathbb{R}^n)$, $s \in \mathbb{R}_+$, given in Definition 2.1.

The definition of Sobolev spaces of fractional or integer order can also be given for functions defined on an open C^∞-bounded domain satisfying certain regularity conditions.

DEFINITION 2.2. Let T be an open C^∞-bounded domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$ and $1 < p < \infty$. We define

$$\overline{H}_p^s(T) = \left\{f \in H_p^s(\mathbb{R}^n);\ \operatorname{supp} f \subseteq \overline{T}\right\}.$$

Again, we shall consider the case $p = 2$, that is, $\overline{H}_2^s(T) = \overline{H}^s(T)$.

We next introduce the definition of subspaces of distributions obtained as restrictions of tempered distributions belonging to the spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

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DEFINITION 2.3. Let T be an open C^∞-bounded domain in $\mathbb{R}^n$. For $s \in \mathbb{R}$, we define

$$H^s(T) = \left\{u \in \mathscr{D}'(T);\ \exists\, U \in H^s(\mathbb{R}^n) \text{ with } u = U_T\right\}$$

where U_T denotes the restriction of U to T. With the quotient norm

$$\|u\|_{H^s(T)} = \inf_{\{U; U_T = u\}} \|U\|_{H^s(\mathbb{R}^n)},$$

$H^s(T)$ is a Hilbert space (Dautray and Lions [3, p. 118]).

The spaces given by Definitions 2.2 and 2.3 are related by duality (see Triebel [15, p. 332]), that is, $[\overline{H}^s(T)]^* = H^{-s}(T)$, $s \in \mathbb{R}_+$, where H^* is the dual space of the Hilbert space H. In the following proposition, we show the relationship between an element $u \in \overline{H}^s(T)$, $s \in \mathbb{R}_+$ (respectively, $f \in H^{-s}(T)$, $s \in \mathbb{R}_+$), and its dual $u^* \in H^{-s}(T)$ (respectively, $f^* \in \overline{H}^s(T)$, $s \in \mathbb{R}_+$), *via* Bessel potentials (see [3] for the integer case).

PROPOSITION 2.2. *The dual of $\overline{H}^s(T)$, the space $H^{-s}(T)$, $s \in \mathbb{R}_+$, is algebraically and topologically equivalent to the space $\mathscr{J}_{2s}(\overline{H}^s(T))$. Also the dual of $H^{-s}(T)$, the space $\overline{H}^s(T)$, $s \in \mathbb{R}_+$, is algebraically and topologically equivalent to the space $\mathscr{J}_{-2s}(H^{-s}(T))$.*

PROOF. See Ruiz-Medina *et al.* [13].

It can also be proved that the quotient space $H^s(T)$, $s \in \mathbb{R}$, can be identified with the orthogonal complement in $H^s(\mathbb{R}^n)$ of the class of distributions $u \in H^s(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$ whose restriction to T, u_T, is the null distribution in $\mathscr{D}'(T)$. Therefore, in the following development, we consider $H^s(T)$ as a subspace of $H^s(\mathbb{R}^n)$ and we also have the following inclusions for Sobolev spaces defined on an open C^∞-bounded domain $T \subseteq \mathbb{R}^n$.

For $s_1 \geq s_2 \geq 0$,

$$\begin{aligned}\mathscr{D}(T) \subseteq \overline{H}^{s_1}(T) \subseteq \overline{H}^{s_2}(T) \subseteq \cdots \subseteq L^2(T)\\ \subseteq \cdots \subseteq H^{-s_2}(T) \subseteq H^{-s_1}(T) \subseteq \mathscr{D}'(T).\end{aligned} \tag{2.2}$$

