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On connection-preserving actions of discrete linear groups

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Abstract. We study actions of arithmetic groups on compact manifolds which preserve a connection.

1. *Introduction*

Let G be a connected semisimple Lie group with finite centre such that $\mathbb{R}$-rank $(G') \geq 2$ for every simple factor G' of G. Let $\mathfrak{g}$ be the Lie algebra of G and $d(\mathfrak{g})$ the minimal dimension of a non-trivial (real) representation of $\mathfrak{g}$. Suppose $\Gamma \subset G$ is a lattice subgroup. In [**9**], [**10**], [**11**] we proved the following result.

THEOREM 1.1. *Let M be a compact manifold with* $\dim(M) < d(\mathfrak{g})$. *Then every smooth action of* Γ *on M which preserves a volume density and a connection must also preserve a Riemannian metric.*

Margulis [**3**] has described the structure of the compact Lie groups K admitting a dense image homomorphism $\Gamma \to K$. (See also [**11**].) Thus theorem 1.1 has the following consequence.

COROLLARY 1.2. *Let* Γ *be a lattice in* $SL(n, \mathbb{R})$ $(n \geq 3)$. *Let M be a compact manifold with* $\dim(M) < n$. *Then any action of* Γ *on M preserving a volume density and a connection is finite, i.e. is an action by a finite quotient of* Γ.

Thus the action of $SL(n, \mathbb{Z})$ on $\mathbb{R}^n/\mathbb{Z}^n$ is a volume preserving affine action which is non-finite of minimal dimension. The point of this note is twofold. First, we examine affine actions of lattices in $SL(n, \mathbb{R})$ on manifolds of dimension exactly n. In this direction we have:

THEOREM 1.3. *Let* $\Gamma \subset SL(n, \mathbb{R})$ $(n \geq 3)$ *be a lattice and M a compact Riemannian manifold with* $\dim(M) = n$. *Suppose* Γ *acts on M by volume preserving affine transformations. If the* Γ *action is not finite, then M is flat and* Γ *is commensurable to* (*a conjugate of*) $SL(n, \mathbb{Z})$. *In particular, if* Γ *is cocompact, any volume preserving affine action on a compact Riemannian n-manifold is finite.*

Secondly, we prove a strengthened version of theorem 1.1 for irreducible lattices. Namely, we have:

THEOREM 1.4. *Let* G, Γ *be as in theorem* 1.1, *with* Γ *irreducible* [**12**]. *Let* $D(\mathfrak{g})$ *be the minimal dimension of a faithful* (*real*) *representation of* $\mathfrak{g}$. *Suppose M is a compact*

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manifold with $\dim(M) < D(\mathfrak{g})$. *Then for every smooth action of* Γ *on* M *preserving a volume density and a connection we have either*:

(1) Γ *also preserves a Riemannian metric on* M; *or*

(2) *there is a non-trivial action of* $\tilde{G}$, *the universal covering of* G, *on* M *by volume preserving affine transformations.*

We remark that by [**10**, theorem 5.5], if M admits a non-trivial volume-preserving $\tilde{G}$-action, then $\dim(M) \geq \dim(G')$ for some simple factor G' of $\tilde{G}$. Thus, theorem 1.4 implies the existence of a Γ-invariant Riemannian metric in a suitable dimension range. In particular, it includes theorem 1.1. As an example of where conclusion (2) can arise, we remark that if H is a simple factor of G, and $\Lambda \subset H$ is a cocompact lattice, then there is an H-invariant connection on H/Λ coming from the invariant pseudo-Riemannian metric defined by the Killing form. By projecting Γ into H, we obtain an action of Γ, and one can clearly arrange matters so that the dimension range $\dim(M) < D(\mathfrak{g})$ is satisfied.

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2. *Proof of theorem* 1.3

We shall freely use (with references) the notions and results discussed in [**11**], and [**12**]. For simplicity, we shall assume that the manifold is orientable so that its volume density defines an $SL(n, \mathbb{R})$-structure on M. The more general case of a structure defined by the matrices with $|\det| = 1$ follows in a similar manner, with only technical changes.

