GENERALIZED LIE ELEMENTS

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Introduction. Let $\lambda(ij)$, $i, j = 1, 2, \ldots, m$, be m^2 elements in a field K of characteristic zero such that $\lambda(ij)\lambda(ji) = 1$ for all i and j, and $x_1, x_2, \ldots, x_m$ non-commutative associative indeterminates over K. Define the elements $[x_{i_1}x_{i_2} \ldots x_{i_n}]$ inductively by $[x_i] = x_i$ and

$$[x_{i_1}x_{i_2} \ldots x_{i_n}] = x_{i_1}[x_{i_2} \ldots x_{i_n}] - \prod_{\nu=2}^{n} \lambda(i_1 i_\nu)[x_{i_2} \ldots x_{i_n}]x_{i_1}.$$

Any linear combination of the elements

$$[x_{i_1}x_{i_2} \ldots x_{i_n}]$$

with coefficients in K will be called a *generalized Lie elememt.* Generalized Lie elements reduce to ordinary Lie elements if $\lambda(ij) = 1$ for all i and j.

The purpose of this paper is to generalize to the generalized Lie elements the following: a theorem of Friedrichs, a theorem of Dynkin-Specht-Wever **(2)**, and the Witt formula on the dimension of the space spanned by homogeneous Lie elements of a fixed degree. The set of all generalized Lie elements will be made into an algebra which generalizes the ordinary free Lie algebra. This algebra turns out to be free in a certain sense. We shall also generalize the algebra associated with shuffles in **(2)**.[1]

1. Generalized Lie algebras. Throughout this paper K will denote a field of characteristic zero. By a *bi-character* in K of an additively written abelian semi-group M we shall mean a map $\chi: M \times M \to K$ satisfying the following:

$$\chi(\rho, \sigma + \tau) = \chi(\rho, \sigma)\chi(\rho, \tau), \chi(\rho + \sigma, \tau) = \chi(\rho, \tau)\chi(\sigma, \tau)$$

for all ρ, σ, τ in M. A bi-character χ will be called *skew-symmetric* if $\chi(\sigma, \tau)$ $\chi(\tau, \sigma) = 1$ for all σ, τ in M. An (associative or non-associative) algebra A over K is said to be *graded* by the semi-group M if A is a direct sum of subspaces A_ρ indexed by $\rho \in M$ such that $f \in A_\rho$ and $g \in A_\sigma$ imply $fg \in A_{\rho+\sigma}$.

Let L be an algebra graded by M, and let χ be a skew-symmetric bi-character of M in K. We shall call L a *generalized Lie algebra of type* χ, or simply a χ-algebra, if $f \in L_\rho$, $g \in L_\sigma$, imply

$$[f, g] + \chi(\rho, \sigma)[g, f] = 0;$$

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[1]The referee remarks that the algebras considered in this paper include, as a special case, the "left Lie algebras" which are used in homological algebra (cf. for example, the exposition by P. Cartier in Séminaire Bourbaki, May, 1955).

$$[f, [g, h]] - \chi(\rho, \sigma)[g, [f, h]] = [[f, g], h],$$

where $[f, g]$ denotes the product in L of f and g. In case χ is trivial, a χ-algebra is clearly an ordinary Lie algebra. Let A be an associative algebra graded by M. Define a new multiplication $[a, b]$ in the vector space A by

$$[a, b] = ab - \chi(\rho, \sigma)ba,$$

where $a \in A_\rho$, $b \in A_\sigma$. Then we obtain a new algebra which we shall denote by $[A]$. It can be seen easily that $[A]$ is a χ-algebra.

