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RESEARCH ARTICLE

Joint transitivity for linear iterates

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Abstract
We establish sufficient and necessary conditions for the joint transitivity of linear iterates in a minimal topological dynamical system with commuting transformations. This result provides the first topological analogue of the classical Berend and Bergelson joint ergodicity criterion in measure-preserving systems.

1. Introduction - Main result

1.1. The joint ergodicity problem

Let $(X, \mathcal{B}, \mu)$ be a standard probability space equipped with an invertible measure-preserving transformation $T : X \to X$ (that is, $\mu(TA) = \mu(A)$ for every $A \in \mathcal{B}$). We say that the quadruple $(X, \mathcal{B}, \mu, T)$ is *a measure-preserving system*. In particular, the latter is called *ergodic* or *weakly mixing* if the transformation T is ergodic (i.e., every T-invariant set $A \in \mathcal{B}$ satisfies $\mu(A) \in \{0, 1\}$) or weakly mixing (i.e., the transformation $T \times T$, acting on the Cartesian square $X^2 := X \times X$, is ergodic), respectively.

Given a weakly mixing measure-preserving system $(X, \mathcal{B}, \mu, T)$ and distinct nonzero integers $a_1, \dots, a_d$, we have the following independence property of the sequences $(T^{a_i n})_n$, $1 \le i \le d$.[1]

Theorem 1.1 [25]. *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing measure-preserving system. Then, for any $d \in \mathbb{N}$, any distinct nonzero integers $a_1, \dots, a_d$, and any $f_1, \dots, f_d \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_1 n} f_1 \cdot \dots \cdot T^{a_d n} f_d = \int_X f_1 \, d\mu \cdot \dots \cdot \int_X f_d \, d\mu, \tag{1.1}$$

where the convergence takes place in $L^2(\mu)$.

This result, in particular its recurrence reformulation on $\mathcal{B}$-measurable sets of positive measure, is a crucial ingredient in Furstenberg's approach in proving Szemerédi's theorem (that is, every subset of natural numbers of positive upper density contains arbitrarily long arithmetic progressions) by recasting it as a recurrence problem.

[1]Throughout this paper, whenever a sequence is written as $(a(n))_n$ without specifying the range of n, it is understood as a $\mathbb{Z}$-sequence $(a(n))_{n\in\mathbb{Z}}$.

Later, in [5], Bergelson extended Theorem 1.1 to essentially distinct integer polynomial iterates. The convergence of general multiple ergodic averages for various classes of iterates to the right-hand side of (1.1), also known as the 'expected limit', developed to be a topic on its own (e.g., see [5, 7, 9, 12, 13, 14, 15, 19, 20, 21, 22, 30, 31, 32, 33, 39] for various results on polynomial and Hardy field functions of polynomial growth); the one of *joint ergodicity*.

Definition 1.2. Let $(X,\mathcal{B},\mu,T_1,\dots,T_d)$ be a measure-preserving system with commuting and invertible transformations[2] and $(a_1(n))_n,\dots,(a_d(n))_n$ be integer-valued sequences. We say that $(T_1^{a_1(n)})_n,\dots,(T_d^{a_d(n)})_n$ are *jointly ergodic (for μ)* if for any $f_1,\dots,f_d\in L^\infty(\mu)$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T_1^{a_1(n)}f_1\cdot\ldots\cdot T_d^{a_d(n)}f_d=\int_X f_1\,d\mu\cdot\ldots\cdot\int_X f_d\,d\mu, \tag{1.2}$$

where the convergence takes place in $L^2(\mu)$ (i.e., the $L^2(\mu)$-limit of the left-hand side of (1.2) exists, and it takes the value of the right-hand side).

The first characterization of joint ergodicity is due to Berend and Bergelson.

Theorem 1.3 [4]. *Let $(X,\mathcal{B},\mu,T_1,\dots,T_d)$ be a measure-preserving system with commuting and invertible transformations. Then $(T_1^n)_n,\dots,(T_d^n)_n$ are jointly ergodic (for μ) if and only if both of the following conditions are satisfied:*

(i) *$T_iT_j^{-1}$ is ergodic (for μ) for all $1\le i,j\le d, i\neq j$; and*
(ii) *$T_1\times\cdots\times T_d$ is ergodic (for $\mu^{\otimes d}$, where $\mu^{\otimes d}:=\mu\times\cdots\times\mu$ is the product measure in X^d).*

There is a plethora of analogous joint ergodicity characterizations for generalized linear functions [6], polynomial functions [12, 15, 23, 24] and Hardy field functions [13, 16].

The objective of this article is to prove a topological counterpart of Theorem 1.3.

1.2. *The joint transitivity problem*

A $\mathbb{Z}^k$*-system* is a tuple $(X,S_1,\dots,S_k)$, where X is a compact metric space and $S_1,\dots,S_k\colon X\to X$ are homeomorphisms with $S_iS_j=S_jS_i$ for all $1\le i,j\le k$. We let $\langle S_1,\dots,S_k\rangle$ denote the group generated by $S_1,\dots,S_k$.

In order to state the result corresponding to Theorem 1.3 in the topological setting, we start with the notion of *joint transitivity*; recall that a G_δ set is a subset of a topological space that is a countable intersection of open sets.

Definition 1.4. Let $(X,S_1,\dots,S_k)$ be a $\mathbb{Z}^k$-system, $T_1,\dots,T_d\in\langle S_1,\dots,S_k\rangle$ and $(a_1(n))_n,\dots,(a_d(n))_n$ be integer-valued sequences. We say that $(T_1^{a_1(n)})_n,\dots,(T_d^{a_d(n)})_n$ are *jointly transitive* if there is a G_δ-dense subset $X_0\subseteq X$ such that for all $x\in X_0$, the set

$$\{(T_1^{a_1(n)}x,\dots,T_d^{a_d(n)}x):n\in\mathbb{Z}\}$$

is dense in $X^d=X\times\cdots\times X$ (d times).[3] In the $d=1$ case, we say that $(T_1^{a_1(n)})_n$ is a *transitive* sequence.

We call a $\mathbb{Z}^k$-system $(X,S_1,\dots,S_k)$ *minimal* if, for any point $x\in X$, its orbit $\{S_1^{m_1}\cdot\ldots\cdot S_k^{m_k}x : (m_1,\dots,m_k)\in\mathbb{Z}^k\}$ is dense in X. A system is *transitive* if there exists a point $x\in X$ whose orbit is dense; we say that any such point x is a *transitive* point (of $(X,S_1,\dots,S_k)$). The pioneer work of Glasner [26] established joint transitivity for sequences given by iterates of powers of a transformation

[2] Naturally, by this we mean that $(X,\mathcal{B},\mu)$ is a standard probability space and $T_1,\dots,T_d : X\to X$ are invertible measure-preserving transformations with $T_iT_j=T_jT_i$ for all $1\le i,j\le d$.

[3] In [29], the authors call the family $(T_1^n,\dots,T_d^n)$ to be Δ*-transitive*. We use the term 'joint transitivity' to emphasize the direct parallelism of Theorem 1.3 to Theorem 1.6 in the measure theoretic setting on 'joint ergodicity'.

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in a minimal and (topologically) weakly mixing $\mathbb{Z}$-system (X, T) (meaning that the product system $(X \times X, T \times T)$ is transitive).

Theorem 1.5 [26]. *Let (X, T) be a minimal weakly mixing $\mathbb{Z}$-system. Then, for any $d \in \mathbb{N}$, and distinct nonzero integers $a_1, \ldots, a_d$, the sequences $(T^{a_1 n})_n, \ldots, (T^{a_d n})_n$ are jointly transitive.*

Theorem 1.5 can be thought as the topological analogue of Theorem 1.1. This result was extended by Huang, Shao and Ye in [29], who obtained topogically joint ergodicity results under weakly mixing assumptions of several transformations but were able to deal with polynomial expressions and nilpotent group actions. Some follow-up works on this line are given in [8, 41].

