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Homological vanishing for the Steinberg representation

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Abstract

For a field k, we prove that the ith homology of the groups $\mathrm{GL}_n(\mathrm{k})$, $\mathrm{SL}_n(\mathrm{k})$, $\mathrm{Sp}_{2n}(\mathrm{k})$, $\mathrm{SO}_{n,n}(\mathrm{k})$, and $\mathrm{SO}_{n,n+1}(\mathrm{k})$ with coefficients in their Steinberg representations vanish for $n \geqslant 2i+2$.

1. Introduction

Let $\mathbf{G}$ be a connected reductive group over a field k. A basic geometric object associated to $\mathbf{G}(\mathrm{k})$ is its Tits building. By definition, this is the simplicial complex $\mathcal{T}_{\mathbf{G}}(\mathrm{k})$ whose i-simplices are increasing sequences

$$0 \subsetneq P_0 \subsetneq \cdots \subsetneq P_i \subsetneq \mathbf{G}(\mathrm{k})$$

of parabolic k-subgroups of $\mathbf{G}(\mathrm{k})$. Letting r be the semisimple k-rank of $\mathbf{G}$, the complex $\mathcal{T}_{\mathbf{G}}(\mathrm{k})$ is $(r-1)$-dimensional, and the Solomon–Tits theorem [Bro89, Theorem IV.5.2] says that in fact $\mathcal{T}_{\mathbf{G}}(\mathrm{k})$ is homotopy equivalent to a wedge of $(r-1)$-dimensional spheres. Letting R be a commutative ring, the *Steinberg representation* of $\mathbf{G}(\mathrm{k})$ over R, denoted $\mathrm{St}_{\mathbf{G}}(\mathrm{k};R)$, is $\widetilde{\mathrm{H}}_{r-1}(\mathcal{T}_{\mathbf{G}}(\mathrm{k});R)$. This is one of the most important representations of $\mathbf{G}(\mathrm{k})$; for instance, if $\mathbf{G}$ is any of the classical groups in Theorem 1.1 below (e.g. $\mathbf{G}=\mathrm{SL}_n$) and k is a finite field of characteristic p, then $\mathrm{St}_{\mathbf{G}}(\mathrm{k};\mathbb{C})$ is the unique irreducible representation of $\mathbf{G}(\mathrm{k})$ whose dimension is a positive power of p (see [MZ01], which proves this aside from three small cases that must be checked by hand). See [Hum87] for a survey of many results concerning the Steinberg representation.

The twisted homology groups $\mathrm{H}_i(\mathbf{G}(\mathrm{k});\mathrm{St}_{\mathbf{G}}(\mathrm{k};R))$ play an interesting role in algebraic K-theory; see [Qui72, Theorem 3]. If $\mathbf{G}(\mathrm{k})$ is a finite group of Lie type, then $\mathrm{St}_{\mathbf{G}}(\mathrm{k};\mathrm{k})$ is a projective $\mathbf{G}(\mathrm{k})$-module (see [Hum87]), and thus the homology groups $\mathrm{H}_i(\mathbf{G}(\mathrm{k});\mathrm{St}_{\mathbf{G}}(\mathrm{k};\mathrm{k}))$ all vanish. However, it is definitely not the case that $\mathrm{St}_{\mathbf{G}}(\mathrm{k};R)$ is projective for a general commutative ring R, and if k is an infinite field then $\mathrm{St}_{\mathbf{G}}(\mathrm{k};\mathrm{k})$ need not be projective. Our main theorem says that, nevertheless, for the classical groups, the homology groups $\mathrm{H}_i(\mathbf{G}(\mathrm{k});\mathrm{St}_{\mathbf{G}}(\mathrm{k};R))$ always vanish in a stable range.

Theorem 1.1. *Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Then for all fields* k *and all commutative rings R, we have $\mathrm{H}_i(\mathbf{G}_n(\mathrm{k});\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R))=0$ for $n \geqslant 2i+2$. Furthermore, there exists a surjection $\mathrm{H}_i(\mathbf{G}_{2i}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{2i}}(\mathrm{k};R)) \to \mathrm{H}_i(\mathbf{G}_{2i+1}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{2i+1}}(\mathrm{k};R))$.*

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Remark 1.2. When $\mathrm{char}(\mathrm{k}) = 2$, the groups $\mathrm{SO}_{n,n}(\mathrm{k})$ and $\mathrm{SO}_{n,n+1}(\mathrm{k})$ in Theorem 1.1 are to be taken naively as the stabilizers of appropriate quadratic forms (see § 2.1 below); we ignore the Dickson invariant.

Theorem 1.1 (and its proof) is motivated by the following conjecture of Church, Farb and Putman [CFP14]. Recall that Borel and Serre [BS73] proved that the virtual cohomological dimension of $\mathrm{SL}_n(\mathbb{Z})$ is $\binom{n}{2}$.

Conjecture 1.3 [CFP14, Conjecture 2]. We have $\mathrm{H}^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0$ for $n \geqslant i + 2$.

In other words, the rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes in codimension i as long as n is sufficiently large. Conjecture 1.3 was proved for $i = 0$ by Lee and Szczarba [LS76] and for $i = 1$ by Church and Putman [CP17]. It is open for $i \geqslant 2$.

To connect Conjecture 1.3 to Theorem 1.1, recall that Borel and Serre [BS73] proved that $\mathrm{SL}_n(\mathbb{Z})$ satisfies a version of Poincaré–Lefschetz duality called virtual Bieri–Eckmann duality. This duality involves a 'dualizing module' that measures the 'homology at infinity'. In our situation, that dualizing module is the Steinberg representation $\mathrm{St}_{\mathrm{SL}_n}(\mathbb{Q}; \mathbb{Q})$ and we have

$$\mathrm{H}^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = \mathrm{H}_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_{\mathrm{SL}_n}(\mathbb{Q}; \mathbb{Q})).$$

Conjecture 1.3 is thus equivalent to the following conjecture, which resembles Theorem 1.1 for $\mathbf{G}_n = \mathrm{SL}_n$.

Conjecture 1.4. We have $\mathrm{H}_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_{\mathrm{SL}_n}(\mathbb{Q}; \mathbb{Q})) = 0$ for $n \geqslant i + 2$.

Remark 1.5. The proofs by Lee and Szczarba [LS76] and Church and Putman [CP17] of special cases of Conjecture 1.3 both start by translating things into the language of Conjecture 1.4.

We now briefly describe our proof of Theorem 1.1. As we will discuss in § 3 below, there is a natural inclusion $\mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$. This induces a stabilization map

$$\mathrm{H}_i(\mathbf{G}_{n-1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k}; R)) \to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)). \tag{1.1}$$

We will show in § 3 that to prove that $\mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) = 0$ for large n, it is enough to prove the seemingly weaker assertion that (1.1) is a surjection for large n. This idea was first introduced by Church, Farb and Putman [CFP14] as a strategy for proving Conjecture 1.4. It was also noticed by Ash in unpublished work.

The surjectivity of (1.1) is a weak form of *homological stability*. There is an enormous literature on homological stability theorems. The basic technique underlying most results in the subject goes back to unpublished work of Quillen. In [Dwy80], Dwyer used these ideas to prove a twisted homological stability theorem for $\mathrm{GL}_n(\mathrm{k})$ with quite general coefficient systems. This work was later generalized by van der Kallen [vdK80] and very recently by Randal-Williams and Wahl [RW15], whose results cover all the classical groups in Theorem 1.1. Unfortunately, the Steinberg representation does *not* satisfy the conditions in any of these known theorems. Indeed, these theorems are general enough that if it did, then this would quickly lead to a proof of Conjecture 1.4. Nevertheless, we are able to use some delicate properties of the Steinberg representation to jury-rig the Quillen machine such that it works to prove that (1.1) is surjective for large n.

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Remark 1.6. Homological stability for a sequence of groups and homomorphisms $X_1 \to X_2 \to \cdots$ states that the induced maps $\mathrm{H}_i(X_n) \to \mathrm{H}_i(X_{n+1})$ are isomorphisms for $n \gg 0$. Alternatively, we can think of each map as 'multiplication by t' and give $\bigoplus_n \mathrm{H}_i(X_n)$ the structure of a $R[t]$-module, where R denotes our coefficient ring. At least when R is a field, this isomorphism would be a consequence of finite generation.

In our setting, with homology twisted by the Steinberg representation, one should instead think of this map as 'multiplication by t' where t is a generator for the exterior algebra in one variable $R[t]/t^2$, so that the groups H_i being 0 for $n \gg 0$ would again be a consequence of finite generation. At least when k is a finite field of size q and R is the field of complex numbers, this is consistent with the idea that $\mathrm{GL}_n(\mathbf{F}_q)$ is a q-analogue of the symmetric group and the Steinberg representation is the q-analogue of its sign representation, which is made more precise via their connection to symmetric functions, see [Mac95, §§ I.7, IV.4].

Outline. We begin in § 2 with some background and notation. Next, in § 3 we reduce Theorem 1.1 to an appropriate homological stability theorem. We then prove a key isomorphism in § 4. We prove Theorem 1.1 in § 5. This proof depends on a calculation which we perform in § 6.

Convention regarding the empty set. If X is the empty set and R is a commutative ring, then we define $\widetilde{\mathrm{H}}_{-1}(X; R) = R$. With this convention, if the semisimple k-rank of $\mathbf{G}$ is 0, then $\mathrm{St}_{\mathbf{G}}(\mathrm{k}; R) = R$ with the trivial $\mathbf{G}(\mathrm{k})$-action.