Let L and L' be two algebras graded by the same M. A linear map $\phi: L \to L'$ will be said to *respect grade* if $f \in L_\rho$ implies $\phi(f) \in L_\rho'$. Let L be a χ-algebra and A an associated algebra both graded by M. A grade-respecting linear map $\phi: L \to A$ will be called a *linearization* of L in A if ϕ is a homomorphism of L into $[A]$, that is, if

$$\phi([f, g]) = \phi(f)\phi(g) - \chi(\rho, \sigma)\phi(g)\phi(f)$$

for all $f \in L_\rho$, $g \in L_\sigma$. The tensor algebra T over the vector space L is graded by M if T_ρ is defined to be the subspace spanned by elements of the form $f_1 \otimes f_2 \otimes \ldots \otimes f_n$, where $f_i \in L_{\rho_i}$ and $\rho_1 + \rho_2 + \ldots + \rho_n = \rho$. Let J be the two-sided ideal of T generated by homogeneous elements of the form $f \otimes g - \chi(\rho, \sigma)g \otimes f - [f, g]$, where $f \in L_\rho. g \in L_\sigma$. Then the algebra $U = T/J$ is also graded by M, and the inclusion map $L \to T$ induces a linearization η of L in U. The algebra U will be called the *universal enveloping algebra* of L; it can be characterized by the property: for any linearization $\phi: L \to A$ of L into an associative algebra A, there exists a grade-respecting homomorphism $\xi: U \to A$ such that $\phi = \xi \circ \eta$.

2. Finitely generated free χ-algebras. From now on we shall consider χ-algebras L satisfying the following conditions (2.1) – (2.4):

(2.1) M is a free abelian group of rank m, with basis elements $\rho_1, \rho_2, \ldots, \rho_m$;

(2.2) $L_\rho = 0$ unless ρ is of the form $\rho = t_1\rho_1 + t_2\rho_2 + \ldots + t_m\rho_m$, where $t_1, t_2, \ldots, t_m$ are non-negative integers not all of which are zero;

(2.3) each L_{ρ_i} $(i = 1, 2, \ldots, m)$ is of dimension 1;

(2.4) L is generated by $L_{\rho_1}, L_{\rho_2}, \ldots, L_{\rho_m}$.

A χ-algebra L satisfying (2.1) – (2.4) above, will be called a *free χ-algebra of rank m* if any χ-algebra satisfying (2.1) – (2.4) is a (grade-respecting) homomorphic image of L. The existence of a free χ-algebra can be seen as follows: let F be the free (non-associative) algebra generated by an m-dimensional vector space E over the field K. If we choose a basis of E over K, then F can be graded in an obvious way by the free abelian group M of rank m. Let J be the two-sided ideal of F generated by homogeneous elements of the forms $fg + \chi(\rho, \sigma)gf$ and $f(gh) - \chi(\rho, \sigma)g(fh) - (fg)h$, where $f \in F_\rho$, $g \in F_\sigma$. Then $L = F/J$ is easily seen to be a free χ-algebra of rank m.

Let U be the universal enveloping algebra of the free χ-algebra L of rank m with the linearization map $\eta: L \to U$, and let A be the free associative algebra over K generated by m free generators $x_1, x_2, \ldots, x_m$. Since L is free, there exists a homomorphism $\phi: L \to [A]$ such that $\phi(f_i) = x_i$, $i = 1, 2, \ldots, m$, that is, ϕ is a linearization of L in A. Then by the definition of U, there exists a grade-respecting homomorphism $\xi: U \to A$ such that $\phi = \xi \circ \eta$. Then ξ must be an isomorphism, since A is free-associative. Thus we may regard U as a free associative algebra with free generators $x_1 = \eta(f_1), \ldots, x_m = \eta(f_m)$. The fact that $\eta(f) = 0$ implies $F = 0$ can also be proved in exactly the same way as in the case of free Lie algebras **(3**, 1–9**)**. Hence we may identify L as the subalgebra of $[U]$ generated by $x_1, \ldots, x_m$. It can be seen easily that L is spanned by the elements

$$[x_{i_1} x_{i_2} \ldots x_{i_n}] = [x_{i_1}[\ldots[x_{i_{n-1}} x_{i_n}] \ldots]]$$

defined in the Introduction by using $\lambda(ij) = \chi(\rho_i, \rho_j)$. Thus we may state