We are now ready to state our main result, which can be regarded as the topological version of Theorem 1.3.

Theorem 1.6. *Let $(X, S_1, \ldots, S_k)$ be a minimal system and $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k \rangle$. Then $(T_1^n)_n, \ldots, (T_d^n)_n$ are jointly transitive if and only if both of the following conditions are satisfied:*

(i) *$(X, T_i^{-1} T_j)$ is transitive for all $1 \leq i, j \leq d$, $i \neq j$; and*
(ii) *$(X^d, T_1 \times \cdots \times T_d)$ is transitive.*

Remark 1.7. As in the measurable case with Theorem 1.3, Theorem 1.6 provides a characterization for all linear iterates.

Indeed, assuming that the iterate of the T_i is $a_i n + b_i$, $a_i \in \mathbb{Z} \setminus \{0\}$, $b_i \in \mathbb{Z}$, $1 \leq i \leq d$, noting that the shifts by the b_i's do not affect the denseness of the orbits, we can use the Theorem 1.6 for the functions T^{a_i} (which still belong to $\langle S_1, \ldots, S_k \rangle$).

It is important to note that the problem in the topological setting differs significantly from the one in the measurable setting as the dense G_δ subset of X might have zero measure. For example, a minimal topologically weakly mixing system (X, T) may exhibit a discrete spectrum with respect to some invariant measure. In such a system, $(T^n)_n, \ldots (T^{dn})_n$ are jointly transitive but not jointly ergodic. In fact, any ergodic measure-preserving system is measurably isomorphic to a minimal and uniquely ergodic topologically (strongly) mixing system (see [35]).

We also want to emphasize that Theorem 1.6 fails without the minimality assumption. Indeed, in [36], Moothathu showed that there exists a nonminimal, strongly mixing shift (X, σ) such that, for every point $x \in X$, the set $\{(\sigma^n x, \sigma^{2n} x) : n \in \mathbb{Z}\}$ fails to be dense in X^2. Because (X, σ) is strongly mixing, the $\mathbb{Z}^2$-system (X, σ, σ^2) satisfies conditions (i) and (ii) of Theorem 1.6, but the sequences $(\sigma^n)_n$, $(\sigma^{2n})_n$ are not jointly transitive. It should be noted that there are no commuting transformations $S_1, \ldots, S_k$ that generate a minimal action and such that $\sigma \in \langle S_1, \ldots, S_k \rangle$. One reason for this is, of course, Theorem 1.6, but this can also be seen directly in Moothathus's example, using the fact that the set of σ-periodic points of a given period have to be preserved by $\langle S_1, \ldots, S_k \rangle$, which prevents minimality.

Due to Theorem 1.6 and recent developments in the theory of topological factors, we believe that there will be numerous results in the joint transitivity problem in the near future.

Problem 1.8. Analogously to Theorem 1.6, obtain joint transitivity characterizations for iterates that come from polynomial and Hardy field of polynomial growth functions, for which the corresponding results in the measure theoretic setting are known.

In particular, we chose to state the following conjecture which is the topological analogue of [15, Theorem 1.4]: a natural extension of the linear case.

Conjecture 1.9. *Let $(X, S_1, \ldots, S_k)$ be a minimal system, $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k \rangle$ and $p \in \mathbb{Z}[t]$. Then $(T_1^{p(n)})_n, \ldots, (T_d^{p(n)})_n$ are jointly transitive if and only if both of the following conditions are satisfied:*

(i) *$((T_i^{-1} T_j)^{p(n)})_n$ is transitive (in the space X) for all $1 \leq i, j \leq d$, $i \neq j$; and*
(ii) *the sequence $((T_1 \times \cdots \times T_d)^{p(n)})_n$ is transitive (in the product space X^d).*

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1.3. Structure of the paper

In Section 2, we recall some notions from the theory of dynamical systems. In particular, we provide equivalent statements to joint transitivity (Lemma 2.1), and we list properties of dynamical cubes and regional proximal relations.

In Section 3, we first characterize (in Theorem 3.1) regional proximal relations for product transformations, and finally, we prove Theorem 1.6 by an inductive argument.

2. Background material and useful facts

Definitions and conventions. For any $\mathbb{Z}^k$-system $(X, S_1, \ldots, S_k)$ and $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, we write $S_m := S_1^{m_1} \cdot \ldots \cdot S_k^{m_k}$. So, we may write a $\mathbb{Z}^k$-system as $(X, (S_m)_{m\in\mathbb{Z}^k})$ whenever we do not need to stress the generators. With this convention, a $\mathbb{Z}^k$-system is minimal if for any point $x \in X$, its orbit $\{S_m x : m \in \mathbb{Z}^k\}$ is dense in X. We adopt a similar notation for subgroups of $\mathbb{Z}^k$. If $G \subseteq \mathbb{Z}^k$ is a subgroup, then $(X, (S_m)_{m\in G})$ denotes the system given by the subaction of G.

We will use ρ to denote the metric on X, and slightly abusing notation, we will use ρ to denote the metric on the product space X^d as well, where $\rho((x_1, \ldots, x_d), (x_1', \ldots, x_d')) = \sup_{1\le i\le d} \rho(x_i, x_i')$.

Let $(X, S_1, \ldots, S_k)$ and $(Y, R_1, \ldots, R_k)$ be two $\mathbb{Z}^k$-systems. We say that Y is a *factor* of X (or that X is an *extension* of Y) if there exists a continuous and onto map $\pi\colon X \to Y$ (called the *factor map* from X to Y) such that $\pi \circ S_i = R_i \circ \pi$ for all $1 \le i \le k$. By slightly abusing the notation, we will sometimes use the same letter to denote the transformations that act on the space X and the factor Y. When π is a homeomorphism, we say that the systems are *topologically conjugate*.

There is a one-to-one correspondence between the factors and the closed invariant equivalence relations on X. Indeed, we can associate a factor map $\pi\colon X \to Y$ with the relation $R_\pi = \{(x, y) : \pi(x) = \pi(y)\}$, and conversely, given a closed invariant equivalence relation R, we can associate it with a factor map $X \to X/R$ being the quotient map. A factor map $\pi\colon X \to Y$ is *almost one-to-one* if there exists a G_δ-dense subset Ω of X such that for any $x \in \Omega$, $\pi^{-1}(\pi(x)) = \{x\}$. A factor map $\pi\colon X \to Y$ is *open* if $\pi(A) \subseteq Y$ is open whenever $A \subseteq X$ is open. Note that the latter implies that for any $(x, x') \in R_\pi$, and $\epsilon > 0$, there exists $\delta > 0$ such that if $\rho(x, y) < \delta$, then there exists $y' \in X$ with $\rho(x', y') < \epsilon$ and $(y, y') \in R_\pi$.

A system $(X, S_1, \ldots, S_k)$ is *equicontinuous* if the family of functions generated by $S_1, \ldots, S_k$ is equicontinuous. Any minimal equicontinuous $\mathbb{Z}^k$-system is topologically conjugate to a rotation on a compact abelian group (see [2, Chapter 2]). The *maximal equicontinuous factor* of a $\mathbb{Z}^k$-system $(X, S_1, \ldots, S_k)$, is the largest equicontinuous factor of it. That is, any equicontinuous factor of $(X, S_1, \ldots, S_k)$ is also a factor of the maximal one. The maximal equicontinuous factor of a minimal $\mathbb{Z}^k$-system is induced by the regionally proximal relation $\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$ (see Section 2.2).

Let $(X, S_1, \ldots, S_k)$ be a $\mathbb{Z}^k$-system. A pair (x, y) is *proximal* if there exists a sequence $(n_i)_i$ in $\mathbb{Z}^k$ such that $\rho(S_{n_i}x, S_{n_i}y) \to 0$ as i goes to infinity. The set of all proximal pairs is denoted by $P(X)$. It is well-known that $P(X) \subseteq \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$ (see, for instance, [38]). A factor map $\pi\colon X \to Y$ is *proximal* if $R_\pi \subseteq P(X)$. Any almost one-to-one factor map between minimal systems is proximal (see [2, Chapter 11]).