2. Background and notation

This section contains some background information and notation needed in the remainder of the paper. It consists of two subsections: § 2.1 introduces some distinguished parabolic subgroups, and § 2.2 gives some background about the Steinberg representations.

Throughout this section, k is a field and $\mathbf{G}_n$ is either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$.

2.1 Parabolic and stabilizer subgroups

Our proof of Theorem 1.1 depends on a careful study of various subgroups of $\mathbf{G}_n(\mathrm{k})$. In this section, we will introduce notation for these subgroups: a certain parabolic subgroup $\mathbf{PG}_n^\ell(\mathrm{k})$, its unipotent radical $\mathbf{UG}_n^\ell(\mathrm{k})$, a Levi component $\mathbf{LG}_n^\ell(\mathrm{k})$ of $\mathbf{PG}_n^\ell(\mathrm{k})$, and another subgroup $\mathbf{FG}_n^\ell(\mathrm{k})$ that lies in $\mathbf{PG}_n^\ell(\mathrm{k})$ and fixes certain vectors.

General and special linear groups. Assume first that $\mathbf{G}_n$ is either GL_n or SL_n. The group $\mathbf{G}_n(\mathrm{k})$ thus acts on the vector space k^n, and the k-parabolic subgroups of $\mathbf{G}_n(\mathrm{k})$ are the stabilizers of flags of subspaces of k^n. Let $(\vec{a}_1, \ldots, \vec{a}_n)$ be the standard basis for k^n. For $1 \leqslant \ell \leqslant n$, the group $\mathbf{PG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{G}_n(\mathrm{k})$-stabilizer of the flag

$$0 \subsetneq \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle.$$

The group $\mathbf{UG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathbf{PG}_n^\ell(\mathrm{k})$ consisting of all $M \in \mathbf{PG}_n^\ell(\mathrm{k})$ that act as the identity on both

$$\langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle \quad \text{and} \quad \mathrm{k}^n / \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle.$$

The group $\mathbf{LG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{PG}_n^\ell(\mathrm{k})$-stabilizer of the flag

$$0 \subsetneq \langle \vec{a}_{\ell+1}, \ldots, \vec{a}_n \rangle.$$

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If $\mathbf{G}_n = \mathrm{GL}_n$ then $\mathbf{LG}_n^\ell(\mathrm{k})$ is the subgroup $\mathrm{GL}_\ell(\mathrm{k}) \times \mathrm{GL}_{n-\ell}(\mathrm{k})$ of $\mathbf{G}_n$, while if $\mathbf{G}_n = \mathrm{SL}_n$ then $\mathbf{LG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathrm{GL}_\ell(\mathrm{k}) \times \mathrm{GL}_{n-\ell}(\mathrm{k})$ consisting of matrices of determinant 1. Finally, define

$$\mathbf{FG}_n^\ell(\mathrm{k}) = \{M \in \mathbf{G}_n(\mathrm{k}) | M(\vec{a}_j) = \vec{a}_j \text{ for } 1 \leqslant j \leqslant \ell\}.$$

We thus have $\mathbf{FG}_n^\ell(\mathrm{k}) \subset \mathbf{PG}_n^\ell(\mathrm{k})$.

Symplectic groups. Now assume that $\mathbf{G}_n = \mathrm{Sp}_{2n}$. Letting $\omega(\cdot,\cdot)$ be the standard symplectic form on k^{2n}, the group $\mathbf{G}_n(\mathrm{k})$ is the subgroup of $\mathrm{GL}_n(\mathrm{k})$ consisting of elements that preserve $\omega(\cdot,\cdot)$. The k-parabolic subgroups of $\mathbf{G}_n(\mathrm{k})$ are the $\mathbf{G}_n(\mathrm{k})$-stabilizers of flags of isotropic subspaces of k^{2n}, that is, subspaces on which $\omega(\cdot,\cdot)$ vanishes identically. Let $(\vec{a}_1,\ldots,\vec{a}_n,\vec{b}_1,\ldots,\vec{b}_n)$ be the standard symplectic basis for k^{2n}, so

$$\omega(\vec{a}_j,\vec{a}_{j'}) = \omega(\vec{b}_j,\vec{b}_{j'}) = 0 \quad \text{and} \quad \omega(\vec{a}_j,\vec{b}_{j'}) = \delta_{jj'}$$

for $1 \leqslant j, j' \leqslant n$, where $\delta_{jj'}$ is the Kronecker delta function. For $1 \leqslant \ell \leqslant n$, the group $\mathbf{PG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{G}_n(\mathrm{k})$-stabilizer of the isotropic flag

$$0 \subsetneq \langle \vec{a}_1,\ldots,\vec{a}_\ell\rangle.$$

The group $\mathbf{UG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathbf{PG}_n^\ell(\mathrm{k})$ consisting of all $M \in \mathbf{PG}_n^\ell(\mathrm{k})$ that act as the identity on both

$$\langle \vec{a}_1,\ldots,\vec{a}_\ell\rangle \quad \text{and} \quad \langle \vec{a}_1,\ldots,\vec{a}_\ell\rangle^\perp/\langle \vec{a}_1,\ldots,\vec{a}_\ell\rangle = \langle \vec{a}_1,\ldots,\vec{a}_\ell,\vec{a}_{\ell+1},\vec{b}_{\ell+1},\ldots,\vec{a}_n,\vec{b}_n\rangle/\langle \vec{a}_1,\ldots,\vec{a}_\ell\rangle.$$

The group $\mathbf{LG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{PG}_n^\ell(\mathrm{k})$-stabilizer of the isotropic flag

$$0 \subsetneq \langle \vec{b}_1,\ldots,\vec{b}_\ell\rangle.$$

The group $\mathbf{LG}_n^\ell(\mathrm{k})$ is thus isomorphic to $\mathrm{GL}_\ell(\mathrm{k}) \times \mathbf{G}_{n-\ell}(\mathrm{k})$. Finally, define

$$\mathbf{FG}_n^\ell(\mathrm{k}) = \{M \in \mathbf{G}_n(\mathrm{k}) \mid M(\vec{a}_j) = \vec{a}_j \text{ for } 1 \leqslant j \leqslant \ell\}.$$

We thus have $\mathbf{FG}_n^\ell(\mathrm{k}) \subset \mathbf{PG}_n^\ell(\mathrm{k})$.

Orthogonal groups. Finally, assume that $\mathbf{G}_n$ is either $\mathrm{SO}_{n,n}$ or $\mathrm{SO}_{n,n+1}$. For an appropriate m, the group $\mathbf{G}_n(\mathrm{k})$ is then the subgroup of $\mathrm{SL}_m(\mathrm{k})$ consisting of elements that preserve a quadratic form $q(\cdot)$ on k^m.

– If $\mathbf{G}_n = \mathrm{SO}_{n,n}$, then let $m = 2n$ and let $(\vec{a}_1,\ldots,\vec{a}_n,\vec{b}_1,\ldots,\vec{b}_n)$ be the standard basis for k^m. The group $\mathbf{G}_n(\mathrm{k})$ is the $\mathrm{SL}_m(\mathrm{k})$-stabilizer of the quadratic form $q(\cdot)$ on k^m defined via the formula

$$q\left(\sum_{j=1}^n (c_j\vec{a}_j + d_j\vec{b}_j)\right) = \sum_{j=1}^n c_j d_j \quad (c_j, d_j \in \mathrm{k}).$$

– If $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$, then let $m = 2n+1$ and let $(\vec{a}_1,\ldots,\vec{a}_n,\vec{b}_1,\ldots,\vec{b}_n,\vec{e})$ be the standard basis for k^m. The group $\mathbf{G}_n(\mathrm{k})$ is the $\mathrm{SL}_m(\mathrm{k})$-stabilizer of the quadratic form $q(\cdot)$ on k^m defined via the formula

$$q\left(\lambda\vec{e} + \sum_{j=1}^n (c_j\vec{a}_j + d_j\vec{b}_j)\right) = \lambda^2 + \sum_{j=1}^n c_j d_j \quad (c_j, d_j, \lambda \in \mathrm{k}).$$

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In both cases, the k-parabolic subgroups of $\mathbf{G}_n(\mathrm{k})$ are the $\mathbf{G}_n(\mathrm{k})$-stabilizers of flags of isotropic subspaces of k^m, that is, subspaces on which $q(\cdot)$ vanishes identically. For $1 \leqslant \ell \leqslant n$ the group $\mathbf{PG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{G}_n(\mathrm{k})$-stabilizer of the isotropic flag

$$0 \subsetneq \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle.$$

For $\mathbf{G}_n = \mathrm{SO}_{n,n}$, the group $\mathbf{UG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathbf{PG}_n^\ell(\mathrm{k})$ consisting of all $M \in \mathbf{PG}_n^\ell(\mathrm{k})$ that act as the identity on both

$$\langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle \quad \text{and} \quad \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle^\perp / \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle = \langle \vec{a}_1, \ldots, \vec{a}_\ell, \vec{a}_{\ell+1}, \vec{b}_{\ell+1}, \ldots, \vec{a}_n, \vec{b}_n \rangle / \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle,$$

while if $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$, then the group $\mathbf{UG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathbf{PG}_n^\ell(\mathrm{k})$ consisting of all $M \in \mathbf{PG}_n^\ell(\mathrm{k})$ that act as the identity on both

$$\langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle \quad \text{and} \quad \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle^\perp / \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle = \langle \vec{a}_1, \ldots, \vec{a}_\ell, \vec{a}_{\ell+1}, \vec{b}_{\ell+1}, \ldots, \vec{a}_n, \vec{b}_n, \vec{e} \rangle / \langle \vec{a}_1, \ldots, \vec{a}_\ell \rangle.$$

The group $\mathbf{LG}_n^\ell(\mathrm{k})$ is defined to be the $\mathbf{PG}_n^\ell(\mathrm{k})$-stabilizer of the isotropic flag

$$0 \subsetneq \langle \vec{b}_1, \ldots, \vec{b}_\ell \rangle.$$

The group $\mathbf{LG}_n^\ell(\mathrm{k})$ is thus isomorphic to $\mathrm{GL}_\ell(\mathrm{k}) \times \mathbf{G}_{n-\ell}(\mathrm{k})$. Finally, define

$$\mathbf{FG}_n^\ell(\mathrm{k}) = \{M \in \mathbf{G}_n(\mathrm{k}) \mid M(\vec{a}_j) = \vec{a}_j \text{ for } 1 \leqslant j \leqslant \ell\}.$$

We thus have $\mathbf{FG}_n^\ell(\mathrm{k}) \subset \mathbf{PG}_n^\ell(\mathrm{k})$.