In the development of this paper, the theory of Riesz potentials also plays a key role. We recall that the Riesz potential is defined by $\mathscr{I}_\gamma = (-\Delta)^{-\gamma/2}$, $0 < \gamma < n$. Then, for $f \in \mathscr{S}(\mathbb{R}^n)$,

$$\mathscr{I}_\gamma(f)(x) = \frac{1}{g(\gamma)} \int_{\mathbb{R}^n} |x-y|^{\gamma-n} f(y)\, dy = (J_\gamma * f)(x),$$

where

$$g(\gamma) = \frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma(n/2 - \gamma/2)}$$

and

$$J_{\gamma}(t) = \frac{|t|^{\gamma - n}}{g(\gamma)}$$

is the Riesz kernel, whose Fourier transform is

$$\widehat{J_{\gamma}}(\lambda) = (2\pi)^{-n/2} |\lambda|^{-\gamma}, \quad \lambda \in \mathbb{R}^n \tag{2.3}$$

(Stein [14, p. 117]; Donoghue [4, p. 292]).

We now define the random field which is characterised by the spectral density (1.5) and determine its covariance operator.

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space and let $\mathscr{L}^2(\Omega, \mathscr{A}, P)$ be the Hilbert space of real-valued zero-mean random variables defined on $(\Omega, \mathscr{A}, P)$ with finite second-order moments and inner product defined by

$$\langle X, Y \rangle_{\mathscr{L}^2(\Omega)} = E[XY], \quad X, Y \in \mathscr{L}^2(\Omega, \mathscr{A}, P).$$

DEFINITION 2.4. For $\alpha \in \mathbb{Q}$, a random function $X_{\alpha}(\cdot)$ from $U_{\alpha} = \overline{H}^{\alpha}(T)$ into $\mathscr{L}^2(\Omega, \mathscr{A}, P)$ is said to be an α-generalised random field (α-GRF) if it is linear and continuous in the mean-square sense with respect to the U_{α}-topology.

We will denote by $H(X_{\alpha})$ the closed span of $\{X_{\alpha}(\varphi) : \varphi \in U_{\alpha}\}$, and by $\mathscr{H}(X_{\alpha})$ the closed span of $\{B_{\alpha}(\phi, \cdot) = E[X_{\alpha}(\phi) X_{\alpha}(\cdot)] : \phi \in U_{\alpha}\}$, which is the reproducing kernel Hilbert space (RKHS) of X_{α}. The topologies in these spaces are defined from the $\mathscr{L}^2(\Omega)$-topology.

The following concept of duality relative to a fractional Sobolev space $U_{\alpha}, \alpha \in \mathbb{Q}$, plays a key role in a study of α-GRFs.

DEFINITION 2.5. For $\alpha \in \mathbb{Q}$, we say that an α−GRF $\widetilde{X}_{\alpha} : [U_{\alpha}]^* \to \mathscr{L}^2(\Omega, \mathscr{A}, P)$ is the dual relative to U_{α} (or α-dual) of the α-GRF $X_{\alpha} : U_{\alpha} \to \mathscr{L}^2(\Omega, \mathscr{A}, P)$ if

(i) $H(X_{\alpha}) = H(\widetilde{X}_{\alpha})$; and

(ii) $\langle X(\phi), \widetilde{X}(g) \rangle_{H(X)} = \langle \phi, g^* \rangle_{U_{\alpha}}$, for $\phi \in U_{\alpha}$ and $g \in [U_{\alpha}]^*$, with g^* being the dual element of g with respect to the U_{α}-topology.

Conversely, the dual of $\widetilde{X}_{\alpha}$ relative to $[U_{\alpha}]^*$ is the GRF X_{α}.

In a parallel way to the case of GRF X_{α}, we consider for its α-dual GRF $\widetilde{X}_{\alpha}$ the definition of the spaces $H(\widetilde{X}_{\alpha})$ and $\mathscr{H}(\widetilde{X}_{\alpha})$ as the closed spans of

$$\{\widetilde{X}_{\alpha}(g) : g \in [U_{\alpha}]^*\} \quad \text{and} \quad \{\widetilde{B}_{\alpha}(g, \cdot) = E[\widetilde{X}_{\alpha}(g) \widetilde{X}_{\alpha}(\cdot)] : g \in [U_{\alpha}]^*\}$$

respectively in the $\mathscr{L}^2(\Omega)$-topology.