2.1. Equivalent definitions for joint transitivity

The following lemma provides a couple of equivalent definitions for joint transitivity. We will use this lemma implicitly throughout this paper. Its proof is a direct generalization of [37, Lemma 2.8] (see also [29, Lemma 2.4] and [36]).

Lemma 2.1. *Let $(X, S_1, \ldots, S_k)$ be a minimal $\mathbb{Z}^k$-system and $(a_1(n))_n, \ldots, (a_d(n))_n$ be sequences with values in $\mathbb{Z}^k$. The following are equivalent:*

(i) *There exists a dense G_δ subset Ω of X such that the set*

$$\{(S_{a_1(n)}x, \dots, S_{a_d(n)}x) \colon n \in \mathbb{Z}\}$$

is dense in X^d for every $x \in \Omega$.

(ii) *There exists some $x \in X$ such that the set*

$$\{(S_{a_1(n)}x, \dots, S_{a_d(n)}x) \colon n \in \mathbb{Z}\}$$

is dense in X^d.

(iii) *For every nonempty open subsets $U, V_1, \dots, V_d$ of X, there is some $n \in \mathbb{Z}$ such that*

$$U \cap S_{-a_1(n)}V_1 \cap \cdots \cap S_{-a_d(n)}V_d \neq \emptyset.$$

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. We next prove that (ii) implies (iii). To this end, let $x \in X$ be such that the set

$$\{(S_{a_1(n)}x, \dots, S_{a_d(n)}x) \colon n \in \mathbb{Z}\}$$

is dense in X^d. Then, for any $m \in \mathbb{Z}^k$, the set

$$X(x, m) := \{(S_{a_1(n)+m}x, \dots, S_{a_d(n)+m}x) \colon n \in \mathbb{Z}\}$$

is also dense in X^d. Now, for any nonempty open subsets $U, V_1, \dots, V_d$ of X, by the minimality of $(X, S_1, \dots, S_k)$, we may find some $m \in \mathbb{Z}^k$ such that $S_m x \in U$, and since $X(x, m)$ is dense in X^d, there exists $n \in \mathbb{Z}$ such that $S_{a_i(n)+m}x \in V_i$ for all $1 \le i \le d$. Therefore, the set

$$U \cap S_{-a_1(n)}V_1 \cap \cdots \cap S_{-a_d(n)}V_d$$

contains the point $T_m x$; hence, it is nonempty.

It remains to show that (iii) implies (i). Let $\mathcal{F}$ be a countable basis of the topology of X and define

$$\Omega := \bigcap_{V_1, \dots, V_d \in \mathcal{F}} \bigcup_{n \in \mathbb{Z}} S_{-a_1(n)}V_1 \cap \cdots \cap S_{-a_d(n)}V_d.$$

It follows from (iii) that Ω is a dense G_δ set. Moreover, the set

$$\{(S_{a_1(n)}x, \dots, S_{a_d(n)}x) \colon n \in \mathbb{Z}\}$$

is dense in X^d for every $x \in \Omega$. □

2.2. Dynamical cubes and regionally proximal relations

The notion of the regionally proximal relation was introduced by Ellis and Gottschalk [18] in the 1960s. Here, we will adapt a few notions from [17] to our setting.

Let $(X, S_1, \dots, S_k)$ be a system and (G_1, G_2) be a pair of subgroups of $\mathbb{Z}^k$. Recalling the definition of S_g for a $g \in \mathbb{Z}^k$ from the beginning of Section 2, we define the *space of* (G_1, G_2)*-cubes* as

$$\boldsymbol{Q}_{G_1,G_2}(X) := \overline{\{(x, S_{g_1}x, S_{g_2}x, S_{g_1+g_2}x) : x \in X, g_1 \in G_1, g_2 \in G_2\}} \subseteq X^4.$$

(We remark that such definitions were initially introduced in [17], for the case where each G_i is generated by a single transformation.) Given $x \in X$, and (G_1, G_2), let

$$\mathcal{F}_{G_1,G_2}(x) := \{(S_{g_1}x, S_{g_2}x, S_{g_1+g_2}x) : g_1 \in G_1, g_2 \in G_2\} \subseteq X^3.$$

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When G_i is generated by a single element g_i, we write $\boldsymbol{Q}_{G_i,G_i}(X)$ simply as $\boldsymbol{Q}_{S_{g_i},S_{g_i}}(X)$; a similar notation is used for $\mathcal{F}_{G_i,G_i}(x)$. Given a single subgroup G of $\mathbb{Z}^k$, we write

$$\mathbf{RP}_G(X) \coloneqq \overline{\{(x, S_g x) : x \in X, g \in G\}} \subseteq X^2$$

(this relation is called the prolongation relation in [3]). Note that if $(X, (S_m)_{m\in G})$ is transitive (meaning that there exists a point x such that $\{S_m x : m \in G\}$ is dense in X), then $\mathbf{RP}_G(X) = X \times X$. Similarly to [17] (or [27] for the case of a $\mathbb{Z}$-action), we define the relation $\mathbf{RP}_{G_1,G_2}(X)$ as the set of points $(x, y) \in X^2$ such that $(x, y, y, y) \in \boldsymbol{Q}_{G_1,G_2}(X)$. It should be noted that if (X, G) is minimal, then $\mathbf{RP}_{G,G}(X)$ is nothing more than the classical regionally proximal relation (see [2, Chapter 9] for more information on this relation).

We need the following lemma.

Lemma 2.2. *Let $(X, S_1, \dots, S_k)$ be a $\mathbb{Z}^k$-system and G_1, G_2 be subgroups of $\mathbb{Z}^k$.*

(i) *Let $\sigma\colon X^4 \to X^4$ be the map with $\sigma(a, b, c, d) \coloneqq (a, c, b, d)$. Then $\sigma(\boldsymbol{Q}_{G_1,G_2}(X)) = \boldsymbol{Q}_{G_2,G_1}(X)$.*

(ii) *Consider the system $(\mathbf{RP}_{G_1}(X), G_2^\Delta)$, where G_2^Δ is the action given by $g(x, y) = (gx, gy)$, for all $g \in G_2$ and $(x, y) \in X^2$.[4] Then $\mathbf{RP}_{G_2^\Delta}(\mathbf{RP}_{G_1}(X)) = \boldsymbol{Q}_{G_1,G_2}(X)$.*

(iii) *If H is a subgroup of $\mathbb{Z}^k$ and $\mathbf{RP}_{G_1}(X) = \mathbf{RP}_{G_2}(X)$, then $\boldsymbol{Q}_{G_1,H}(X) = \boldsymbol{Q}_{G_2,H}(X)$.*

(iv) *If G_1', G_2' and G_1, G_2 are subgroups of $\mathbb{Z}^k$ such that $G_1' \subseteq G_1$, and $G_2' \subseteq G_2$, then $\boldsymbol{Q}_{G_1',G_2'}(X) \subseteq \boldsymbol{Q}_{G_1,G_2}(X)$.*

Proof. (i) and (iv) follow directly from the definitions.