2.2 Facts about the Steinberg representation

Let R be a commutative ring. The following theorem of Reeder [Ree91] will play an important role in our proof of Theorem 1.1.

THEOREM 2.1 [Ree91, Proposition 1.1]. *Let $\mathbf{G}$ be a connected reductive group defined over a field* k*, let $\mathbf{PG}(\mathrm{k})$ be a* k*-parabolic subgroup of $\mathbf{G}(\mathrm{k})$, and let $\mathbf{LG}(\mathrm{k})$ be a Levi component of $\mathbf{PG}(\mathrm{k})$. Then there exists an $\mathbf{LG}(\mathrm{k})$-equivariant map*

$$\mathrm{St}_{\mathbf{LG}}(\mathrm{k}; R) \longrightarrow \mathrm{Res}^{\mathbf{G}(\mathrm{k})}_{\mathbf{LG}(\mathrm{k})} \mathrm{St}_{\mathbf{G}}(\mathrm{k}; R)$$

such that the induced map

$$\mathrm{Ind}^{\mathbf{PG}(\mathrm{k})}_{\mathbf{LG}(\mathrm{k})} \mathrm{St}_{\mathbf{LG}}(\mathrm{k}; R) \to \mathrm{Res}^{\mathbf{G}(\mathrm{k})}_{\mathbf{PG}(\mathrm{k})} \mathrm{St}_{\mathbf{G}}(\mathrm{k}; R)$$

is an isomorphism.

Remark 2.2. The map in Theorem 2.1 is not unique; for instance, it can be post-composed with any element of the unipotent radical of $\mathbf{PG}(\mathrm{k})$. The paper [Ree91] contains a specific construction of this map, and whenever we refer to the map in Theorem 2.1 we mean the one constructed in [Ree91].

We wish to apply this to the distinguished parabolic subgroups $\mathbf{PG}_n^\ell(\mathrm{k})$ that we introduced in § 2.1. To do this, we need to identify $\mathrm{St}_{\mathbf{LG}_n^\ell}(\mathrm{k}; R)$.

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Lemma 2.3. *Let* $\mathbf{G}_n$ *be either* GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, *or* $\mathrm{SO}_{n,n+1}$. *Then for all fields* k *and all commutative rings* R, *we have*

$$\mathrm{St}_{\mathbf{LG}_n^\ell}(\mathrm{k};R) = \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)$$

for $1 \leqslant \ell \leqslant n$.

Proof. For $\mathbf{G}_n \neq \mathrm{SL}_n$, this follows from the decomposition $\mathbf{LG}_n^\ell(\mathrm{k}) = \mathrm{GL}_\ell(\mathrm{k}) \times \mathbf{G}_{n-\ell}(\mathrm{k})$. For $\mathbf{G}_n = \mathrm{SL}_n$, we instead have that $\mathbf{LG}_n^\ell(\mathrm{k})$ is the subgroup of $\mathrm{GL}_\ell(\mathrm{k}) \times \mathrm{GL}_{n-\ell}(\mathrm{k})$ consisting of matrices of determinant 1. The lemma in this case follows from two facts.

- There is a bijection between k-parabolic subgroups of $\mathrm{GL}_{n-\ell}(\mathrm{k})$ and $\mathrm{SL}_{n-\ell}(\mathrm{k})$, and thus an $\mathrm{SL}_{n-\ell}(\mathrm{k})$-equivariant isomorphism between $\mathrm{St}_{\mathrm{GL}_{n-\ell}}(\mathrm{k};R)$ and $\mathrm{St}_{\mathrm{SL}_{n-\ell}}(\mathrm{k};R)$.
- There is a bijection between k-parabolic subgroups of $\mathrm{GL}_\ell(\mathrm{k}) \times \mathrm{GL}_{n-\ell}(\mathrm{k})$ and $\mathbf{LG}_n^\ell(\mathrm{k})$, and thus an $\mathbf{LG}_n^\ell(\mathrm{k})$-equivariant isomorphism between $\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathrm{GL}_{n-\ell}}(\mathrm{k};R)$ and $\mathrm{St}_{\mathbf{LG}_n^\ell}(\mathrm{k};R)$.

Both of these bijections come from taking intersections. □

These two results allow us to make the following definition.

Definition 2.4. Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Also, let k be a field and R be a commutative ring. For $1 \leqslant \ell \leqslant n$, the *Reeder product map* is the map

$$\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R) \longrightarrow \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)$$

obtained by combining Lemma 2.3 and Theorem 2.1.

Remark 2.5. Identifying $\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)$ with its image in $\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)$ under the Reeder product map, one way of viewing Theorem 2.1 is that it asserts that

$$\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R) = \bigoplus_{u \in \mathbf{UG}_n^\ell(\mathrm{k})} u \cdot (\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)).$$

We will also need the following lemma, which is precisely the case $i = 0$ of Theorem 1.1. It generalizes [LS76, Theorem 4.1]. Recall that if G is a group and M is a G-module, then the coinvariants M_G are the largest quotient of M on which G acts trivially. The coinvariants M_G are isomorphic to $\mathrm{H}_0(G;M)$.

Lemma 2.6. *Let* $\mathbf{G}_n$ *be either* GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, *or* $\mathrm{SO}_{n,n+1}$. *Then for all fields* k *and all commutative rings* R, *we have* $(\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R))_{\mathbf{G}_n(\mathrm{k};R)} = 0$ *for* $n \geqslant 2$.

Proof. Theorem 2.1 implies that

$$\mathrm{Ind}_{\mathbf{LG}_n^2(\mathrm{k})}^{\mathbf{PG}_n^2(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R) \cong \mathrm{Res}_{\mathbf{PG}_n^2(\mathrm{k})}^{\mathbf{G}_n(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R).$$

It is thus enough to prove that

$$(\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R))_{\mathbf{LG}_n^2(\mathrm{k})} = 0.$$

Whatever $\mathbf{G}_n$ is, the group $\mathbf{LG}_n^2(\mathrm{k})$ contains the subgroup $\mathrm{SL}_2(\mathrm{k}) \times 1$. It is thus enough to prove that

$$(\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R))_{\mathrm{SL}_2(\mathrm{k})} = 0.$$

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This is an easy exercise using the fact that

$$\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R) = \widetilde{\mathrm{H}}_0(\mathcal{T}_2(\mathrm{k});R) = \widetilde{\mathrm{H}}_0(\mathbb{P}^1(\mathrm{k});\mathbb{Q}),$$

where $\mathbb{P}^1(\mathrm{k})$ is the projective line over k, regarded as a discrete set of points. For details, see [LS76, Theorem 4.1]. □

3. Reduction to stability

Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Let k be a field and R be a commutative ring. In this section, we reduce Theorem 1.1 to an appropriate homological stability theorem.

Fix some $i \geqslant 0$ and some $n \geqslant 2$. The *stabilization map* for $\mathrm{H}_i(\mathbf{G}_{n-1}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R))$ is the map

$$\mathrm{H}_i(\mathbf{G}_{n-1}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R)) \to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k});\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)) \tag{3.1}$$

induced by the following two maps.

- The group homomorphism $\mathbf{G}_{n-1}(\mathrm{k}) \to \mathbf{G}_n(\mathrm{k})$ obtained as follows. The group $\mathbf{LG}_n^1(\mathrm{k})$ is a subgroup of $\mathrm{GL}_1(\mathrm{k}) \times \mathbf{G}_{n-1}(\mathrm{k})$ that contains the subgroup $1 \times \mathbf{G}_{n-1}(\mathrm{k})$. In fact, $\mathbf{LG}_n^1(\mathrm{k}) = \mathrm{GL}_1(\mathrm{k}) \times \mathbf{G}_{n-1}(\mathrm{k})$ except when $\mathbf{G}_n = \mathrm{SL}_n$. We can thus define a homomorphism $\mathbf{G}_{n-1}(\mathrm{k}) \to \mathbf{G}_n(\mathrm{k})$ via the composition
$$\mathbf{G}_{n-1}(\mathrm{k}) = 1 \times \mathbf{G}_{n-1}(\mathrm{k}) \hookrightarrow \mathbf{LG}_n^1(\mathrm{k}) \hookrightarrow \mathbf{G}_n(\mathrm{k}).$$
- The map $\mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)$ that equals the composition
$$\mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R) \cong R \otimes \mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R) \cong \mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k};R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R),$$
where the final arrow is the Reeder product map. Here we are using the convention regarding the empty set discussed at the end of the introduction which implies that $\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R) = R$; this convention is compatible with Theorem 2.1.

The main result of this section is then as follows.