The existence of the β-dual GRF $\widetilde{X}_\beta$, for some $\beta \geq \alpha$, allows us to derive the covariance factorisation of X_α on the space U_β, where the dual GRF exists (Ruiz-Medina *et al.* [13]). We denote by $\widetilde{X}_{\widetilde{\alpha}}$ the $\widetilde{\alpha}$-dual GRF of X_α with $\widetilde{\alpha} \geq \alpha$ being the minimum fractional order for which the dual GRF of X_α relative to U_β with $\beta \geq \alpha$ exists. We call this order $\widetilde{\alpha}$ the *minimum fractional duality order* of X_α.

This covariance factorisation is the basis for an abstract representation of $X_{\widetilde{\alpha}}$ in terms of generalised white noise.

DEFINITION 2.6. A generalised random field $\varepsilon(\cdot)$ defined on a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ is called a generalised white noise (GWN) relative to H if

$$\langle \varepsilon(u), \varepsilon(v) \rangle_{H(\varepsilon)} = \langle u, v \rangle_H \quad \forall u, v \in H.$$

DEFINITION 2.7. A generalised random field X defined on a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ is said to have a weak-sense abstract representation if there exists an isomorphism $L : H \to H$ such that

$$\langle XL(u), XL(v) \rangle_{H(X)} = \langle u, v \rangle_H \quad \forall u, v \in H,$$

that is, if $\varepsilon \equiv XL$ is a GWN relative to H. This abstract representation is written as

$$X(Lu) = \varepsilon(u) \quad \forall u \in H.$$

PROPOSITION 2.3. *Assuming the existence of the $\widetilde{\alpha}$-dual GRF $\widetilde{X}_{\widetilde{\alpha}}$ of the α-GRF X_α, with $\widetilde{\alpha}$ being the minimum fractional duality order of X_α, the restriction $X_{\widetilde{\alpha}}$ to $U_{\widetilde{\alpha}}$ of X_α has a weak-sense abstract representation, which is unique except for isometric isomorphisms.*

PROOF. See Ruiz-Medina *et al.* [13].

The above abstract representation is written as

$$X_{\widetilde{\alpha}}(L\phi) = \varepsilon_{\widetilde{\alpha}}(\phi) \quad \forall \phi \in U_{\widetilde{\alpha}}. \tag{2.4}$$

Then

$$\langle X_{\widetilde{\alpha}}(L\phi), X_{\widetilde{\alpha}}(L\psi) \rangle_{H(X_{\widetilde{\alpha}})} = \langle \varepsilon_{\widetilde{\alpha}}(\phi), \varepsilon_{\widetilde{\alpha}}(\psi) \rangle_{H(X_{\widetilde{\alpha}})} = \langle \phi, \psi \rangle_{U_{\widetilde{\alpha}}} \quad \forall \phi, \psi \in U_{\widetilde{\alpha}}. \tag{2.5}$$

Since

$$\langle \phi, \psi \rangle_{U_{\widetilde{\alpha}}} = \langle \mathscr{I}_{-\widetilde{\alpha}}(\phi), \mathscr{I}_{-\widetilde{\alpha}}(\varphi) \rangle_{L^2(T)} \quad \forall \phi, \psi \in U_{\widetilde{\alpha}},$$

where $\mathscr{I}_{-\widetilde{\alpha}}$ is the inverse of the Bessel potential $\mathscr{I}_{\widetilde{\alpha}}$ defined earlier, we also obtain the following interpretation of the representation (2.4):

$$X_{\widetilde{\alpha}}(L\varphi) = \varepsilon_{L^2(T)} \mathscr{I}_{-\widetilde{\alpha}}(\varphi) \quad \forall \varphi \in U_{\widetilde{\alpha}}, \tag{2.6}$$

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where $\varepsilon_{L^2(T)}(\cdot)$ is a GWN relative to $L^2(T)$. The right-hand side of (2.4) represents the weak-sense derivative of fractional order $\widetilde{\alpha}$ of a GWN in $L^2(T)$.