To show (ii), first note that for all $x \in X$, $g_1 \in G_1$, and $g_2 \in G_2$, the point $(x, S_{g_1}x, S_{g_2}x, S_{g_1+g_2}x)$ belongs to $\mathbf{RP}_{G_2^\Delta}(\mathbf{RP}_{G_1}(X))$ (here, we naturally identify this point with $((x, S_{g_1}x), (S_{g_2}x, S_{g_1+g_2}x))$). Therefore, $\boldsymbol{Q}_{G_1,G_2}(X) \subseteq \mathbf{RP}_{G_2^\Delta}(\mathbf{RP}_{G_1}(X))$. For the converse inclusion, it suffices to show that for any $(x, y) \in \mathbf{RP}_{G_1}(X)$ and any $g_2 \in G_2$, we have $(x, y, S_{g_2}x, S_{g_2}y) \in \boldsymbol{Q}_{G_1,G_2}(X)$. Let $\epsilon > 0$ and choose $0 < \delta < \epsilon$ so that if $z, z' \in X$ and $\rho(z, z') < \delta$, then $\rho(S_{g_2}z, S_{g_2}z') < \epsilon$. We can find $x' \in X$ and $g_1 \in G_1$ such that $\rho((x', S_{g_1}x'), (x, y)) < \delta$. It follows that $(x', S_{g_1}x', S_{g_2}x', S_{g_1+g_2}x')$ is at distance at most ϵ of $(x, y, S_{g_2}x, S_{g_2}y)$. Since $\epsilon > 0$ is arbitrary, we conclude that $(x, y, S_{g_2}x, S_{g_2}y) \in \boldsymbol{Q}_{G_1,G_2}(X)$ as desired.

(iii) follows immediately from (ii). □

Corollary 2.3. *Let $(X, S_1, \dots, S_k)$ be a $\mathbb{Z}^k$-system and let $G \subseteq \mathbb{Z}^k$ be a subgroup such that $(X, (S_m)_{m\in G})$ is transitive. Then, for any subgroup H of $\mathbb{Z}^k$, we have $\boldsymbol{Q}_{G,H}(X) = \boldsymbol{Q}_{\mathbb{Z}^k,H}(X)$. In particular, $\mathbf{RP}_{G,G}(X) = \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$.*

Proof. Note that by Lemma 2.2 (iv), the inclusion $\boldsymbol{Q}_{G,H}(X) \subseteq \boldsymbol{Q}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$ always holds. In addition, if G is transitive, since $\mathbf{RP}_G(X) = X \times X = \mathbf{RP}_{\mathbb{Z}^k}(X)$, Lemma 2.2 (iii) implies that $\boldsymbol{Q}_{G,H}(X) = \boldsymbol{Q}_{\mathbb{Z}^k,H}(X)$. Using (i) and (iii) of Lemma 2.2, we get $\boldsymbol{Q}_{G,G}(X) = \boldsymbol{Q}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$, from where $\mathbf{RP}_{G,G}(X) = \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$. □

We remark that Corollary 2.3 could also be deduced by using a proof similar to that of [17, Lemma 6.13].

Veech [40] proved that the regionally proximal relation is an equivalence relation for a minimal system and an abelian action. The first part of the following, now classical, theorem can be found, for example, in [2, Chapter 9], while the second one can be found in [38].

Theorem 2.4. *Let $(X, S_1, \dots, S_k)$ be a minimal system. Then $\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$ is an equivalence relation, the system $X/\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}$ is the maximal equicontinuous factor of X, and this factor is topologically conjugate to a rotation on a compact abelian group.*

Furthermore, if $\pi\colon X \to Y$ is a factor map between minimal $\mathbb{Z}^k$-systems, then $\pi \times \pi(\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)) = \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(Y)$.

[4]Note that $\mathbf{RP}_{G_1}(X)$ is invariant under this action since G_1 and G_2 commute.

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2.3. *The O-diagram*

The following is a classical theorem in the structural theory of topological dynamical systems and will be very useful for our purposes. We state a version for $\mathbb{Z}^k$, giving only the information we need for our work. We note that this theorem is valid for general group actions. For further details, the interested reader may consult [10, Chapter VI, Section 3] or [2, Chapter 14].

Theorem 2.5. *Let $(X, S_1, \ldots, S_k)$ and $(Y, R_1, \ldots, R_k)$ be two $\mathbb{Z}^k$-minimal systems and $\pi\colon X \to Y$ a factor map. Then there exist two $\mathbb{Z}^k$-minimal systems $(\tilde{X}, \tilde{S}_1, \ldots, \tilde{S}_k)$ and $(\tilde{Y}, \tilde{R}_1, \ldots, \tilde{R}_k)$ and factor maps $\tilde{\pi}\colon \tilde{X} \to X$, $\tilde{\sigma}\colon \tilde{X} \to X$, $\tilde{\tau}\colon \tilde{Y} \to Y$ such that the following diagram (which is called the O-diagram)*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ \tilde{Y} \arrow[r, "\tilde{\tau}"] & Y \end{tikzcd}$$

is commutative, $\tilde{\sigma}$ and $\tilde{\tau}$ are almost one-to-one, and $\tilde{\pi}$ is open.

Theorem 2.5 says that, modulo almost one-to-one extensions, we may assume that the factor map is open.

3. The proof of the main result

3.1. *A characterization for the regional proximal relation for product transformations*

The following is the main tool we use in the proof of Theorem 1.6 and can be interpreted as a topological analogue of seminorm control in product spaces (see [15, Lemma 5.2] or [12, Lemma 3.4] for analogous statements in the measurable setting).

Theorem 3.1. *Let $(X, S_1, \ldots, S_k)$ be a minimal $\mathbb{Z}^k$-system and $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k\rangle$. Let $(Y_i, S_1, \ldots, S_k)$, $1 \le i \le d$ be factors of X, such that for all i, the factor map $\pi_i\colon X \to Y_i$ is open, and $R_{\pi_i} \subseteq \mathbf{RP}_{T_i,T_i}(X)$. For all $1 \le i \le d$, let $(x_i, y_i) \in R_{\pi_i}$. Then*

$$((x_1, \ldots, x_d), (y_1, \ldots, y_d)) \in \mathbf{RP}_{T_1\times\cdots\times T_d}(X^d).$$

Roughly speaking, Theorem 3.1 can be interpreted as saying that $(X^d/\mathbf{RP}_{T_1\times\cdots\times T_d}(X^d), T_1\times\cdots\times T_d)$ is a factor of $(X/\mathbf{RP}_{T_1,T_1}(X)\times\cdots\times X/\mathbf{RP}_{T_d,T_d}(X), T_1\times\cdots\times T_d)$ (assuming that all the quotient spaces are well defined).

The main ingredient in proving Theorem 3.1 is to show that if $(x_i, y_i) \in \mathbf{RP}_{T_i,T_i}(X)$ for some fixed $1 \le i \le d$, then we may find a common time n such that a neighborhood of x_j returns to itself under T_j^n for all j, while a neighborhood of x_i visits a neighborhood of y_i under T_i^n. This is done in Lemma 3.3. We then use Lemma 3.3 repeatedly to move the point $(x_1, \ldots, x_d)$ to $(y_1, \ldots, y_d)$ by changing one coordinate at a time.

In order to prove Lemma 3.3, we need the following lemma which shows that the set of visiting times of x to a neighborhood of y under T^n, when $(x, y, x) \in \overline{\mathcal{F}_{T,T}}(x)$, contains the difference set of an infinite sequence. The proof can be deduced from the proof of [28, Theorem 7.3.2] or by using arguments from [40]. We give a proof for completeness.