Let $P(M)$ be the principal bundle of special frames on M with respect to the volume form. We can measurably trivialize $P(M) \to M$, i.e. choose a measurable isomorphism of bundles $P(M) \cong M \times SL(n, \mathbb{R})$. Under this isomorphism, the natural Γ action on $P(M)$ can be described on $M \times SL(n, \mathbb{R})$ via a measurable cocycle $\alpha: \Gamma \times M \to SL(n, \mathbb{R})$. (See [**10**], [**11**]] for discussion.) Let (E, μ) be an ergodic component of the Γ action on M. Since Γ preserves a volume density on M, we can suppose $E \subset M$ is a Γ-invariant Borel set and that μ is a Γ-invariant probability measure. Let $H \subset SL(n, \mathbb{R})$ be the algebraic hull of $\alpha | \Gamma \times E$ [**12**]. (Thus on E, α is equivalent to a cocycle into H, but not into any proper algebraic subgroup of H.) By passing to an ergodic finite extension $p: E' \to E$, we can assume that the algebraic hull of $\alpha': \Gamma \times E' \to SL(n, \mathbb{R})$, given by $\alpha'(\gamma, y) = \alpha(\gamma, p(y))$, is H^0, the Zariski connected component of H [**12**, prop. 9.2.6]. Let δ be a cocycle equivalent to α' and taking all values in H^0. Write $H^0 = L \ltimes U$ where L is reductive and U is the unipotent radical. Let $\beta: \Gamma \times E' \to L/Z(L)$ be the composition of δ with the projection of H^0 onto $L/Z(L)$, where $Z(L)$ is the centre of L. Let L' be a simple factor of $L/Z(L)$ and β' the projection of β onto L'. Since the algebraic hull of β' is L', the super-rigidity theorem for cocycles [**12**], [**7**], [**8**] implies that either L' is compact or β' is equivalent to a cocycle of the form $(\gamma, m) \mapsto \pi(\gamma)$ where $\pi: SL(n, \mathbb{R}) \to L'$ is an $\mathbb{R}$-rational homomorphsim. In the latter case (since $\dim(L') \leq \dim(SL(n, \mathbb{R}))$

by construction), we must clearly have that H^0 is locally isomorphic to SL $(n, \mathbb{R})$, and that $H^0 \to L'$ is simply a (possibly trivial) covering. In case L' is compact for each simple factor L' of $L/Z(L)$, H^0 will be amenable. However, Γ has Kazhdan's property, and the algebraic hull of α' is H^0. Therefore, H^0 is itself compact [**12**, theorem 9.1.1]. We may therefore deduce that either:

(i) the algebraic hull of α' is compact; or

(ii) there is a cocycle $\theta : \Gamma \times E' \to \mathrm{PSL}(n, \mathbb{R})$ such that $\pi \circ \alpha' \sim \theta$ and such that $\theta(\gamma, y) = \pi(A(\gamma))$ for all $\gamma, y \in \Gamma \times E'$, where $\pi : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$ is the covering, and A is an automorphism of SL $(n, \mathbb{R})$. Let P' be the pullback of $P(M) \to E$ to a bundle over E'. Under condition (i), it is clear that there is a Γ-invariant probability measure on P' (cf. [**10**]), and hence via projection, a Γ-invariant probability measure on $P(M)$. However, since Γ preserves a connection, this implies that Γ acts isometrically with respect to some Riemannian metric on M [**9**], [**10**], [**11**]. If K is the full group of isometries of this metric, then K is a compact Lie group of dimension at most $n(n+1)/2$. However, via results of Margulis [**3**], any compact Lie group K admitting a dense range homomorphism $\Gamma \to K$ where Γ is a lattice in SL $(n, \mathbb{R})$ must satisfy $\dim(K) \geq \dim(\mathrm{SL}(n, \mathbb{R})) = n^2 - 1$. (See [**11**] for a proof.) Thus, since $n \geq 3$, the Γ action will be finite. To prove theorem 1.3, it therefore remains to consider case (ii) above.