In a similar fashion, the condition of Proposition 2.3 also implies a unique abstract representation for the $\widetilde{\alpha}$-dual GRF $\widetilde{X}_{\widetilde{\alpha}}$:

$$\widetilde{X}_{\widetilde{\alpha}}\left(\widetilde{L}g\right) = \widetilde{\varepsilon}_{\widetilde{\alpha}}(g) \quad \forall g \in [U_{\widetilde{\alpha}}]^* \tag{2.7}$$

with $\widetilde{L} = R_{\widetilde{\alpha}} L I_{[U_{\widetilde{\alpha}}]^*}$, $R_{\widetilde{\alpha}}$ being the covariance operator of $X_{\widetilde{\alpha}}$, L the isomorphism defining the abstract representation of $X_{\widetilde{\alpha}}$, and $I_{[U_{\widetilde{\alpha}}]^*} : [U_{\widetilde{\alpha}}]^* \to U_{\widetilde{\alpha}}$ the isometric isomorphism defined by the Riesz representation theorem (Ruiz-Medina *et al.* [13]).

Similarly to (2.6), we can also write (2.7) alternatively as

$$\widetilde{X}_{\widetilde{\alpha}}\left(\widetilde{L}g\right) = \widetilde{\varepsilon}_{L^2(T)}\mathscr{I}_{\widetilde{\alpha}}(g) \quad \forall g \in [U_{\widetilde{\alpha}}]^*. \tag{2.8}$$

The right-hand side of (2.8) is interpreted as a fractional integral in the weak sense of a GWN relative to $L^2(T)$.

The abstract representation (2.8) can be equivalently expressed as

$$\widetilde{X}_{\widetilde{\alpha}}(g) = \widetilde{\varepsilon}_{L^2(T)}\left[\mathscr{I}_{\widetilde{\alpha}}\left(\widetilde{L}^{-1}g\right)\right] \quad \forall g \in [U_{\widetilde{\alpha}}]^*.$$

The generalised covariance function $\widetilde{B}_{\widetilde{\alpha}}$ of the $\widetilde{\alpha}$-GRF $\widetilde{X}_{\widetilde{\alpha}}$ then takes the form

$$\begin{aligned}\widetilde{B}_{\widetilde{\alpha}}(g,h) &= E\left[\widetilde{X}_{\widetilde{\alpha}}(g)\widetilde{X}_{\widetilde{\alpha}}(h)\right] = \left\langle \mathscr{I}_{\widetilde{\alpha}}\widetilde{L}^{-1}(g)(\cdot), \mathscr{I}_{\widetilde{\alpha}}\widetilde{L}^{-1}(h)(\cdot)\right\rangle_{L^2(T)} \\ &= \left\langle g^*(\cdot), \left[\widetilde{L}^{-1}\right]^* \mathscr{I}_{\widetilde{\alpha}}^* \mathscr{I}_{\widetilde{\alpha}}\widetilde{L}^{-1}(h)\right\rangle_{U_{\widetilde{\alpha}}} = \left\langle g^*(\cdot), \widetilde{R}_{\widetilde{\alpha}}(h)\right\rangle_{U_{\widetilde{\alpha}}} \qquad \forall g,h \in [U_{\widetilde{\alpha}}]^*.\end{aligned}$$

Consequently, the covariance operator $\widetilde{R}_{\widetilde{\alpha}}$ of $\widetilde{X}_{\widetilde{\alpha}}$ is given by

$$\widetilde{R}_{\widetilde{\alpha}} = \left[\mathscr{I}_{\widetilde{\alpha}}\widetilde{L}^{-1}\right]^*\left[\mathscr{I}_{\widetilde{\alpha}}\widetilde{L}^{-1}\right]. \tag{2.9}$$

We consider $\widetilde{\alpha} = \alpha + \gamma$, $\alpha, \gamma \in \mathbb{Q}$, in Proposition 2.3. Then the covariance operator of (2.9) becomes

$$\widetilde{R}_{\alpha+\gamma} = \left[\mathscr{I}_{\alpha+\gamma}\widetilde{L}^{-1}\right]^*\left[\mathscr{I}_{\alpha+\gamma}\widetilde{L}^{-1}\right]. \tag{2.10}$$