LEMMA 3.1. *Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Let k be a field and let R be a commutative ring. Assume that the stabilization map (3.1) is a surjection for $n \geqslant N$. Then $\mathrm{H}_i(\mathbf{G}_n(\mathrm{k});\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)) = 0$ for $n \geqslant N+1$.*

Proof. Consider $n \geqslant N+1$. By assumption, the map

$$\mathrm{H}_i(\mathbf{G}_{n-2}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R)) \to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k});\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)) \tag{3.2}$$

obtained by iterating the stabilization map twice is surjective. It is thus enough to show that the image of this map is 0. We can factor this map as

$$\begin{aligned}\mathrm{H}_i(\mathbf{G}_{n-2}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R)) &\to \mathrm{H}_i(\mathbf{G}_{n-2}(\mathrm{k});\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R))\\ &\to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k});\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)).\end{aligned}$$

Regard $\mathrm{SL}_2(\mathrm{k})$ as a subgroup of $\mathbf{G}_n(\mathrm{k})$ via the composition

$$\mathrm{SL}_2(\mathrm{k}) = \mathrm{SL}_2(\mathrm{k}) \times 1 \hookrightarrow \mathbf{LG}_n^2(\mathrm{k}) \hookrightarrow \mathbf{G}_n(\mathrm{k}).$$

The subgroup $\mathrm{SL}_2(\mathrm{k})$ of $\mathbf{G}_n(\mathrm{k})$ commutes with the image of $\mathbf{G}_{n-2}(\mathrm{k})$ in $\mathbf{G}_n(\mathrm{k})$ under the map used to define (3.2). Inner automorphisms act trivially on homology, even with twisted

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coefficients; see [Bro94, Proposition III.8.1]. It follows that to show that the image of (3.2) is 0, it is enough to prove that

$$(\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R))_{\mathrm{SL}_2(\mathrm{k})} = 0.$$

This is equivalent to

$$(\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R))_{\mathrm{SL}_2(\mathrm{k})} = 0,$$

which is one case of Lemma 2.6. □

4. The stabilizer subgroups

This section constructs an isomorphism (Lemma 4.1 below) that will play a fundamental role in our proof of Theorem 1.1.

Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Let k be a field and R be a commutative ring. Fix some $1 \leqslant \ell \leqslant n$. There is a map

$$\mathrm{H}_i(1 \times \mathbf{G}_{n-\ell}(\mathrm{k}); \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R)) \to \mathrm{H}_i(\mathbf{FG}_n^\ell(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \tag{4.1}$$

induced by the following two maps:

- the inclusion map $1 \times \mathbf{G}_{n-\ell}(\mathrm{k}) \to \mathbf{FG}_n^\ell(\mathrm{k})$ (here we are regarding $\mathbf{FG}_n^\ell(\mathrm{k})$ as a subgroup of $\mathbf{PG}_n^\ell(\mathrm{k})$ that contains $1 \times \mathbf{G}_{n-\ell}(\mathrm{k}) \subset \mathbf{LG}_n^\ell(\mathrm{k}) \subset \mathbf{PG}_n^\ell(\mathrm{k})$);
- the Reeder product map $\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$.

Our main result is as follows.

Lemma 4.1. *Let* k *be a field, let* R *be a commutative ring, let* $1 \leqslant \ell \leqslant n$, *and let* $i \geqslant 0$. *Then the map (4.1) is an isomorphism.*

Proof. Shapiro's Lemma [Bro94, Proposition III.6.2] gives an isomorphism

$$\begin{aligned}&\mathrm{H}_i(1 \times \mathbf{G}_{n-\ell}(\mathrm{k}); \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R))\\ &\quad\cong \mathrm{H}_i(\mathbf{FG}_n^\ell(\mathrm{k}); \mathrm{Ind}^{\mathbf{FG}_n^\ell(\mathrm{k})}_{1\times\mathbf{G}_{n-\ell}(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R)).\end{aligned}$$

Below we will prove that there is an isomorphism

$$\mathrm{Ind}^{\mathbf{FG}_n^\ell(\mathrm{k})}_{1\times\mathbf{G}_{n-\ell}(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R) \cong \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R) \tag{4.2}$$

of $\mathbf{FG}_n^\ell(\mathrm{k})$-representations. Combined with the above, this will yield an isomorphism between the left- and right-hand sides of (4.1), which is easily seen to be the map in (4.1).

It remains to construct the isomorphism (4.2). Since $\mathbf{FG}_n^\ell(\mathrm{k}) \subset \mathbf{PG}_n^\ell(\mathrm{k})$, we can restrict the isomorphism given by Theorem 2.1 and Lemma 2.3 to obtain an isomorphism

$$\mathrm{Res}^{\mathbf{PG}_n^\ell(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})} \mathrm{Ind}^{\mathbf{PG}_n^\ell(\mathrm{k})}_{\mathbf{LG}_n^\ell(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R) \cong \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R). \tag{4.3}$$

The unipotent radical of $\mathbf{PG}_n^\ell(\mathrm{k})$ is contained in $\mathbf{FG}_n^\ell(\mathrm{k})$. This implies that there is a single $(\mathbf{FG}_n^\ell(\mathrm{k}), \mathbf{LG}_n^\ell(\mathrm{k}))$-double coset in $\mathbf{PG}_n^\ell(\mathrm{k})$. Also, $\mathbf{FG}_n^\ell(\mathrm{k}) \cap \mathbf{LG}_n^\ell(\mathrm{k}) = \mathbf{G}_{n-\ell}(\mathrm{k})$. The double coset formula [Bro94, Proposition III.5.6b] therefore implies that the left side of (4.3) is canonically isomorphic to

$$\mathrm{Ind}^{\mathbf{FG}_n^\ell(\mathrm{k})}_{1\times\mathbf{G}_{n-\ell}(\mathrm{k})} \mathrm{Res}^{\mathbf{LG}_n^\ell(\mathrm{k})}_{1\times\mathbf{G}_{n-\ell}(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R) = \mathrm{Ind}^{\mathbf{FG}_n^\ell(\mathrm{k})}_{1\times\mathbf{G}_{n-\ell}(\mathrm{k})} \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k}; R),$$

as desired. □

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The following alternate version of Lemma 4.1 will be useful.

Corollary 4.2. *Let* k *be a field, let* R *be a commutative ring, let* $1 \leqslant \ell \leqslant n$, *and let* $i \geqslant 0$. *Then there exists an isomorphism*

$$\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{H}_i(\mathbf{G}_{n-\ell}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)) \cong \mathrm{H}_i(\mathbf{FG}_n^\ell(\mathrm{k});\mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})}\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)).$$

Proof. Since $\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R)$ is a free R-module, we have

$$\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{H}_i(\mathbf{G}_{n-\ell}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)) \cong \mathrm{H}_i(1 \times \mathbf{G}_{n-\ell}(\mathrm{k});\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)).$$

The corollary now follows from Lemma 4.1. □

We will also need an explicit inverse

$$\mathrm{H}_i(\mathbf{FG}_n^\ell(\mathrm{k});\mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^\ell(\mathrm{k})}\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)) \to \mathrm{H}_i(1 \times \mathbf{G}_{n-\ell}(\mathrm{k});\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)) \tag{4.4}$$

to the isomorphism (4.1). The map (4.4) will be induced by the following two maps:

- the homomorphism $\mathbf{FG}_n^\ell(\mathrm{k}) \to 1 \times \mathbf{G}_{n-\ell}(\mathrm{k})$ obtained by restricting the projection $\mathbf{PG}_n^\ell(\mathrm{k}) \to \mathbf{LG}_n^\ell(\mathrm{k})$ to $\mathbf{FG}_n^\ell(\mathrm{k})$;
- the map
$$\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R) \to \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)$$
which equals the composition
$$\begin{aligned}\mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R) &= \bigoplus_{u \in \mathbf{UG}_n^\ell(\mathrm{k})} u \cdot (\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)) \\ &\to \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R),\end{aligned}$$
where the first equality comes from a combination of Theorem 2.1 and Lemma 2.3 (see Remark 2.5) and the last arrow takes $u \cdot x \in u \cdot (\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R))$ to $x \in \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)$ (we will call this map the *Reeder projection map*).

It is clear that these define a map of the form (4.4). The following lemma says that this is an inverse to (4.1).

Lemma 4.3. *Let* k *be a field, let* R *be a commutative ring, let* $1 \leqslant \ell \leqslant n$, *and let* $i \geqslant 0$. *Then the map (4.4) is an inverse to the map (4.1).*

Proof. The proof follows immediately from the fact that the compositions

$$1 \times \mathbf{G}_{n-\ell}(\mathrm{k}) \to \mathbf{FG}_n^\ell(\mathrm{k}) \to 1 \times \mathbf{G}_{n-\ell}(\mathrm{k})$$

and

$$\mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R) \to \mathrm{St}_{\mathrm{GL}_\ell}(\mathrm{k};R) \otimes \mathrm{St}_{\mathbf{G}_{n-\ell}}(\mathrm{k};R)$$

of the maps used to define (4.1) and (4.4) equal the identity. □

5. Vanishing

This section is devoted to the proof of Theorem 1.1. The actual proof is in § 5.3. This is preceded by two sections of preliminaries.

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5.1 Equivariant homology

In our proof of Theorem 1.1, we will need some basic facts about equivariant homology. A basic reference is [Bro94, ch. VII.7].