Let W be the space of pairs of vectors in $\mathbb{R}^n$, with two pairs (v, w), (v', w') identified if $v' = -v$ and $w' = -w$. Let W_M be the bundle of such pairs with the corresponding identifications in the tangent bundle of M. Thus W_M is a bundle over M with fibre W. Let W'_M be the pullback of $W_M \to E$ to a bundle over E'. We note that the natural SL $(n, \mathbb{R})$ action on W factors to an action of PSL $(n, \mathbb{R})$. Thus, the Γ-action on W'_M over E' can be described via the cocycle θ in (ii) by the action of Γ on $E' \times W$ given by

$$\gamma \cdot (y, w) = (\gamma y, \theta(\gamma, y) w) = (\gamma y, \pi(A(\gamma)) w).$$

The curvature of the Riemannian connection on M is a horizontal sl $(n, \mathbb{R})$-valued 2-form R on $P(M)$. Since the connection is Riemannian, for each $X, Y \in TP(M)_y$, $R(X, Y)$ is in the Ad (SL $(n, \mathbb{R})$)-orbit of a skew symmetric matrix. If $\varphi : \mathrm{sl}(n, \mathbb{R}) \to \mathbb{R}$ is a homogeneous Ad (SL $(n, \mathbb{R})$)-invariant polynomial, then $\varphi \circ R$ is a Γ-invariant $\mathbb{R}$-valued function on $TM \otimes TM$ (vanishing at 0), and in particular defines a Γ-invariant function $f_\varphi : W_M \to \mathbb{R}$. If the Γ-action on W_M over each ergodic component E is still ergodic, then we will clearly have $f_\varphi = 0$ a.e. on each ergodic component. Since $\varphi(R(X, Y)) = \varphi(B)$ for some skew symmetric B in the same Ad (SL $(n, \mathbb{R})$)-orbit as $R(X, Y)$, it follows that $R = 0$ a.e. over each ergodic component, and hence by continuity, this would imply flatness. Thus M would be finitely covered by a flat torus. It then follows readily that Γ is commensurable to (a conjugate of) SL $(n, \mathbb{Z})$. Therefore, to prove the theorem, it suffices to show that Γ acts ergodically on W_M over E. To see this, it clearly suffices to see that Γ acts ergodically on W'_M, i.e. on $E' \times W$. Since SL $(n, \mathbb{R})$ acts transitively on an open conull set in W, and has a non-compact stabilizer on this orbit (recall $n \geq 3$), it suffices to verify the following lemma.

LEMMA 2.1. *Let G be a connected simple Lie group with finite centre, $\Gamma \subset G$ a lattice, $H \subset G$ a non-compact closed subgroup, and X an ergodic Γ-space with finite invariant measure. Then Γ acts ergodically (via the diagonal action) on $X \times G/H$.*

Proof. Let Y be the G-space induced from the Γ-space $(X \times G/H)$, i.e. $Y = (X \times G/H \times G)/\Gamma$. (See [**5**], [**12**] for discussion.) Then as a G-space $Y \cong ((X \times G)/\Gamma) \times G/H$. The Γ-action on $X \times G/H$ is ergodic if and only if the G-action on Y is ergodic, and this in turn is equivalent to ergodicity of the H-action on $(X \times G)/\Gamma$ (cf. [**12**, prop. 2.2.2]). However, the space $(X \times G)/\Gamma$ is an ergodic G-space with finite invariant measure, and since H is non-compact, ergodicity of H on this space follows by Moore's ergodicity theorem [**12**]. This completes the proof.

3. *Proof of theorem* 1.4

We shall need the notion of an amenable action of a locally compact group for which we refer the reader to [**12**] as a general reference. We collect a number of facts we shall need.

LEMMA 3.1. *Suppose a locally compact group Γ acts properly on a locally compact second countable space. Then the action is amenable with respect to any Γ-quasi-invariant measure.*

We recall that an action is proper if for any compact sets A, B, $\{g \in \Gamma \mid gA \cap B \neq \varnothing\}$ is precompact. Any such action clearly has compact (in particular, amenable) stabilizers, and has locally closed orbits. (In the terminology of [**1**], it defines a type I equivalence relation.) The lemma then follows by standard arguments.

The next result about amenability follows easily from the results of [**6**].

LEMMA 3.2. *Suppose Γ, Q are locally compact groups and that $\Gamma \times Q$ acts on a measure space (X, μ), leaving μ quasi-invariant. Suppose further that Q is an amenable group and that the Q-action is smooth in the sense of ergodic theory, i.e. that the quotient space X/Q is countably separated. (See* [**12**, chapter 2].) *Then the action of Γ on X is amenable if and only if the action of Γ on X/Q is amenable.*

Proof. Since Q is amenable, there is a relatively Γ-invariant mean for $X \to X/Q$ in the sense of [**6**]. The lemma then follows from results in [**6**].

We also recall the main result of [**13**].