Here

$$\widetilde{L} = R_{\alpha+\gamma} L I_{[U_{\alpha+\gamma}]^*} = S_{\alpha+\gamma}\mathscr{I} I_{[U_{\alpha+\gamma}]^*},$$

with

$$S_{\alpha+\gamma} : H(X_{\alpha+\gamma}) \to [U_{\alpha+\gamma}]^*, \qquad \mathscr{I} : U_{\alpha+\gamma} \to H(X_{\alpha+\gamma}) = H(\widetilde{X}_{\alpha+\gamma})$$

and

$$I_{[U_{\alpha+\gamma}]^*} : U_{\alpha+\gamma}^* \to U_{\alpha+\gamma}$$

being isomorphisms. Therefore $\widetilde{L}^{-1} = I^{-1}_{[U_{\alpha+\gamma}]^*}\mathscr{I}^{-1}S^{-1}_{\alpha+\gamma}$. Also, it follows from Proposition 2.2 that $I^{-1}_{[U_{\alpha+\gamma}]^*} = \mathscr{I}_{-(2\alpha+2\gamma)}$. Thus $\widetilde{L}^{-1} = \mathscr{I}_{-(2\alpha+2\gamma)}\mathscr{I}^{-1}S^{-1}_{\alpha+\gamma}$.

In particular, we shall consider

$$S^{-1}_{\alpha+\gamma} = \mathscr{I}\mathscr{I}_{2\alpha+\gamma}\mathscr{J}^*_{\gamma}, \tag{2.11}$$

with $\mathscr{J}_{\gamma}(g)$, $g \in U_\alpha$ and $\mathscr{I}_{2\alpha+\gamma}(\varphi)$, $\varphi \in [U_{\alpha+\gamma}]^*$, respectively defined as

$$\mathscr{J}_{\gamma}(g)(\phi) = (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\widehat{g}(\lambda)\widehat{\phi}(\lambda)}{|\lambda|^{\gamma}}\,d\lambda \quad \forall\phi \in [U_{\alpha+\gamma}]^*$$

and

$$\mathscr{I}_{2\alpha+\gamma}(\varphi)(\psi) = (2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\widehat{\varphi}(\lambda)\widehat{\psi}(\lambda)}{(1+|\lambda|^2)^{\alpha+\gamma/2}}\,d\lambda \quad \forall\psi \in U_\alpha.$$

With the above choice of $S_{\alpha+\gamma}$, we have

$$\widetilde{R}_{\alpha+\gamma} = [\mathscr{I}_\alpha\mathscr{J}^*_\gamma]^*\mathscr{I}_\alpha\mathscr{J}^*_\gamma = \mathscr{J}_\gamma\mathscr{I}^*_\alpha\mathscr{I}_\alpha\mathscr{J}^*_\gamma, \tag{2.12}$$

which, in view of (2.1) and (2.3), is understood in the following weak sense:

$$\int_{\mathbb{R}^n}\widetilde{R}_{\alpha+\gamma}(h)(x)\overline{g(x)}dx = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\frac{\widehat{h}(\lambda)}{|\lambda|^{2\gamma}(1+|\lambda|^2)^{\alpha}}\,\widehat{g}(\lambda)\,d\lambda \tag{2.13}$$

for $h, g \in H^{-(\alpha+\gamma)}(T)$.

From (2.13), the spectral density of the random field resultant from the above selection has the form

$$f(\lambda) = \frac{1}{(2\pi)^{2n}}\frac{1}{|\lambda|^{2\gamma}}\frac{1}{\left(1+|\lambda|^2\right)^{\alpha}}, \quad 0 < \gamma < n,\ \alpha \geq 0, \tag{2.14}$$

interpreted in the weak sense.

Since this form involves the Fourier transforms of the Riesz kernel and the Bessel kernel, these random fields are named fractional Riesz-Bessel motion (fRBm). We shall prove the following proposition in the next section.