Let G be a group, let X be a semisimplicial set on which G acts, let R be a ring, and let M be an $R[G]$-module. Let EG be a contractible semisimplicial set on which G acts freely and let $BG = EG/G$, so BG is a classifying space for G. Denote by $EG \times_G X$ the quotient of $EG \times X$ by the diagonal action of G. This is known as the *Borel construction*. The homotopy type of $EG \times_G X$ does not depend on the choice of EG. The projection $EG \times_G X \to EG/G = BG$ induces a homomorphism $\pi_1(EG \times_G X) \to \pi_1(BG) = G$. Via this homomorphism, we can regard M as a local coefficient system on $EG \times_G X$. The *G-equivariant homology groups* of X with coefficients in M, denoted $\mathrm{H}_*^G(X; M)$, are the homology groups of $EG \times_G X$ with respect to the local coefficient system M.

Lemma 5.1. *If X is ℓ-connected, then the above map $EG \times_G X \to EG/G = BG$ induces an isomorphism $\mathrm{H}_i^G(X; M) \cong \mathrm{H}_i(G; M)$ for $0 \leqslant i \leqslant \ell$ and a surjection $\mathrm{H}_{\ell+1}^G(X; M) \to \mathrm{H}_{\ell+1}(X; M)$.*

Proof. The group G acts freely on $EG \times X$ and $EG \times X$ is ℓ-connected. Viewing $EG \times X$ as a CW-complex, we can make $EG \times X$ contractible by adding cells of dimension at least $(\ell+2)$. We conclude that there exists a classifying space for G whose $(\ell+1)$-skeleton equals the $(\ell+1)$-skeleton of $EG \times_G X$. The lemma follows. □

Our main tool for understanding $\mathrm{H}_*^G(X; M)$ is the following spectral sequence, which is constructed in [Bro94, Equation VII.7.7].

Lemma 5.2. *For all $p \geqslant 0$, let Σ_p be a set containing exactly one representative for each orbit of the action of G on the p-simplices of X. For $\sigma \in \Sigma_p$, let G_σ be the stabilizer of σ. Then there is a first quadrant spectral sequence*

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} \mathrm{H}_q(G_\sigma; \mathrm{Res}^G_{G_\sigma} M) \Longrightarrow \mathrm{H}^G_{p+q}(X; M).$$

Remark 5.3. In [Bro94, Equation VII.7.7], the action of G_σ on M is twisted by an 'orientation character'; however, this is unnecessary in our situation, since we are working with semisimplicial sets rather than ordinary simplicial complexes (the point being that in the geometric realization, the setwise stabilizer of a simplex stabilizes the simplex pointwise).

5.2 Complexes of partial bases

Let k be a field and let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. To prove Theorem 1.1, we will need to construct a highly connected space $\mathbf{CG}_n(\mathrm{k})$ on which $\mathbf{G}_n(\mathrm{k})$ acts. The definition of this complex is as follows.

- If $\mathbf{G}_n = \mathrm{GL}_n$ or $\mathbf{G}_n = \mathrm{SL}_n$, then define $\mathbf{CG}_n(\mathrm{k})$ to be the *complex of partial bases* for k^n, i.e. the semisimplicial complex whose ℓ-simplices are ordered sequences $[\vec{v}_0, \ldots, \vec{v}_\ell]$ of linearly independent elements of k^n.
- If $\mathbf{G}_n = \mathrm{Sp}_{2n}$ or $\mathbf{G}_n = \mathrm{SO}_{n,n}$ or $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$ and k^m is the vector space upon which $\mathbf{G}_n(\mathrm{k})$ acts (so m is either $2n$ or $2n+1$), then define $\mathbf{CG}_n(\mathrm{k})$ to be the *complex of partial isotropic bases* for k^m, i.e. the semisimplicial complex whose ℓ-simplices are ordered sequences $[\vec{v}_0, \ldots, \vec{v}_\ell]$ of linearly independent elements of k^m that span an isotropic subspace.

The following theorem summarizes the properties of $\mathbf{CG}_n(\mathrm{k})$.

Theorem 5.4. *Let* k *be a field and let* $\mathbf{G}_n$ *be either* GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, *or* $\mathrm{SO}_{n,n+1}$. *The following then hold.*

(1) *The group* $\mathbf{G}_n(\mathrm{k})$ *acts transitively on the* ℓ*-cells of* $\mathbf{CG}_n(\mathrm{k})$ *for all* $0 \leqslant \ell < n-1$.

(2) *The space* $\mathbf{CG}_n(\mathrm{k})$ *is* $f(n)$*-connected where* $f(n)$ *is given by:*

 (a) $f(n) = n-2$ *if* $\mathbf{G}_n$ *is either* GL_n *or* SL_n;

 (b) $f(n) = (n-3)/2$ *if* $\mathbf{G}_n$ *is either* Sp_{2n}, $\mathrm{SO}_{n,n}$, *or* $\mathrm{SO}_{n,n+1}$.

Proof. The first assertion is well known (and also holds for $\ell = n-1$ except when $\mathbf{G}_n = \mathrm{SL}_n$). As for the second, Maazen proved in his thesis [Maa79] that $\mathbf{CG}_n(\mathrm{k})$ is $(n-2)$-connected for $\mathbf{G}_n = \mathrm{GL}_n$ and $\mathbf{G}_n = \mathrm{SL}_n$. See [vdK80] for a published proof of a more general result. Friedrich proved in [Fri16, Theorem 3.23] that $\mathbf{CG}_n(\mathrm{k})$ is $(n-3)/2$-connected for $\mathbf{G}_n = \mathrm{Sp}_{2n}$ and $\mathbf{G}_n = \mathrm{SO}_{n,n}$ and $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$ (for Sp_{2n} and $\mathrm{SO}_{n,n}$, this was proven earlier in [Mv02, Theorem 7.3]). To apply the cited result of Friedrich to our situation, we need the fact that the unitary stable rank of a field is 1 (see, e.g., [Mv02, Example 6.5]). □

5.3 The proof of Theorem 1.1

Let us first recall the statement of the theorem. Let $\mathbf{G}_n$ be either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$. Also, let k be a field and R be a commutative ring. Our goal is to prove that $\mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) = 0$ for $n \geqslant 2i+2$ and that there exists a surjection

$$\mathrm{H}_i(\mathbf{G}_{2i}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{2i}}(\mathrm{k}; R)) \to \mathrm{H}_i(\mathbf{G}_{2i+1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{2i+1}}(\mathrm{k}; R)). \tag{5.1}$$

Of course, this surjection will be induced by the stabilization map defined in § 3.

The proof is by induction on i. We begin with the base case $i = 0$. Lemma 2.6 says that $\mathrm{H}_0(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) = 0$ for $n \geqslant 2$, so we only need to show that the map (5.1) is a surjection for $i = 0$. For the domain, $\mathbf{G}_0(\mathrm{k})$ is the trivial group. By our convention regarding the empty set discussed at the end of the introduction, we thus have $\mathrm{St}_{\mathbf{G}_0}(\mathrm{k}; R) = R$, and hence $\mathrm{H}_i(\mathbf{G}_0(\mathrm{k}); \mathrm{St}_{\mathbf{G}_0}(\mathrm{k}; R)) = R$. To simplify the codomain, we have several cases.

- $\mathbf{G}_1 = \mathrm{GL}_1$ or $\mathbf{G}_1 = \mathrm{SO}_{1,1}$. In fact, these groups are isomorphic and are commutative, so $\mathrm{St}_{\mathbf{G}_1}(\mathrm{k}; R) = R$ in these cases and (5.1) is an isomorphism.
- $\mathbf{G}_1 = \mathrm{SL}_1$. The group SL_1 is the trivial group and thus $\mathrm{St}_{\mathbf{G}_1}(\mathrm{k}; R) = R$ and (5.1) is an isomorphism.
- $\mathbf{G}_1 = \mathrm{Sp}_2 \cong \mathrm{SL}_2$ or $\mathbf{G}_1 = \mathrm{SO}_{2,1} \cong \mathrm{PSL}_2$. These groups have isomorphic Steinberg representations and the action of $\mathrm{SL}_2(\mathrm{k})$ on $\mathrm{St}_{\mathrm{SL}_2}(\mathrm{k}; R)$ factors through $\mathrm{PSL}_2(\mathrm{k})$. This case thus follows from Lemma 2.6, which says that $\mathrm{H}_0(\mathrm{SL}_2(\mathrm{k}); \mathrm{St}_{\mathrm{SL}_2}(\mathrm{k}; R)) = 0$.

This completes the base case.

Assume now that $i > 0$ and that the desired result is true for all smaller values of i. We will prove that the stabilization map

$$\mathrm{H}_i(\mathbf{G}_{n-1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k}; R)) \to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \tag{5.2}$$

is surjective for $n \geqslant 2i+1$. Lemma 3.1 will then imply that $\mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) = 0$ for $n \geqslant 2i+2$, and the theorem will follow.

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Fix some $n \geqslant 2i+1$ and let k^m be the standard vector space representation of $\mathbf{G}_n(\mathrm{k})$ (so m is either n, $2n$, or $2n+1$). Let $\{\vec{a}_1, \dots, \vec{a}_n\}$ be the vectors in k^m such that

$$\mathbf{FG}_n^{\ell}(\mathrm{k}) = \{M \in \mathbf{G}_n(\mathrm{k}) \mid M(\vec{a}_j) = \vec{a}_j \text{ for } 1 \leqslant j \leqslant \ell\}$$

for $1 \leqslant \ell \leqslant n$. Combining the second conclusion of Theorem 5.4 with Lemma 5.1, we have a surjection

$$\mathrm{H}_i^{\mathbf{G}_n(\mathrm{k})}(\mathbf{CG}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \to \mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)). \tag{5.3}$$

We will analyze $\mathrm{H}_i^{\mathbf{G}_n(\mathrm{k})}(\mathbf{CG}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))$ using the spectral sequence from Lemma 5.2. To calculate its E^1-page, observe that the first conclusion of Theorem 5.4 says that $\mathbf{G}_n(\mathrm{k})$ acts transitively on the p-simplices of $\mathbf{CG}_n(\mathrm{k})$ for $0 \leqslant p < n-1$. The stabilizer of the $(p-1)$-simplex $[\vec{a}_1, \dots, \vec{a}_p]$ is $\mathbf{FG}_n^p(\mathrm{k})$, so the spectral sequence in Lemma 5.2 has

$$E^1_{p,q} = \mathrm{H}_q(\mathbf{FG}_n^{p+1}(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^{p+1}(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)), \tag{5.4}$$

for $0 \leqslant p < n-1$.