Lemma 3.2. *Let (X, T) be a $\mathbb{Z}$-system and let $x, y \in X$ be such that $(x, y, x) \in \overline{\mathcal{F}_{T,T}}(x)$. Then for any open neighborhood U of y, there is a sequence $(a_i)_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ of integers taking infinitely many values such that the set $\{n \in \mathbb{Z} : T^n x \in U\}$ contains $\{a_j - a_i : j > i\}$.*

Proof. Let $\epsilon > 0$ be such that $B(y, \epsilon) \subseteq U$. For $i \in \mathbb{N}$, set $\epsilon_i = \epsilon/2^i$. Construct a sequence $(\delta_i)_{i\in\mathbb{N}}$, with $0 < \delta_i < \epsilon_i$ and a sequence $(m_i, n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}\times\mathbb{Z}$ as follows: Let n_1, m_1 be such that $\rho(T^{n_1}x, x) < \epsilon_1$, $\rho(T^{m_1}x, y) < \epsilon_1$, and $\rho(T^{n_1+m_1}x, x) < \epsilon_1$. Pick $0 < \delta_2 < \epsilon_2$ such that $\rho(z, z') < \delta_2$ implies that

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$\rho(T^a z, T^a z') < \epsilon_2$ for all $|a| \le |n_1| + |m_1|$. Take n_2, m_2 such that $\rho(T^{n_2}x, x) < \delta_2$, $\rho(T^{m_2}x, y) < \delta_2$ and $\rho(T^{n_2+m_2}x, x) < \delta_2$. (We highlight here that the numbers n_2, m_2 can be taken to be arbitrarily large.) Note that the definition of δ_2 implies that

$$\rho(T^{n_2+n_1}x, x) < \epsilon_1 + \epsilon_2,\ \rho(T^{n_2+n_1+m_1}x, x) < \epsilon_1 + \epsilon_2,\ \rho(T^{n_2+m_2+n_1}x, x) < \epsilon_1 + \epsilon_2,$$
$$\rho(T^{n_2+m_2+n_1+m_1}x, x) < \epsilon_1 + \epsilon_2,\ \rho(T^{n_2+m_1}x, y) < \epsilon_1 + \epsilon_2,\ \text{ and }\ \rho(T^{n_2+m_2+m_1}x, y) < \epsilon_1 + \epsilon_2.$$

So, if we set $R_1 = \{n_1, n_1 + m_1\}$, $P_1 = \{m_1\}$, $R_2 = \{n_2, n_2 + m_2\}$, we have that $\rho(T^{a+b}x, x) < \epsilon_1 + \epsilon_2$ for all $a \in R_2$ and $b \in R_1$, and $\rho(T^{a+c}x, y) < \epsilon_1 + \epsilon_2$ for all $a \in R_2$ and $c \in P_1$ (here, R stands for 'return' and P for 'passage').

The idea of the proof is that return times associated with large indices are compatible with return times and passages associated with smaller indices. More precisely, inductively, suppose that we have defined δ_i, m_i and n_i for all $1 \le i \le l$ for some $l \in \mathbb{N}$, and for the set $R_i = \{n_i, n_i + m_i\}$ and $P_i = \{m_i\}$, we have that if $a = r_{j_1} + \cdots + r_{j_l}$, with $r_{j_k} \in R_{j_k}$, $j_1 < \ldots < j_l$, then $\rho(T^a x, x) < \sum_{t=1}^{l} \epsilon_{j_t}$, and $\rho(T^{a+c}x, y) < \epsilon_k + \sum_{t=1}^{l} \epsilon_{j_t}$ if $c \in P_k$, $k < j_1$.

Let $0 < \delta_{i+1} < \epsilon_{i+1}$ be such that $\rho(z, z') < \delta_{i+1}$ implies that $\rho(T^a z, T^a z') < \epsilon_{i+1}$ for all $|a| \le |n_1| + |m_1| + \cdots + |n_i| + |m_i|$. Then choose n_{i+1} and m_{i+1} such that $\rho(T^{n_{i+1}}x, x) < \delta_{i+1}$, $\rho(T^{n_{i+1}+m_{i+1}}x, x) < \delta_{i+1}$, and $\rho(T^{m_{i+1}}x, y) < \delta_{i+1}$, and set $R_{i+1} = \{n_{i+1}, n_{i+1} + m_{i+1}\}$, $P_{i+1} = \{m_{i+1}\}$.

We claim that if $a = r_{j_1} + r_{j_2} + \cdots + r_{j_l}$, with $r_{j_k} \in R_{j_k}$, $j_1 < \ldots < j_l \le i+1$, then $\rho(T^a x, x) < \sum_{t=1}^{l} \epsilon_{j_t}$, and $\rho(T^{a+c}x, y) < \epsilon_k + \sum_{t=1}^{l} \epsilon_{j_t}$ if $c \in P_k$, $k < j_1$.

We only need to check the case $j_l = i + 1$. Assume that $r_{i+1} = n_{i+1}$ (the case $r_{i+1} = n_{i+1} + m_{i+1}$ is identical). Since $\rho(T^{n_{i+1}}x, x) < \delta_{i+1}$, and $|a - n_{i+1}| \le |n_1| + |m_1| + \cdots + |n_i| + |m_i|$, we get $\rho(T^a x, T^{a-n_{i+1}}x) < \epsilon_{i+1}$. By induction, $\rho(T^{a-n_{i+1}}x, x) < \sum_{t=1}^{l-1} \epsilon_{j_t}$, so the triangle inequality implies that $\rho(T^a x, x) < \sum_{t=1}^{l} \epsilon_{j_t}$, as desired. The estimate of $\rho(T^{a+c}x, y)$ follows in a similar way.

Now set $a_i = n_{i+1} + \sum_{j=1}^{i}(n_j + m_j)$. Note that we may further require the sequence $(a_i)_{i\in\mathbb{N}}$ to take infinitely many values (by choosing the n_i's and m_i's to go to infinity) and $a_{i+l} - a_i = n_{i+l+1} + \sum_{j=i+2}^{i+l}(n_j + m_j) + m_{i+1}$. Hence, we can rewrite this as $a_{i+l} - a_i = r_{i+l+1} + \sum_{j=i+2}^{i+l} r_j + c$, where $r_j \in R_j$ and $c \in P_{i+1}$. It follows from the construction of the sequence $\{n_i, m_i : i \in \mathbb{N}\}$ that $\rho(T^{a_{i+l}-a_i}x, y) < \sum_{t=i+1}^{i+l+1} \epsilon_i < \epsilon$, which implies that $T^{a_{i+l}-a_i}x \in U$, as was to be shown. ◻

Lemma 3.3. *Let $(X, S_1, \ldots, S_k)$ be a minimal $\mathbb{Z}^k$-system, $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k\rangle$ and $\pi\colon X \to Y$ an open factor map with $R_\pi \subseteq \mathbf{RP}_{T_i,T_i}(X)$ for some $1 \le i \le d$. Let $x_1, \ldots, x_d, x_i' \in X$ with $(x_i, x_i') \in R_\pi$, and U_z be a neighborhood of z for $z = x_1, \ldots, x_d, x_i'$. There exist $n \in \mathbb{Z}$ such that $T_i^n U_{x_i} \cap U_{x_i'} \ne \emptyset$ and $T_j^n U_{x_j} \cap U_{x_j} \ne \emptyset$ for all $1 \le j \le d$.*

Proof. The set Ω_i of $\tilde{x} \in X$ such that the set $\{(a, b, c) : (\tilde{x}, a, b, c) \in \boldsymbol{Q}_{T_i,T_i}(X)\}$ equals $\overline{\mathcal{F}_{T_i,T_i}}(\tilde{x})$ is a dense G_δ set of points (see, for instance, [26, Lemma 4.5]). Since π is open, we can find $\tilde{x} \in U_{x_i} \cap \Omega_i$ and $\tilde{y} \in U_{x_i'}$ with $(\tilde{x}, \tilde{y}) \in R_\pi$. Because $R_\pi \subseteq \mathbf{RP}_{T_i,T_i}(X)$, we have that $(\tilde{x}, \tilde{x}, \tilde{y}, \tilde{x}) \in \boldsymbol{Q}_{T_i,T_i}(X)$, and since $\tilde{x} \in \Omega$, we obtain $(\tilde{x}, \tilde{y}, \tilde{x}) \in \overline{\mathcal{F}_{T_i,T_i}}(\tilde{x})$. By Lemma 3.2, the set $\{n \in \mathbb{Z} : T_i^n \tilde{x} \in U_{x_i'}\}$ contains a set of the form $\{a_r - a_{r'} : r > r'\}$ for some $\mathbb{Z}$-valued sequence $(a_i)_{i\in\mathbb{N}}$ taking infinitely many values. In particular, the same is true for the set $\{n \in \mathbb{Z} : T_i^n U_{x_i} \cap U_{x_i'} \ne \emptyset\}$. Let μ be a $\mathbb{Z}^k$-invariant measure on X. By the minimality of $\langle S_1, \ldots, S_k\rangle$, μ has full support. Consider the product system $(X_1 \times \cdots \times X_d, \mathcal{B}(X)^{\otimes d}, \mu^{\otimes d}, T_1 \times \cdots \times T_d)$ and $U = U_{x_1} \times \cdots \times U_{x_d}$. Then $\mu^{\otimes d}(U) > 0$, and so by the proof of the Poincaré recurrence theorem, the set $\{n \in \mathbb{Z} : \mu^{\otimes d}(U \cap (T_1 \times \cdots \times T_d)^{-n}U) > 0\}$ must intersect nontrivially every infinite set of the form $\{a_r - a_{r'} : r > r'\}$. This implies that it has nonempty intersection with $\{n \in \mathbb{Z} : T_i^n U_{x_i} \cap U_{x_i'} \ne \emptyset\}$. Picking now an $n \in \mathbb{Z}$ in the intersection, we get that $T_i^n U_{x_i} \cap U_{x_i'} \ne \emptyset$ and $T_j^n U_{x_j} \cap U_{x_j} \ne \emptyset$ for all $1 \le j \le d$, as desired. ◻