THEOREM 2.5. *Let K be a field of characteristic zero, $x_1, x_2, \ldots, x_m$ non-commutative associative indeterminates over K, and $\lambda(ij)$, $i, j = 1, 2, \ldots, m$, be m^2 elements in K such that $\lambda(ij)\lambda(ji) = 1$ for all i and j. Then the vector space over K spanned by the elements*

$$[x_{i_1} x_{i_2} \ldots x_{i_n}]$$

defined above forms a free χ-algebra L with respect to the multiplication

$$[[x_{i_1} \ldots x_{i_p}], [x_{j_1} \ldots x_{j_q}]] = [x_{i_1} \ldots x_{i_p}][x_{j_1} \ldots x_{j_q}] - \prod_{\mu=1}^{p} \prod_{\nu=1}^{q} \lambda(i_\mu j_\nu)[x_{j_1} \ldots x_{j_q}][x_{i_1} \ldots x_{i_p}].$$

The universal enveloping algebra of L is isomorphic to the free associative algebra with m free generators.

It should be understood in the above theorem that L is graded by M as follows: for $\rho = t_1\rho_1 + t_2\rho_2 + \ldots + t_m\rho_m$, L_ρ consists of linear combinations of elements of the form

$$[x_{i_1} x_{i_2} \ldots x_{i_n}]$$

in which, for each i, x_i appears t_i times. Also, χ is defined by $\chi(\rho_i, \rho_j) = \lambda(ij)$.

3. A generalization of a Witt formula. Let L be as in Theorem 2.5. An element in L will be called a *homogeneous element of degree n* if it is a linear combination of elements of the form

$$[x_{i_1} x_{i_2} \ldots x_{i_n}].$$

In this section we shall compute the dimension of the space spanned by all homogeneous elements of degree n, following a method given by Witt **(4)**. By the same method one may be able to compute the dimension of each L_ρ.

Let A and B be two associative algebras both graded by M, and $A \otimes B$ the tensor product of A and B regarded as vector spaces over K. Using a bi-character χ of M, define a multiplication in the vector space $A \otimes B$ by

$$(a \otimes b)(a' \otimes b') = \chi(\sigma, \rho')(aa' \otimes bb')$$

where $b \in B_\sigma$, $a' \in A_\rho'$. The algebra obtained in this way is easily seen to be associative, and will be denoted simply by $A \otimes B$. It will be used in the proof of (3.1), below, as well as in the formulation of a generalization of a theorem of Friedrichs.

Now, for the skew-symmetric bi-character χ of M, we have $\chi(\rho, \rho) = \pm 1$ for any $\rho \in M$. The subspace L_ρ of the free χ-algebra L will be called *positive* or *negative* according as $\chi(\rho, \rho) = 1$ or $\chi(\rho, \rho) = -1$. Choose a basis for each positive L_ρ and let the union of these basis elements be $P_1, P_2, P_3 \ldots$. Also, choose a basis for each negative L_ρ and let the union of these basis elements be $Q_1, Q_2, Q_3 \ldots$. Let η: $L \to U$ be the linearization of L into its universal enveloping algebra U. Then we have

THEOREM 3.1. *The elements*

$$\eta(P_1)^{s1}\, \eta(P_2)^{s2} \ldots \eta(P_k)^{sk}\, \eta(Q_1)^{t_1}\, \eta(Q_2)^{t_2} \ldots \eta(Q_n)^{t_n}$$

form a basis of the universal enveloping algebra U of the free χ-algebra L. Here the indices run as follows: $s_1, s_2 \ldots$ are non-negative integers; each of t_i is either 0 *or* 1; $k, n = 0, 1, 2 \ldots$.