We will prove that all of the terms on the $p+q=i$ line of the E^∞-page of our spectral sequence vanish except for possibly the term $E^\infty_{0,i}$. To do this, consider $p, q \geqslant 0$ with $p+q=i$ and $p \geqslant 1$. The case $p=1$ and $n=2i+1$ is exceptional and must be treated separately. To avoid getting bogged down here, we postpone this calculation until § 6 below, where it appears as Lemma 6.1.[1]

We thus can assume that either $p \geqslant 2$ or that $n \geqslant 2i+2$. Since $n \geqslant 2i+1$, we certainly have $p < n-1$, so $E^1_{p,q}$ is in the regime where the above description of the E^1-page holds. Applying Corollary 4.2 to (5.4), we see that

$$E^1_{p,q} = \mathrm{St}_{\mathrm{GL}_{p+1}}(\mathrm{k}; R) \otimes \mathrm{H}_q(\mathbf{G}_{n-p-1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-p-1}}(\mathrm{k}; R)).$$

To see that this vanishes, it is enough to show that $\mathrm{H}_q(\mathbf{G}_{n-p-1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-p-1}}(\mathrm{k}; R)) = 0$. This is a consequence of our inductive hypothesis; to see that it applies, observe that if $p \geqslant 2$ then

$$n-p-1 \geqslant (2i+1)-p-1 = 2(p+q)-p = 2q+p \geqslant 2q+2,$$

while if $p=1$ and $n \geqslant 2i+2$ then

$$n-p-1 \geqslant (2i+2)-1-1 = 2(p+q) = 2q+2.$$

This implies that $E^1_{p,q} = 0$, and thus that $E^\infty_{p,q} = 0$.

The $p+q=i$ line of the E^∞-page of our spectral sequence thus only has a single potentially nonzero entry, namely $E^\infty_{0,i}$, and this is a quotient of

$$E^1_{0,i} = \mathrm{H}_i(\mathbf{FG}_n^1(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^1(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)).$$

This entry thus surjects onto $\mathrm{H}_i^{\mathbf{G}_n(\mathrm{k})}(\mathbf{CG}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))$. Combining this with the surjection (5.3), we obtain a surjection

$$\mathrm{H}_i(\mathbf{FG}_n^1(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^1(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \longrightarrow \mathrm{H}_i(\mathbf{G}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)). \tag{5.5}$$

[1] This exceptional case could be avoided at the cost of only proving that $\mathrm{H}_i(\mathbf{G}_n; \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) = 0$ for $n \geqslant 3i+2$ instead of for $n \geqslant 2i+2$.

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Examining the construction of our spectral sequence in [Bro94, ch. VII.7], it is easy to see that this comes from the map induced by the inclusion $\mathbf{FG}_n^1(\mathrm{k}) \hookrightarrow \mathbf{G}_n(\mathrm{k})$. Combining (5.5) with the isomorphism

$$\mathrm{H}_i(\mathbf{G}_{n-1}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-1}}(\mathrm{k}; R)) \xrightarrow{\cong} \mathrm{H}_i(\mathbf{FG}_n^1(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^1(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))$$

given by the $\ell = 1$ case of Corollary 4.2, we conclude that (5.2) is a surjection, as desired.

6. Killing the exceptional term in the spectral sequence

This section is devoted to proving the vanishing result postponed from the proof of Theorem 1.1 in § 5.3. The notation in this section is thus identical to that in § 5.3:

- $\mathbf{G}_n$ is either GL_n, SL_n, Sp_{2n}, $\mathrm{SO}_{n,n}$, or $\mathrm{SO}_{n,n+1}$;
- k is a field and R is a commutative ring;
- $i > 0$ and $n = 2i + 1$ (the only case that remained in that section);
- k^m is the standard vector space representation of $\mathbf{G}_n(\mathrm{k})$ (so m is either n, $2n$, or $2n + 1$);
- $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is the set of vectors in k^m such that

$$\mathbf{FG}_n^\ell(\mathrm{k}) = \{M \in \mathbf{G}_n(\mathrm{k}) \mid M(\vec{a}_j) = \vec{a}_j \text{ for } 1 \leqslant j \leqslant \ell\}$$

  for $1 \leqslant \ell \leqslant n$.
- $E^r_{p,q}$ is the spectral sequence from Lemma 5.2 converging to $\mathrm{H}_i^{\mathbf{G}_n(\mathrm{k})}(\mathbf{CG}_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))$.

What we must prove is as follows.

Lemma 6.1. *Let the notation be as above, and assume that the stabilization map*

$$\mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R)) \to \mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R)) \qquad (6.1)$$

is surjective. Then the differential $E^1_{2,i-1} \to E^1_{1,i-1}$ is surjective, and thus $E^\infty_{1,i-1} = 0$.

The proof of Lemma 6.1 is divided into five sections.

- In § 6.1, we give an explicit form for the differential $E^1_{2,i-1} \to E^1_{1,i-1}$.
- In § 6.2, we translate that explicit form into one involving the stabilization map (6.1).
- In § 6.3, we summarize what remains to be proved.
- In § 6.4, we give some needed background information about apartments.
- In § 6.5, we finish off the proof of Lemma 6.1.

6.1 Identifying the differential

The notation is as in the beginning of § 6. In this section, we identify the differential $E^1_{2,i-1} \to E^1_{1,i-1}$. Since $n = 2i + 1$ and $i > 0$, we have $1 < n - 1$, so $E^1_{1,i-1}$ is as described in (5.4), i.e.

$$E^1_{1,i-1} \cong \mathrm{H}_{i-1}(\mathbf{FG}_n^2(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^2(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)).$$

If $i = 1$, then we do not have $2 < n - 1$, so in this case $E^1_{2,i-1}$ is not as described in (5.4). The issue is that $\mathbf{G}_n(\mathrm{k})$ might not act transitively on the 2-simplices of $\mathbf{CG}_n(\mathrm{k})$ (this is actually only a problem for $\mathbf{G}_n = \mathrm{SL}_n$). However, for all values of i it is still the case that $E^1_{2,i-1}$ contains

$$\mathrm{H}_{i-1}(\mathbf{FG}_n^3(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}_n^3(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))$$

as a summand. The restriction of the differential $E^1_{2,i-1} \to E^1_{1,i-1}$ to this summand is a map

$$\partial\colon \mathrm{H}_{i-1}(\mathbf{FG}^3_n(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}^3_n(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \longrightarrow \mathrm{H}_{i-1}(\mathbf{FG}^2_n(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}^2_n(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)). \tag{6.2}$$

To prove Lemma 6.1, it is enough to prove that ∂ is surjective.

We can describe ∂ using the recipe described in [Bro94, ch. VII.8]. Recall that $\mathbf{FG}^3_n(\mathrm{k})$ is the $\mathbf{G}_n(\mathrm{k})$-stabilizer of the ordered sequence of vectors $\sigma = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$. For $1 \leqslant m \leqslant 3$, let σ_m be the ordered sequence obtained by deleting $\vec{a}_m$ from σ and let $(\mathbf{G}_n(k))_{\sigma_m}$ denote the $\mathbf{G}_n(k)$-stabilizer of σ_m. We then have $\partial = \partial_1 - \partial_2 + \partial_3$, where ∂_m is the composition

$$\begin{aligned} &\mathrm{H}_{i-1}(\mathbf{FG}^3_n(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}^3_n(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \\ &\quad \xrightarrow{\partial'_m} \mathrm{H}_{i-1}((\mathbf{G}_n(\mathrm{k}))_{\sigma_m}; \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{(\mathbf{G}_n(\mathrm{k}))_{\sigma_m}} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \\ &\quad \xrightarrow{\partial''_m} \mathrm{H}_{i-1}(\mathbf{FG}^2_n(\mathrm{k}); \mathrm{Res}^{\mathbf{G}_n(\mathrm{k})}_{\mathbf{FG}^2_n(\mathrm{k})} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)) \end{aligned}$$

of the following two maps.

- The map ∂'_m is induced by the inclusion $\mathbf{FG}^3_n(\mathrm{k}) \hookrightarrow (\mathbf{G}_n(\mathrm{k}))_{\sigma_m}$.
- Define $\kappa_m \in \mathbf{G}_n(\mathrm{k})$ as follows. First, $\kappa_3 = \mathrm{id}$. For $m \in \{1, 2\}$, we do the following.
  * If $\mathbf{G}_n = \mathrm{GL}_n$ or $\mathbf{G}_n = \mathrm{SL}_n$, then $\kappa_m \in \mathrm{SL}_n(\mathrm{k})$ is the map $\mathrm{k}^n \to \mathrm{k}^n$ that takes $\vec{a}_m$ to $\vec{a}_3$, takes $\vec{a}_3$ to $-\vec{a}_m$, and fixes all the other basis vectors.
  * If $\mathbf{G}_n = \mathrm{Sp}_{2n}$ or $\mathbf{G}_n = \mathrm{SO}_{n,n}$ or $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$, then $\kappa_m \in \mathbf{G}_n(\mathrm{k})$ is the map $\mathrm{k}^m \to \mathrm{k}^m$ defined as follows. Let $\vec{b}_1, \ldots, \vec{b}_n$ be the standard basis vectors for k^m that pair with the $\vec{a}_j$ (there is one additional standard basis vector if $\mathbf{G}_n = \mathrm{SO}_{n,n+1}$). Then κ_m takes $\vec{a}_m$ to $\vec{a}_3$, takes $\vec{a}_3$ to $-\vec{a}_m$, takes $\vec{b}_m$ to $\vec{b}_3$, takes $\vec{b}_3$ to $-\vec{b}_m$, and fixes all the other basis vectors.