Proof of Theorem 3.1. Our goal here is to find, for every $\epsilon > 0$, a point $(z_1, \ldots, z_d)$ and an integer n so that $(x_1, \ldots, x_d)$ is close to $(z_1, \ldots, z_d)$ and $(y_1, \ldots, y_d)$ is close to $(T^n z_1, \ldots, T^n z_d)$. We do so by induction on the coordinates. To this end, fix $\epsilon > 0$ and set $\epsilon_d := \epsilon$. Suppose that we have constructed

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$\epsilon_{r+1}, \ldots, \epsilon_d > 0$ for some $1 \leq r \leq d-1$. We let $0 < \epsilon_r < \epsilon_{r+1}/2$ to be a number such that for any $z_{r+1} \in X$ with $\rho(y_{r+1}, z_{r+1}) < \epsilon_r$, there exists $x'_{r+1} \in X$ with $\rho(x'_{r+1}, x_{r+1}) < \epsilon_{r+1}/2$ such that $(x'_{r+1}, z_{r+1}) \in R_{\pi_{r+1}}$. The existence of such ϵ_r follows from the assumptions that $(x_{r+1}, y_{r+1}) \in R_{\pi_{r+1}}$, and that the map $X \mapsto Y_{r+1} = X/R_{\pi_{r+1}}$ is open.

For $1 \leq r \leq d$, we say that *Property r* holds if there exist $z_1, \ldots, z_d \in X$ and $n \in \mathbb{Z}$ such that

- $\rho(x_i, z_i) < \epsilon_r$ for all $1 \leq i \leq r$;
- $\rho(y_i, z_i) < \epsilon_r$ for all $r+1 \leq i \leq d$;
- $\rho(y_i, T_i^n z_i) < \epsilon_r$ for all $1 \leq i \leq d$.

By Lemma 3.3, setting $x_j = y_j, 1 \leq j \leq d, x'_1 = x_1$, and $U_z = B(z, \epsilon_1)$ for $z = y_1, \ldots, y_d, x_1$, for $-n$, we have that Property 1 holds. Now suppose that Property r holds for some $1 \leq r \leq d-1$. Since $(x_{r+1}, y_{r+1}) \in R_{\pi_{r+1}}$ and $\rho(y_{r+1}, z_{r+1}) < \epsilon_r$, by the construction of ϵ_r, there exists $x'_{r+1} \in X$ with $\rho(x'_{r+1}, x_{r+1}) < \epsilon_{r+1}/2$ such that $(x'_{r+1}, z_{r+1}) \in R_{\pi_{r+1}}$. Let $\delta' := \min\{\epsilon_{r+1}/2, \epsilon_{r+1} - \epsilon_r\}$. Take $0 < \delta < \delta'$ such that for all $x, y \in X$, if $\rho(x, y) < \delta$, then $\rho(T^n x, T^n y) < \delta'$ (n is the one from Property r above). By Lemma 3.3, there exist $z'_1, \ldots, z'_d \in X$ and $n' \in \mathbb{Z}$ such that

- $\rho(z_i, z'_i) < \delta$ for all $1 \leq i \leq d, i \neq r+1$;
- $\rho(z_i, T_i^{n'} z'_i) < \delta$ for all $1 \leq i \leq d, i \neq r+1$;
- $\rho(x'_{r+1}, z'_{r+1}) < \delta$;
- $\rho(z_{r+1}, T_{r+1}^{n'} z'_{r+1}) < \delta$.

Then for all $1 \leq i \leq r$,

$$\rho(x_i, z'_i) \leq \rho(x_i, z_i) + \rho(z_i, z'_i) < \epsilon_r + \delta \leq \epsilon_{r+1}.$$

For all $r+2 \leq i \leq d$,

$$\rho(y_i, z'_i) \leq \rho(y_i, z_i) + \rho(z_i, z'_i) < \epsilon_r + \delta \leq \epsilon_{r+1}.$$

Moreover,

$$\rho(x_{r+1}, z'_{r+1}) \leq \rho(x_{r+1}, x'_{r+1}) + \rho(x'_{r+1}, z'_{r+1}) < \epsilon_{r+1}/2 + \delta \leq \epsilon_{r+1}.$$

However, for all $1 \leq i \leq d, i \neq r+1$, since $\rho(z_i, T_i^{n'} z_i) < \delta$, we have $\rho(T_i^n z_i, T_i^{n+n'} z_i) < \delta'$ and so

$$\rho(y_i, T_i^{n+n'} z_i) \leq \rho(y_i, T_i^n z_i) + \rho(T_i^n z_i, T_i^{n+n'} z_i) < \epsilon_r + \delta' \leq \epsilon_{r+1}.$$

Finally, since $\rho(z_{r+1}, T_{r+1}^{n'} z_{r+1}) < \delta$, we have that $\rho(T_{r+1}^n z_{r+1}, T_{r+1}^{n+n'} z_{r+1}) < \delta'$ and so

$$\rho(y_{r+1}, T_{r+1}^{n+n'} z_{r+1}) \leq \rho(y_{r+1}, T_{r+1}^n z_{r+1}) + \rho(T_{r+1}^n z_{r+1}, T_{r+1}^{n+n'} z_{r+1}) < \epsilon_r + \delta' \leq \epsilon_{r+1}.$$

In conclusion, we have that that Property $r+1$ holds.

So it follows from induction that Property d holds, which means that there exist $(z_1, \ldots, z_d) \in X^d$ and $n \in \mathbb{Z}$ such that

$$\rho((x_1, \ldots, x_d), (z_1, \ldots, z_d)) < \epsilon \text{ and } \rho((y_1, \ldots, y_d), (T_1^n z_1, \ldots, T_d^n z_d)) < \epsilon.$$

Since ϵ is arbitrary, we have that $((x_1, \ldots, x_d), (y_1, \ldots, y_d)) \in \mathbf{RP}_{T_1 \times \cdots \times T_d}(X^d)$. □

As a consequence of Theorem 3.1, we have the following:

Proposition 3.4. *Let $(X, S_1, \ldots, S_k)$ be a minimal $\mathbb{Z}^k$-system and $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k \rangle$. Suppose that $(X, T_1), \ldots, (X, T_d)$ are transitive. Then $(X^d, T_1 \times \cdots \times T_d)$ is transitive if and only if $(Y^d, T_1 \times \cdots \times T_d)$ is transitive, where $Y = X/\mathbf{RP}_{\mathbb{Z}^k, \mathbb{Z}^k}(X)$.*

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Proof. The 'only if' part is straightforward. Now assume that $(Y^d, T_1 \times \cdots \times T_d)$ is transitive. By the O-diagram (Theorem 2.5), we may consider almost one-to-one extensions $\tilde{X}, \tilde{Y}$ of X and Y, respectively, such that the projection $\tilde{\pi}\colon \tilde{X} \to \tilde{Y}$ is open. Note that $(\tilde{X}, \tilde{T}_1), \ldots, (\tilde{X}, \tilde{T}_d)$ and $(\tilde{Y}^d, \tilde{T}_1 \times \cdots \times \tilde{T}_d)$ are also transitive because this property is preserved under almost one-to-one extensions (see [1]). We now show that $(\tilde{X}^d, \tilde{T}_1 \times \cdots \times \tilde{T}_d)$ is transitive, which implies that $(X^d, T_1 \times \cdots \times T_d)$ is transitive.