LEMMA 3.3 [**13**, corollary 1.2]. *Suppose H is a connected semisimple Lie group and that $\Gamma \subset H$ is a countable dense subgroup. Then the action of Γ on H is not amenable.*

The last fact about amenability we require is the following. A proof is straightforward given the techniques of [**12**, § 4.3].

LEMMA 3.4. *Suppose that X, Y are G-spaces (with quasi-invariant measure), and suppose that the measure on X is finite and invariant. Then the action on Y is amenable if and only if the diagonal action on $X \times Y$ is amenable.*

We now proceed to the proof of the theorem. The connection on M defines in a natural way a Riemannian metric on the manifold $P(M)$ [**11**, § 2], where the latter

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is, as in the preceding section, the bundle of special frames on M. Let A be the full isometry group of $P(M)$ with this metric. It is well known that A has the structure of a Lie group acting smoothly and properly on $P(M)$. (See [2].) Every volume preserving affine transformation of M defines an element of A, and in particular we have a natural homomorphism $\Gamma \to A$. Let $\bar{\Gamma}$ be the closure of the image of Γ in A and $B = (\bar{\Gamma})_0$ its connected component. The elements of $\bar{\Gamma}$ can be identified with volume preserving affine transformations of M. The subgroup $\Gamma \cap B \subset \Gamma$ is normal, and hence by a result of Margulis [4] (see also [12]), is either finite or of finite index. If it is of finite index, it is itself an irreducible lattice in G. In this case, since B is a connected Lie group, Margulis superrigidity [3], [12] implies that either:

(1) B contains a subgroup locally isomorphic to a factor of $\tilde{G}$; or

(2) B is amenable.

In case (1) we of course have a non-trivial volume preserving affine action of $\tilde{G}$ on M. In case (2), since $\Gamma \cap B$ has Kazhdan's property and is dense in B, B must be compact [12, cor. 7.1.10.]. Therefore, in this case, we have the existence of an invariant Riemannian metric on M.

It therefore remains to show that the situation in which the normal subgroup $\Gamma \cap B$ is finite cannot actually occur. In this case, Γ will be a discrete subgroup of A. Since A acts properly on $P(M)$, it follows from lemma 3.1 that the Γ action on $P(M)$ is amenable with respect to any Γ-quasi-invariant measure on $P(M)$. Now let $E \subset M$, $\alpha: \Gamma \times E \to \mathrm{SL}(n, \mathbb{R})$, H, H^0, L, U, E', α', δ, β be defined as in the proof of theorem 1.3. We further set α_0 to be a cocycle equivalent to α taking all values in H. The action of Γ on $E \times H$ given by $\gamma \cdot (x, h) = (\gamma x, \alpha_0(\gamma, x)h)$ is (measurably) isomorphic to the action of Γ on an invariant subset of $P(M)$, and hence is amenable. It follows readily that the action of Γ on $E' \times H^0$ given by $\gamma \cdot (y, h) = (\gamma y, \alpha'(\gamma, y)h)$ is also amenable. By lemma 3.2, we deduce that the action of Γ on $E' \times L/Z(L)$ given by $\gamma \cdot (y, a) = (\gamma y, \beta(\gamma, y)a)$ is amenable as well. Let L' be the product of the non-compact simple factors of $L/Z(L)$. By lemma 3.2 again, the corresponding action of Γ on $E' \times L'$ (defined by the cocycle obtained by projecting β onto L'), is amenable. However, the algebraic hull of this cocycle is L' itself. It therefore follows from the super-rigidity theorem for cocycles [12] that this cocycle is equivalent to one of the form $(\gamma, y) \mapsto \pi(\gamma)$ for some $\mathbb{R}$-rational epimorphsim $\pi: G \to L'$. Therefore the action of Γ on $E' \times L'$ given by $\gamma \cdot (y, a) = (\gamma y, \pi(\gamma)a)$ is amenable. By lemma 3.4, the Γ-action on L' defined by the homomorphism π is amenable. However, the Lie algebra of L' admits a faithful representation on $\mathbb{R}^n$. Since $n < D(\mathfrak{g})$, the homomorphism π has a kernel of positive dimension. Since Γ is an irreducible lattice in G, it follows that $\pi(\Gamma)$ is dense in L'. However, by lemma 3.3, this implies that the Γ action on L' is not amenable. This contradiction completes the proof.

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