PROPOSITION 2.4. *For $0 < \gamma < n$, $\alpha \geq 0$, the operator $\widetilde{R}_{\alpha+\gamma}$ defined by (2.12) is a continuous isomorphism from $H^{-(\alpha+\gamma)}(T)$ to $\overline{H}^{\alpha+\gamma}(T)$ whose inverse $\widetilde{R}^{-1}_{\alpha+\gamma}$ is also continuous.*

REMARK 2.1. From the remark after (9.5.10), p. 660, and Theorem 9.5.6(a), p. 658, of Edwards [5], $\mathscr{J}_\gamma$ is compact for $n/2 < \gamma < n$. Therefore, from Lemma 3.2 below, $\mathscr{I}_\alpha\mathscr{J}^*_\gamma$ is continuous for $0 < \gamma < n, \alpha \geq 0$, and since $\mathscr{I}_\alpha$ is continuous for $\alpha \geq 0$, the operator $\widetilde{R}^{-1}_{\alpha+\gamma} = \mathscr{J}_\gamma\,\mathscr{I}^*_\alpha\mathscr{I}_\alpha\mathscr{J}^*_\gamma$ is compact for $n/2 < \gamma < n$ and $\alpha \geq 0$.

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REMARK 2.2. Proposition 2.4 implies that, for any $g \in \overline{H}^{\alpha+\gamma}(T)$, (1.4) has a unique solution in $H^{-(\alpha+\gamma)}(T)$ and this solution depends continuously on $g \in \overline{H}^{\alpha+\gamma}(T)$ in the norm of $H^{-(\alpha+\gamma)}(T)$. Furthermore, the problem of solving (1.4) in $H^{-(\alpha+\gamma)}(T)$ is well-posed since the operator $R_{\alpha+\gamma}^{-1} : \overline{H}^{\alpha+\gamma}(T) \to H^{-(\alpha+\gamma)}(T)$ is defined on all of $\overline{H}^{\alpha+\gamma}(T)$ and is continuous there.

3. Proof of the main result

Proof of Proposition 2.4 is given in the following lemmas.

LEMMA 3.1. *For $0 < \gamma < n$, the Riesz potential $\mathscr{J}_\gamma = (-\Delta)^{-\gamma/2}$ is a continuous operator on $L^2(T)$.*

PROOF. See [5, Theorem 9.5.10(a), p. 660].

LEMMA 3.2. *For $\alpha \geq 0$ and $0 < \gamma < n$, the operators $\widetilde{R}_{\alpha+\gamma} : H^{-(\alpha+\gamma)}(T) \to \overline{H}^{\alpha+\gamma}(T)$ and $\widetilde{R}_{\alpha+\gamma}^{-1} : \overline{H}^{\alpha+\gamma}(T) \to H^{-(\alpha+\gamma)}(T)$ are continuous.*

PROOF. We prove the continuity of $\mathscr{I}_\alpha \mathscr{J}_\gamma^*$, as an operator from $H^{-(\alpha+\gamma)}(T)$ into $L^2(T)$, and its inverse $(\mathscr{I}_\alpha \mathscr{J}_\gamma^*)^{-1}$, which from (2.12) implies the statement of the lemma.

It follows from (2.13) that

$$\left\| \mathscr{I}_\alpha \mathscr{J}_\gamma^*(h) \right\|_{L^2(T)}^2 = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \frac{|\widehat{h}(\lambda)|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^\alpha} \, d\lambda, \tag{3.1}$$

for $h \in H^{-(\alpha+\gamma)}(T)$.

Since

$$|\lambda|^{2\gamma} \leq \left(1+|\lambda|^2\right)^\gamma \quad \forall \lambda \in \mathbb{R}^n,$$

the inequality

$$\|h\|_{H^{-(\alpha+\gamma)}(T)}^2 = \int_{\mathbb{R}^n} \frac{\left|\widehat{h}(\lambda)\right|^2}{\left(1+|\lambda|^2\right)^{\alpha+\gamma}} \, d\lambda \leq \int_{\mathbb{R}^n} \frac{\left|\widehat{h}(\lambda)\right|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^\alpha} \, d\lambda = \left\| \mathscr{I}_\alpha \mathscr{J}_\gamma^*(h) \right\|_{L^2(T)}^2$$

holds for $h \in H^{-(\alpha+\gamma)}(T)$. That is, the operator $(\mathscr{I}_\alpha \mathscr{J}_\gamma^*)^{-1}$is continuous.