Note that since $\tilde{X}$ is an almost one-to-one extension of X, we have that $\tilde{X}/\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$ and $X/\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$ are conjugate, so we have that $R_{\tilde{\pi}}$ is a subset of $\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$. To see this, by only assuming that $\tilde{\sigma}$ is proximal (which covers the almost one-to-one case), let q be the projection from $\tilde{X}$ to $X/\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$. It suffices to show that $R_q \subseteq \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$. Let $\tilde{x}, \tilde{x}' \in \tilde{X}$ with $q(\tilde{x}) = q(\tilde{x}')$. Then $(\tilde{\sigma}(\tilde{x}), \tilde{\sigma}(\tilde{x}')) \in \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$. By the second part of Theorem 2.4, we can find $(\tilde{y}, \tilde{y}') \in \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$ such that $(\tilde{\sigma}(\tilde{y}), \tilde{\sigma}(\tilde{y}')) = (\tilde{\sigma}(\tilde{x}), \tilde{\sigma}(\tilde{x}'))$. It follows that $(\tilde{x}, \tilde{y}), (\tilde{x}', \tilde{y}') \in P(\tilde{X})$ (the proximal relation on $\tilde{X}$). Since $P(\tilde{X}) \subseteq \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$, and this is an equivalence relation, we conclude $(\tilde{x}, \tilde{x}') \in \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X})$.[5]

Let U, V be nonempty open subsets of $\tilde{X}^d$. Our goal is to find $m \in \mathbb{N}$ such that $U \cap (\tilde{T}_1 \times \cdots \times \tilde{T}_d)^{-m} V \neq \emptyset$. Then $\tilde{\pi}^{\times d}(U)$ and $\tilde{\pi}^{\times d}(V)$ are nonempty open sets, where $\tilde{\pi}^{\times d} := \pi \times \cdots \times \pi$ (d-times). Since $(\tilde{Y}^d, \tilde{T}_1 \times \cdots \times \tilde{T}_d)$ is transitive, there exist $(x_1, \ldots, x_d) \in U$ and $n \in \mathbb{Z}$ such that $\tilde{\pi}^{\times d}(\tilde{T}_1^n x_1, \ldots, \tilde{T}_d^n x_d) \in \tilde{\pi}^{\times d}(V)$. That is, there exists $(x_1', \ldots, x_d') \in V$ such that $(\tilde{T}_i^n x_i, x_i') \in R_{\tilde{\pi}}$ for all $1 \leq i \leq d$. Let $\epsilon > 0$ be such that $B((x_1, \ldots, x_d), \epsilon) \subseteq U$ and $B((x_1', \ldots, x_d'), \epsilon) \subseteq V$. Take $0 < \delta < \epsilon$ so that $\rho(a, b) < \delta$ implies $\rho(\tilde{T}_i^{-n} a, \tilde{T}_i^{-n} b) < \epsilon$ for $1 \leq i \leq d$. Thanks to the transitivity of $\tilde{T}_i$, using Corollary 2.3, we get that $\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(\tilde{X}) = \mathbf{RP}_{\tilde{T}_i,\tilde{T}_i}(\tilde{X})$ and thus $R_{\tilde{\pi}} \subseteq \mathbf{RP}_{\tilde{T}_i,\tilde{T}_i}(\tilde{X})$ for all $1 \leq i \leq d$. By Theorem 3.1, we obtain $((\tilde{T}_1^n x_1, \ldots, \tilde{T}_d^n x_d), (x_1', \ldots, x_d')) \in \mathbf{RP}_{\tilde{T}_1 \times \cdots \times \tilde{T}_d}(\tilde{X}^d)$. Therefore, there exist $(y_1, \ldots, y_d) \in \tilde{X}^d$ and $m \in \mathbb{Z}$ such that $\rho((y_1, \ldots, y_d), (\tilde{T}_1^n x_1, \ldots, \tilde{T}_d^n x_d)) < \delta$ and $\rho((\tilde{T}_1^m y_1, \ldots, \tilde{T}_d^m y_d), (x_1', \ldots, x_d')) < \delta$. It follows that $(\tilde{T}_1^{-n} y_1, \ldots, \tilde{T}_d^{-n} y_d) \in U$ and $(\tilde{T}_1^{m+n} \tilde{T}_1^{-n} y_1, \ldots, \tilde{T}_d^{m+n} \tilde{T}_d^{-n} y_d) \in V$. Therefore, $U \cap (\tilde{T}_1 \times \cdots \times \tilde{T}_d)^{-(m+n)} V \neq \emptyset$. We conclude that $(\tilde{X}^d, \tilde{T}_1 \times \cdots \times \tilde{T}_d)$ is transitive. □

3.2. *The proof of Theorem 1.6*

In this last subsection we prove Theorem 1.6. We start with its forward direction, which is almost straightforward.

Proof of the forward direction of Theorem 1.6. We use Lemma 2.1 implicitly throughout the proof. Assume that $(T_1^n)_n, \ldots, (T_d^n)_n$ are jointly transitive. Equivalently, for all $V_0, \ldots, V_d$ nonempty and open subsets of X, there exists $n \in \mathbb{Z}$ such that

$$V_0 \cap T_1^{-n} V_1 \cap \cdots \cap T_d^{-n} V_d \neq \emptyset. \tag{3.1}$$

To show (i), pick any $i \neq j$, and let U_0, U_1 be nonempty opens sets. Setting $V_i = U_0, V_j = U_1$ and $V_k = X$ for all $k \in \{0, \ldots, d\} \setminus \{i, j\}$, it follows from (3.1) that $T_i^{-n} V_i \cap T_j^{-n} V_j \neq \emptyset$, or $U_0 \cap (T_i^{-1} T_j)^{-n} U_1 \neq \emptyset$. To show (ii), pick any point x for which $\{(T_1^n x, \ldots, T_d^n x) : n \in \mathbb{Z}\}$ is dense in X. Then the point $(x, \ldots, x)$ is a transitive point for $T_1 \times \cdots \times T_d$. □

It remains to show the inverse direction of Theorem 1.6. To this end, we first need a couple of statements. In particular, the first one will allow us to run an inductive argument.

Proposition 3.5. *Let $(X, S_1, \ldots, S_k)$ be a minimal $\mathbb{Z}^k$-system, $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k \rangle$, and $R_2 := T_2 T_1^{-1}, \ldots, R_d := T_d T_1^{-1}$. If $(X, R_2), \ldots, (X, R_d)$ and $(X^d, T_1 \times \cdots \times T_d)$ are transitive, then $(X^{d-1}, R_2 \times \cdots \times R_d)$ is transitive.*

Proof. Since $(X, R_2), \ldots, (X, R_d)$ are transitive, by Corollary 2.3, we have $\mathbf{RP}_{R_2,R_2}(X) = \ldots = \mathbf{RP}_{R_d,R_d}(X) = \mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$, which is an equivalence relation since $(X, S_1, \ldots, S_k)$ is minimal.

[5] This should be a well-known result; we chose to present its short proof (which simplifies the one of [11, Lemma 5.3] that covers the $\mathbb{Z}$ case and almost one-to-one extensions) for completeness.