  Then ∂''_m is induced by the map $(\mathbf{G}_n(\mathrm{k}))_{\sigma_m} \to \mathbf{FG}^2_n(\mathrm{k})$ that takes $g \in (\mathbf{G}_n(\mathrm{k}))_{\sigma_m}$ to $\kappa_m g \kappa_m^{-1}$ and the map $\mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$ that takes $x \in \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$ to $\kappa_m(x) \in \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$. We remark that easier choices of κ_m (without the signs) could be used for $\mathbf{G}_n \neq \mathrm{SL}_n$, but we chose the ones above to make our later formulas more uniform.

This is summarized in the following lemma.

Lemma 6.2. *Let the notation be as above. Then the map ∂ in (6.2) equals $\partial_1 - \partial_2 + \partial_3$, where ∂_m is induced by the map $\mathbf{FG}^3_n(\mathrm{k}) \to \mathbf{FG}^2_n(\mathrm{k})$ defined via the formula*

$$g \mapsto \kappa_m g \kappa_m^{-1} \quad (g \in \mathbf{FG}^3_n(\mathrm{k}))$$

and the map $\mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$ defined via the formula

$$x \mapsto \kappa_m(x) \quad (x \in \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)).$$

6.2 Bringing in the stabilization map

The notation is as in the beginning of § 6. Fix some $1 \leqslant m \leqslant 3$, and let ∂_m and κ_m be as in Lemma 6.2. Applying the isomorphism in Corollary 4.2 to the domain and codomain of ∂_m, we obtain a homomorphism

$$\begin{aligned} \widehat{\partial}_m\colon &\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \otimes \mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R)) \\ &\to \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R)). \end{aligned}$$

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Our goal in this section is to prove that $\widehat{\partial}_m$ is the tensor product of the stabilization map

$$\mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R)) \to \mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R)) \tag{6.3}$$

with the map $\zeta_m\colon \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \to \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R)$ defined as follows. Let $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ be the standard basis for k^3. Let $\widehat{\kappa}_3 = \mathrm{id} \in \mathrm{SL}_3(\mathrm{k})$, and for $m \in \{1, 2\}$ let $\widehat{\kappa}_m \in \mathrm{SL}_3(\mathrm{k})$ be the element that takes $\vec{a}_m$ to $\vec{a}_3$, takes $\vec{a}_3$ to $-\vec{a}_m$, and fixes all the other basis vectors. Then ζ_m is the composition

$$\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \xrightarrow{\widehat{\kappa}_m} \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathrm{GL}_1}(\mathrm{k}; R) \cong \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R),$$

where the second arrow is the Reeder projection map (see § 4) and the final isomorphism comes from the fact that $\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k}; R) = R$.

The main result of this section is then as follows.

Lemma 6.3. *Let the notation be as above. Then $\widehat{\partial}_m$ is the tensor product of ζ_m with the stabilization map (6.3).*

Proof. By construction, $\widehat{\partial}_m$ equals the composition

$$\begin{aligned}
&\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \otimes \mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R))\\
&\xrightarrow{\cong} \mathrm{H}_{i-1}(1 \times \mathbf{G}_{n-3}(\mathrm{k}); \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R))\\
&\xrightarrow{\cong} \mathrm{H}_{i-1}(\mathbf{FG}^3_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))\\
&\xrightarrow{\partial_m} \mathrm{H}_{i-1}(\mathbf{FG}^2_n(\mathrm{k}); \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R))\\
&\xrightarrow{\cong} \mathrm{H}_{i-1}(1 \times \mathbf{G}_{n-2}(\mathrm{k}); \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R))\\
&\xrightarrow{\cong} \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k}); \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R)),
\end{aligned}$$

where the various maps are as follows.

- The first and last arrows use the fact that $\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R)$ and $\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R)$ are free R-modules (cf. the proof of Corollary 4.2).
- The second arrow is the map described in Lemma 4.1, that is, the map induced by the inclusion $1 \times \mathbf{G}_{n-3}(\mathrm{k}) \hookrightarrow \mathbf{FG}^3_n(\mathrm{k})$ and the Reeder product map $\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$.
- The third arrow is the map ∂_m described in Lemma 6.2, that is, the map induced by the map $\mathbf{FG}^3_n(\mathrm{k}) \to \mathbf{FG}^2_n(\mathrm{k})$ given by conjugation by κ_m and the map $\mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R) \to \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)$ induced by κ_m.
- The fourth arrow is the map described in Lemma 4.3, that is, the map induced by the projection $\mathbf{FG}^2_n(\mathrm{k}) \to 1 \times \mathbf{G}_{n-2}(\mathrm{k})$ together with the Reeder projection map $\mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R) \to \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R)$.

We must show that this composition equals the indicated tensor product of maps. This will take some work.

Define Ψ to be the composition

$$\begin{aligned}
\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k}; R) &\longrightarrow \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)\\
&\xrightarrow{\kappa_m} \mathrm{St}_{\mathbf{G}_n}(\mathrm{k}; R)\\
&\longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k}; R),
\end{aligned}$$

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where the first map is the Reeder product map and the last map is the Reeder projection map. Also, define Φ to be the composition

$$\begin{aligned}\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R) &\longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)\\ &\longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R),\end{aligned}$$

where the maps are as follows.

- The first map is the tensor product of the Reeder projection map $\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)\to\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)$ and the identity map $\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)\to\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)$.
- The second map is the tensor product of the identity map $\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\to\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)$ and the Reeder product map $\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)\to\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R)$.

By the above, it is enough to prove that $\Psi=\Phi\circ(\widehat{\kappa}_m\otimes\mathrm{id})$.

Define Ψ' to be the composition

$$\begin{aligned}\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R) &\longrightarrow \mathrm{St}_{\mathbf{G}_n}(\mathrm{k};R)\\ &\longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R),\end{aligned}$$

where the first map is the Reeder product map and the second map is the Reeder projection map. From its definition, we see that $\Psi=\Psi'\circ(\widehat{\kappa}_m\otimes\mathrm{id})$. We thus see that it is enough to prove that $\Psi'=\Phi$.

Define $U=\mathbf{UGL}_3^2(\mathrm{k})$ to be the unipotent radical of the parabolic subgroup $\mathbf{PGL}_3^2(\mathrm{k})$ of $\mathrm{GL}_3(\mathrm{k})$ (despite the bad notation, this is not the projective general linear group). Using Theorem 2.1 as in Remark 2.5, we see that

$$\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)=\bigoplus_{u\in U}u\cdot(\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)).$$

Consider $u\in U$ and $x\in\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)$ and $y\in\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)$ and $z\in\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)$. Examining the definition of Ψ', we see that

$$\Psi'((u\cdot(x\otimes y))\otimes z)=x\otimes(y\otimes z),$$

where $y\otimes z\in\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k};R)\otimes\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R)$ is identified with an element of $\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R)$ using the Reeder product map. But this equals $\Phi((u\cdot(x\otimes y))\otimes z)$, as desired. □

6.3 Summary of where we are

The notation is as in the beginning of § 6. Recall that Lemma 6.1 asserts that the differential $E^1_{2,i-1}\to E^1_{1,i-1}$ is surjective. Let ∂ be as in § 6.1. Also, let ζ_m and $\widehat{\kappa}_m$ be as in § 6.2. Define

$$\zeta\colon\ \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)\to\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)$$

via the formula $\zeta=\zeta_1-\zeta_2+\zeta_3$. Combining Lemmas 6.2 and 6.3, we see that to prove Lemma 6.1, it is enough to show that the map

$$\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k};R)\otimes\mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R))\to\mathrm{St}_{\mathrm{GL}_2}(\mathrm{k};R)\otimes\mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R))$$

obtained as the tensor product of ζ and the stabilization map

$$\mathrm{H}_{i-1}(\mathbf{G}_{n-3}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-3}}(\mathrm{k};R))\to\mathrm{H}_{i-1}(\mathbf{G}_{n-2}(\mathrm{k});\mathrm{St}_{\mathbf{G}_{n-2}}(\mathrm{k};R))$$

is surjective. One of the assumptions in Lemma 6.1 is that this stabilization map is surjective. To prove that lemma, it is thus enough to prove the following.

Lemma 6.4. *Let the notation be as above. Then ζ is surjective.*

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6.4 Apartments

Before we prove Lemma 6.4, we need to discuss some background material on the Steinberg representation. Unlike the previous sections, in this section $n \geqslant 1$ is arbitrary. Recall that $\mathrm{St}_{\mathrm{GL}_n}(\mathrm{k}; R) = \widetilde{\mathrm{H}}_{n-2}(\mathcal{T}_{\mathrm{GL}_n}(\mathrm{k}); R)$, where $\mathcal{T}_{\mathrm{GL}_n}(\mathrm{k})$ is the Tits building associated to $\mathrm{GL}_n(\mathrm{k})$. This building can be described as the simplicial complex whose r-simplices are flags

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_r \subsetneq \mathrm{k}^n$$

of nonzero proper subspaces of k^n.