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To prove the continuity of $\mathscr{I}_\alpha \mathscr{J}_\gamma^*$, we decompose the integral in (3.1) as

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\left|\widehat{h}(\lambda)\right|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^\alpha}\,d\lambda &= \int_{\mathbb{R}^n} \frac{\left(1+|\lambda|^2\right)^\gamma \left|\widehat{h}(\lambda)\right|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^{\alpha+\gamma}}\,d\lambda \\ &= \int_{\varepsilon_M(0)} \frac{\left(1+|\lambda|^2\right)^\gamma \left|\widehat{h}(\lambda)\right|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^{\alpha+\gamma}}\,d\lambda + \int_{\mathbb{R}^n-\varepsilon_M(0)} \frac{\left(1+|\lambda|^2\right)^\gamma \left|\widehat{h}(\lambda)\right|^2}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^{\alpha+\gamma}}\,d\lambda \\ &= I_1 + I_2,\end{aligned}$$

where $\varepsilon_M(0) = \{\lambda \in \mathbb{R}^n : |\lambda| \le M\}$, $M > 0$.

From Lemma 3.1, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{\left|\widehat{g}(\lambda)\right|^2}{|\lambda|^{2\gamma}}\,d\lambda \le C \int_{\mathbb{R}^n} \left|\widehat{g}(\lambda)\right|^2 d\lambda,$$

for $g \in L^2(T)$. Hence,

$$I_1 \le (1+M^2)C \int_{\mathbb{R}^n} \frac{\left|\widehat{h}(\lambda)\right|^2}{\left(1+|\lambda|^2\right)^{\alpha+\gamma}}\,d\lambda = M' \|h\|^2_{H^{-(\alpha+\gamma)}(T)}. \tag{3.2}$$

Now, as there exists $M'' > 0$ such that

$$\left(\frac{1+|\lambda|^2}{|\lambda|^2}\right)^\gamma \le M''$$

for $\lambda \in \mathbb{R}^n - \varepsilon_M(0)$, we have

$$I_2 \le M'' \int_{\mathbb{R}^n-\varepsilon_M(0)} \frac{\left|\widehat{h}(\lambda)\right|^2}{\left(1+|\lambda|^2\right)^{\alpha+\gamma}}\,d\lambda \le M'' \|h\|^2_{H^{-(\alpha+\gamma)}(T)}. \tag{3.3}$$

The continuity of $\mathscr{I}_{-\alpha} \mathscr{J}_\gamma^*$ follows from (3.2) and (3.3).

LEMMA 3.3. *$\widetilde{R}_{\alpha+\gamma} : H^{-(\alpha+\gamma)}(T) \to \overline{H}^{\alpha+\gamma}(T)$ is an onto mapping for $\alpha \ge 0$ and $0 < \gamma < n$.*

PROOF. By definition, $\widetilde{R}_{\alpha+\gamma}$ is a self-adjoint operator on $L^2(T)$. Also, the range of $\widetilde{R}_{\alpha+\gamma}$ coincides with the orthogonal complement of the null space of $\widetilde{R}^*_{\alpha+\gamma} = \widetilde{R}_{\alpha+\gamma}$ (see [7, Theorem 6.5.10(i), p. 164]). Since $\widetilde{R}_{\alpha+\gamma}$ is injective, we have Range$(\widetilde{R}_{\alpha+\gamma}) = L^2(T)$. Additionally, in view of (2.2), $\overline{H}^{\alpha+\gamma}(T) \subseteq L^2(T) \subseteq H^{-(\alpha+\gamma)}(T)$. Hence, $R_{\alpha+\gamma}(H^{-(\alpha+\gamma)}(T)) \supseteq R_{\alpha+\gamma}(L^2(T)) = L^2(T) \supseteq \overline{H}^{\alpha+\gamma}(T)$.