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By Proposition 3.4, it suffices to show that $(Y^{d-1}, R_2 \times \cdots \times R_d)$ is transitive, where $Y = X/\mathbf{RP}_{\mathbb{Z}^k,\mathbb{Z}^k}(X)$. By Theorem 2.4, we have that $(Y, S_1, \ldots, S_k)$ is a rotation on a compact abelian group, and so we may write $T_i(y) = y + \alpha_i$ for $\alpha_i \in Y$ for $1 \le i \le d$. We can choose a metric ρ on Y that is compatible with its topology, such that $S_1, \ldots, S_k$ act as isometries on Y. Since $(X^d, T_1 \times \cdots \times T_d)$ is transitive, we get that $(Y^d, T_1 \times \cdots \times T_d)$ is transitive, and hence minimal. (This holds because rotations are distal, and in this class transitivity and minimality are equivalent conditions – for example, see [2, Chapters 2 and 5].)

Take $y \in Y$ and $(y_2, \ldots, y_d) \in Y^{d-1}$. Since $(Y^d, T_1 \times \cdots \times T_d)$ is minimal, given $\epsilon > 0$, there exists n such that $\rho(y + n\alpha_1, y) < \epsilon$, and $\rho(y + n\alpha_i, y_i) < \epsilon$, for all $2 \le i \le d$. It follows that

$$\rho(y + n(\alpha_i - \alpha_1), y_i) \le \rho(y + n(\alpha_i - \alpha_1), y + n\alpha_i) + \rho(y + n\alpha_i, y_i) = \rho(y, y + n\alpha_1) + \rho(y + n\alpha_i, y_i) < 2\epsilon$$

for all $2 \le i \le d$. As $\epsilon > 0$ is arbitrary, we get that $(y_2, \ldots, y_d)$ belongs to the orbit closure of $(y, \ldots, y) \in X^{d-1}$ under $R_2 \times \cdots \times R_d$. Since $y_2, \ldots, y_d$ are arbitrary, this orbit closure is all of Y^{d-1}. We get that $(Y^{d-1}, R_2 \times \cdots \times R_d)$ is minimal, as it is the orbit closure of a point in an equicontinuous system. Proposition 3.4 allows us to conclude. □

The following lemma is a generalization of [8, Lemma 2.9] (see also [34, Lemma 3]).

Lemma 3.6. *Let $(X, S_1, \ldots, S_k)$ be a $\mathbb{Z}^k$-system and $T_1, \ldots, T_d \in \langle S_1, \ldots, S_k \rangle$. Let $(R_j)_{1 \le j \le N}$ be a finite sequence of continuous maps from X to X. Assume that $(X^d, T_1 \times \cdots \times T_d)$ is transitive. Then for all nonempty open sets $V_1, \ldots, V_d$, there exists $n_j \in \mathbb{Z}, 1 \le j \le N$, and for each $1 \le i \le d$ a nonempty open subset $\tilde{V}_i$ of V_i such that*

$$T_i^{-n_j} R_j^{-1} \tilde{V}_i \subseteq V_i \text{ for all } 1 \le j \le N, 1 \le i \le d.$$

Proof. We use induction on N. Since $(X^d, T_1 \times \cdots \times T_d)$ is transitive, there exists $n_1 \in \mathbb{Z}$ such that $T_i^{-n_1} R_1^{-1} V_i \cap V_i \ne \emptyset$ for all $1 \le i \le d$. Set $\tilde{V}_i = V_i^{(1)} := T_i^{-n_1} R_1^{-1} V_i \cap V_i$. This completes the proof for the case $N = 1$.

Now assume that for some $N \ge 2$, we have constructed $n_1, \ldots, n_{N-1} \in \mathbb{N}$ with $n_1 < \ldots < n_{N-1}$, and for each $1 \le i \le d$ a sequence of nonempty open sets $V_i \supseteq V_i^{(1)} \supseteq \ldots \supseteq V_i^{(N-1)}$ such that

$$T_i^{-n_j} R_j^{-1} V_i^{(m)} \subseteq V_i \text{ for all } 1 \le j \le m, 1 \le m \le N-1, 1 \le i \le d.$$

Let $U_i := R_N^{-1} V_i^{(N-1)}$. Since $(X^d, T_1 \times \cdots \times T_d)$ is transitive, there exists $n_N \in \mathbb{N}$ with $n_N > n_{N-1}$ such that $T_i^{-n_N} U_i \cap V_i \ne \emptyset$ for all $1 \le i \le d$. This implies that

$$V_i \cap T_i^{-n_N} R_N^{-1} V_i^{(N-1)} = V_i \cap T_i^{-n_{N-1}} U_i \ne \emptyset.$$

Let

$$V_i^{(N)} := V_i^{(N-1)} \cap (T_i^{-n_N} R_N^{-1})^{-1} V_i.$$

Then $V_i^{(N)} \subseteq V_i^{(N-1)}$ is a nonempty open set and $T_i^{-n_N} R_N^{-1} V_i^{(N)} \subseteq V_i$. Since $V_i^{(N)} \subseteq V_i^{(N-1)}$, we also have that

$$T_i^{-n_j} R_j^{-1} V_i^{(N)} \subseteq V_i \text{ for all } 1 \le j \le N-1, 1 \le i \le d.$$

This completes the induction step, and we are done by setting $\tilde{V}_i := V_i^{(N)}$. □

Proof of the inverse direction of Theorem 1.6. There is nothing to prove when $d = 1$. Now we assume that Theorem 1.6 holds for $d - 1$ for some $d \ge 2$, and we prove it for d. By Proposition 3.5, conditions (i) and (ii) imply that $(X^{d-1}, T_2 T_1^{-1} \times \cdots \times T_d T_1^{-1})$ is transitive. However, we have that

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$(T_iT_1^{-1})^{-1}(T_jT_1^{-1}) = T_i^{-1}T_j$ is transitive for all $1 \le i, j \le d, i \ne j$. So, by induction hypothesis, we have that $((T_2T_1^{-1})^n)_n, \dots, ((T_dT_1^{-1})^n)_n$ are jointly transitive.

For $m = (m_1, \dots, m_k) \in \mathbb{Z}^k$, recall that $S_m = S_1^{m_1} \cdot \ldots \cdot S_k^{m_k}$. Let $U, V_1, \dots, V_d$ be open and nonempty. We wish to show that there exists $n \in \mathbb{Z}$ such that

$$U \cap T_1^{-n}V_1 \cap \cdots \cap T_d^{-n}V_d \ne \emptyset.$$

Since $(X, S_1, \dots, S_k)$ is minimal, there exists a finite set $F \subseteq \mathbb{Z}^k$ such that $X = \bigcup_{r \in F} S_rU$. By assumption (ii) and Lemma 3.6, there exist nonempty open sets $\tilde{V}_1, \dots, \tilde{V}_d$ and for all $r \in F$, some $n_r \in \mathbb{Z}$, such that

$$(T_1 \times \cdots \times T_d)^{n_r}(S_r \times \cdots \times S_r(\tilde{V}_1 \times \cdots \times \tilde{V}_d)) \subseteq V_1 \times \cdots \times V_d$$

for all $r \in F$. Since $(T_2T_1^{-1})^n, \dots, (T_dT_1^{-1})^n$ are jointly transitive, we can find $m = m(F) \in \mathbb{Z}$ such that

$$\tilde{V}_1 \cap (T_2T_1^{-1})^{-m}\tilde{V}_2 \cap \cdots \cap (T_dT_1^{-1})^{-m}\tilde{V}_d \ne \emptyset.$$

Take $x \in \tilde{V}_1$ such that $T_i^mT_1^{-m}x \in \tilde{V}_i$ for all $2 \le i \le d$, and write $y = T_1^{-m}x$. Let $r \in F$ be such that $z := S_ry \in U$. Then $T_i^{m+n_r}z = T_i^{m+n_r}S_ry = T_i^{n_r}S_r(T_iT_1^{-1})^mx \in T_i^{n_r}S_r(\tilde{V}_i) \subseteq V_i$ for all $1 \le i \le d$. It follows that $z \in U \cap T_1^{-(m+n_r)}V_1 \cap \cdots \cap T_d^{-(m+n_r)}V_d$. □

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