The Solomon–Tits theorem [Bro89, Theorem IV.5.2] says that the R-module $\mathrm{St}_{\mathrm{GL}_n}(\mathrm{k}; R)$ is spanned by *apartment classes*, which are defined as follows. Consider an $n \times n$ matrix B with entries in k^n none of whose columns are identically 0. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be the columns of B. Let S_n be the simplicial complex whose r-simplices are chains

$$0 \subsetneq I_0 \subsetneq \cdots \subsetneq I_r \subsetneq \{1, \ldots, n\}.$$

The complex S_n is isomorphic to the barycentric subdivision of the boundary of an $(n-1)$-simplex; in particular, S_n is homeomorphic to an $(n-2)$-sphere. There is a simplicial map $f\colon S_n \to \mathcal{T}_{\mathrm{GL}_n}(\mathrm{k})$ defined via the formula

$$f(I) = \langle \vec{v}_i \mid i \in I \rangle \quad (\emptyset \subsetneq I \subsetneq \{1, \ldots, n\}).$$

The *apartment class* corresponding to B, denoted $\|B\|$, is the image of the fundamental class $[S_n] \in \widetilde{\mathrm{H}}_{n-2}(S_n; R) = R$ under the map $f_*\colon \widetilde{\mathrm{H}}_{n-2}(S_n; R) \to \widetilde{\mathrm{H}}_{n-2}(\mathcal{T}_{\mathrm{GL}_n}(\mathrm{k}); R) = \mathrm{St}_{\mathrm{GL}_n}(\mathrm{k}; R)$.

Remark 6.5. We have $\|B\| = 0$ if the $\vec{v}_i$ do not form a basis for k^n, i.e. if B is not invertible.

Permuting the columns of B changes $\|B\|$ by the sign of the permutation, and multiplying a column of B by a nonzero scalar does not change $\|B\|$. The apartment classes also satisfy the following more interesting relation.

Lemma 6.6. *Let* k *be a field, let R be a commutative ring, and let $n \geqslant 2$. Let B be an $n \times (n+1)$-matrix with entries in* k. *Assume that none of the columns of B are identically* 0. *Ordering the columns of B from* 0 *to n, for $0 \leqslant m \leqslant n$ let B_m be the result of deleting the mth column from B. Then $\|B_0\| - \|B_1\| + \|B_2\| - \cdots + (-1)^n \|B_n\| = 0$.*

Proof. The simplices forming the apartment classes $\|B_i\|$ cancel in pairs; see Figure 1. □

The Solomon–Tits theorem [Bro89, Theorem IV.5.2] gives the following basis for $\mathrm{St}_{\mathrm{GL}_n}(\mathrm{k}; R)$.

Theorem 6.7 (Solomon–Tits). *Let* k *be a field, let R be a commutative ring, and let $n \geqslant 1$. Then $\mathrm{St}_{\mathrm{GL}_n}(\mathrm{k}; R)$ is a free R-module on the basis consisting of all $\|B\|$ such that B is an upper unitriangular matrix in $\mathrm{GL}_n(\mathrm{k})$.*

6.5 The proof of Lemma 6.4

We finally prove Lemma 6.4, which as discussed in § 6.3 suffices to prove Lemma 6.1. First, we recall its statement. For $1 \leqslant m \leqslant 3$, let ζ_m and $\widehat{\kappa}_m$ be as in § 6.2. Define

$$\zeta\colon\ \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \to \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R)$$

via the formula $\zeta = \zeta_1 - \zeta_2 + \zeta_3$. Our goal is to prove that ζ is surjective.

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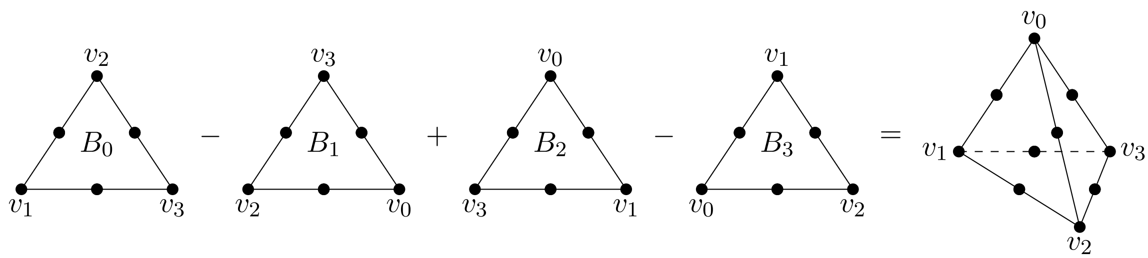

Figure 1. As we illustrate here in the case $n = 3$, the apartment classes corresponding to the B_i can be placed on the boundary of an n-dimensional simplex such that their simplices cancel in pairs. In the picture, the vertices labeled with the vectors $\vec{v}_i$ are taken to the lines spanned by the $\vec{v}_i$ while the unlabeled vertices are taken to the two-dimensional subspaces spanned by the vectors on their two neighbors.

Before we do that, we introduce some formulas. Let $\pi\colon \mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \to \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R)$ be the composition

$$\mathrm{St}_{\mathrm{GL}_3}(\mathrm{k}; R) \longrightarrow \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R) \otimes \mathrm{St}_{\mathrm{GL}_1}(\mathrm{k}; R) \xrightarrow{\cong} \mathrm{St}_{\mathrm{GL}_2}(\mathrm{k}; R),$$

where the first arrow is the Reeder projection map and the second arrow comes from the fact that $\mathrm{St}_{\mathrm{GL}_1}(\mathrm{k}; R) = R$. From its definition, we see that

$$\pi\left(\left\|\begin{matrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{matrix}\right\|\right) = \left\|\begin{matrix} 1 & x \\ 0 & 1 \end{matrix}\right\|$$

for all $x, y, z \in \mathrm{k}$. What is more, for all 3×3 matrices B none of whose columns are identically 0 we have

$$\zeta(\|B\|) = \pi(\widehat{\kappa}_1(B)) - \pi(\widehat{\kappa}_2(B)) + \pi(\widehat{\kappa}_3(B)).$$

Here the $\widehat{\kappa}_m$ act on B via matrix multiplication.

We now turn to proving that ζ is surjective. Consider $a \in \mathrm{k}$, and set

$$A_a = \left\|\begin{matrix} 1 & a \\ 0 & 1 \end{matrix}\right\|.$$

By Theorem 6.7, it is enough to prove that $A_a \in \mathrm{Im}(\zeta)$. We have

$$\begin{aligned} \zeta\left(\left\|\begin{matrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) &= \pi\left(\left\|\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & a & 0 \end{matrix}\right\|\right) - \pi\left(\left\|\begin{matrix} 1 & a & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}\right\|\right) + \pi\left(\left\|\begin{matrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) \\ &= \pi\left(-\left\|\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{matrix}\right\|\right) - \pi\left(-\left\|\begin{matrix} 1 & 0 & a \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) + \pi\left(\left\|\begin{matrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) \\ &= \pi\left(-\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{matrix}\right\|\right) - \pi\left(-\left\|\begin{matrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) + \pi\left(\left\|\begin{matrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\|\right) \\ &= -\pi\left(\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{matrix}\right\|\right) + \left\|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right\| + \left\|\begin{matrix} 1 & a \\ 0 & 1 \end{matrix}\right\|, \end{aligned} \tag{6.4}$$

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where the second equality uses the fact that permuting the columns of a matrix changes the associated apartment by the sign of the permutation and the third equality uses the fact that multiplying a column by a nonzero scalar does not change the associated apartment.

If $a = 0$, then the right-hand side of (6.4) simplifies to

$$-\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| = \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| = A_0,$$

so $A_0 \in \operatorname{Im}(\zeta)$. Assume now that $a \neq 0$. Plugging the matrix

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & a & 1\end{pmatrix}$$

into Lemma 6.6, we get the relation

$$\begin{aligned} 0 &= \left\|\begin{matrix}0 & 0 & 0\\ 1 & 1 & 0\\ 0 & a & 1\end{matrix}\right\| - \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & a & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\| - \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & a\end{matrix}\right\| \\ &= 0 - \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & a & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\| - \left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & a^{-1}\\ 0 & 0 & 1\end{matrix}\right\|, \end{aligned}$$

where the equality uses the fact that the columns of the first matrix are not linearly independent and the fact that multiplying a column of a matrix by a nonzero scalar does not change the associated apartment. Plugging this relation into (6.4), we see that the right-hand side of (6.4) equals

$$\begin{aligned} &-\left(\pi\left(\left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\|\right) - \pi\left(\left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & a^{-1}\\ 0 & 0 & 1\end{matrix}\right\|\right)\right) + \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & a\\ 0 & 1\end{matrix}\right\| \\ &= -\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\| + \left\|\begin{matrix}1 & a\\ 0 & 1\end{matrix}\right\| \\ &= A_0 + A_a. \end{aligned}$$

Since we have already seen that $A_0 \in \operatorname{Im}(\zeta)$, we deduce that $A_a \in \operatorname{Im}(\zeta)$, as desired.

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Avner Ash ashav@bc.edu

Department of Mathematics, Boston College, Chestnut Hill, MA 02467-3806, USA

Andrew Putman andyp@nd.edu

Department of Mathematics, University of Notre Dame, 279 Hurley Hall, Notre Dame, IN 46556, USA

Steven V Sam svs@math.wisc.edu

Department of Mathematics, University of Wisconsin, 480 Lincoln Dr., Madison, WI 53706-1325, USA