Proposition 2.4 now follows from Lemmas 3.1–3.3.

4. Approximate solution

For the given equation

$$Rh = g, \tag{4.1}$$

let us consider an approximation using least squares.

Let $\{\psi_j, j = 1, 2, 3, \dots\}$ be a complete system of linearly independent elements of $H^{-(\alpha+\gamma)}(T)$ and let $M_m = \overline{Sp}\{\psi_1, \dots, \psi_m\}$. Also let $h_m \in M_m$ be a solution to

$$\|Rh_m - g\|_{H^{-(\alpha+\gamma)}(T)} = \min. \tag{4.2}$$

By the least squares principle and since M_m is finite-dimensional, (4.2) has a unique solution in M_m. Also,

$$\|h_m - h\|_{H^{-(\alpha+\gamma)}(T)} \to 0 \quad \text{as} \quad m \to \infty, \tag{4.3}$$

where $h = R^{-1}g \in H^{-(\alpha+\gamma)}(T)$. In fact, since R^{-1} is bounded, there exists a finite constant c such that

$$\|h_m - h\|_{H^{-(\alpha+\gamma)}(T)} \le c\|Rh_m - g\|_{\overline{H}^{\alpha+\gamma}(T)}. \tag{4.4}$$

Now, the completeness of $\{\psi_j\}$ in $H^{-(\alpha+\gamma)}(T)$ implies that $\{R\psi_j\}$ is complete in $\overline{H}^{\alpha+\gamma}(T)$ since R is an isomorphism between $H^{-(\alpha+\gamma)}(T)$ and $\overline{H}^{\alpha+\gamma}(T)$. Hence the right-hand side of (4.4) tends to 0 as $m \to \infty$.

EXAMPLE 4.1. Let us consider the case $n = 1$, $\alpha + \gamma = 1$, $H_0 = L^2(-1, 1)$. Let p be an isometric isomorphism between H_0 and $\overline{H}^1(-1, 1)$, and $q = p^{-1}$. For a Schauder basis $\{\phi_j\}$ of H_0, put $h_m = \sum_{j=1}^m c_j^{(m)}\phi_j$. We shall consider an approximate solution of the form

$$h_m^A = h_m + c_{-1}\delta(x+1) + c_{-2}\delta(x-1). \tag{4.5}$$

The constants $c_j^{(m)}$, $j = -2, -1, 0, \dots, m$, are obtained by solving the problem

$$\|Rh_m^A - g\|_{\overline{H}^1(-1,1)} = \min. \tag{4.6}$$

The variational problem (4.6) can be written as

$$\varepsilon_m = \int_{-1}^1 \left[\left| \sum_{j=-2}^m c_j^{(m)}\psi_j(x) - g(x) \right|^2 + \left| \sum_{j=-2}^m c_j^{(m)}\psi_j'(x) - g'(x) \right|^2 \right] dx = \min,$$

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where $\psi_j(x) = R\phi_j, j = -2, -1, 0, \dots, m$. The linear system

$$\frac{\partial \varepsilon_m}{\partial c_j^{(m)}} = 0, \quad j = -2, -1, 0, \dots, m,$$

for $c_j^{(m)}$ is then

$$\sum_{j=-2}^{m} M_{ij} c_j = b_i, \quad i = -2, -1, 0, \dots, m, \tag{4.7}$$

where $M_{ij} = \langle \psi_j, \psi_i \rangle_{\overline{H}^1}$, $b_i = \langle g, \psi_i \rangle_{\overline{H}^1}$, $i, j = -2, -1, 0, \dots, m$. The linearly independent system $\{\psi_j\}$ is complete in $\overline{H}^1$. Also, the matrix M_{ij} is symmetric and positive definite for any m. Hence the system (4.7) is uniquely solvable for each m.

Liu and Anh [8] propose the use of Filon's method with parameter as a suitable numerical integration scheme for computing the matrix M and the right-hand side vector b of (4.7). The numerical results reported in [8] indicate a good performance of the method.

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