Proof. Since, for each i,

$$\eta([Q_i, Q_i]) = \eta(Q_i)^2 - \chi(\rho, \rho)\eta(Q_i)^2 = 2\eta(Q_i)^2,$$

it follows that $\eta(Q_i)^2$ is a linear combination of some $\eta(P_j)$'s and some $\eta(Q_k)$'s. Then by the definition of the linearization, it is clear that U is spanned by the given elements. Thus it remains to show that the given elements are linearly independent. For this purpose, let U' be a replica of U with grade-respecting isomorphism ι: $U \to U'$, and let $\eta' = \iota \circ \eta$. Let $U \otimes U'$ be the tensor product of U and U' with respect to χ. Then $U \otimes U'$ is also graded by M in an obvious way, and the map $\bar{\eta}$: $L \to U \otimes U'$ defined by

$$\bar{\eta}(f) = \eta(f) \otimes 1 + 1 \otimes \eta'(f)$$

is easily seen to be a linearization of L into $U \otimes U'$. Therefore there exists a homomorphism ξ: $U \to U \otimes U'$ such that $\xi \circ \eta = \bar{\eta}$. Using ξ, one may now prove the linear independence of the given elements in exactly the same way as in the case of ordinary Lie algebras (**3**, pp. 1–8). We omit the details.

Now, let the free χ-algebra L given in (2.5) be graded by M as in the remark following (2.5). Let the basis elements $\rho_1, \rho_2, \ldots, \rho_m$ be such that

$$L_{\rho 1}, \ldots, L_{\rho_p}$$

are positive while

$$L_{\rho_{p+1}}, \ldots, L_{\rho_{p+q}}.$$

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($p + q = m$) are negative. Since, for $\rho = t_1\rho_1 + \ldots + t_m\rho_m$,

$$\chi(\rho, \rho) = \prod_{i,j} \chi(\rho_i, \rho_j)^{t_i t_j} = \prod_i \chi(\rho_i, \rho_i)^{t_i^2} = (-1)^t,$$

where $t = t_{p+1} + \ldots + t_{p+q}$, it follows that

$$[x_{i_1}x_{i_2} \ldots x_{i_n}]$$

belongs to a positive L_ρ if and only if its degree with respect to $x_{p+1}, \ldots, x_{p+q}$ is even. Denote by p_n and q_n, respectively, the numbers of P_i's of degree n and the numbers of Q_i's of degree n, and consider the formal power series

$$F(x, \lambda) = \prod_{d=1}^{\infty} (1 + x^d + x^{2d} + \ldots)^{p_d}(1 + \lambda x^d)^{q_d}$$

with a parameter λ. The coefficient $c_n(\lambda)$ of x^n in $F(x)$ is a polynomial in λ with integral coefficients. By (3.1), $c_n(1)$ is equal to the dimension of the subspace of U spanned by all homogeneous elements of degree n; $c_n(1) = (p + q)^n$. On the other hand, also by (3.1), $c_n(-1) = a_n - b_n$, where a_n denotes the dimension of the subspace A_n of U spanned by all homogeneous elements which are of even degrees with respect to $x_{p+1}, \ldots, x_{p+q}$, and where b_n denotes the dimension of the subspace B_n of U spanned by all homogeneous elements which are of odd degrees with respect to $x_{p+1}, \ldots, x_{p+q}$. Since U is free associative, A_n (resp. B_n) is spanned by elements

$$x_{i_1}x_{i_2} \ldots x_{i_n}$$

of even (resp. odd) degree with respect to $x_{p+1}, \ldots, x_{p+q}$. Thus

$$a_n = C_{n,0}p^n + C_{n,2}p^{n-2}q^2 + \ldots,$$

$$b_n = C_{n,1}p^{n-1}q + C_{n,3}p^{n-3}q^3 + \ldots,$$

where $C_{n,r}$ are binomial coefficients. Hence $a_n - b_n = (p - q)^n$, and we have

$$F(x, 1) = 1 + (p + q)x + (p + q)^2x^2 + \ldots,$$

$$F(x, 1) = -1 + (p - q)x + (p - q)^2x^2 + \ldots.$$

Taking logarithms of both sides, and comparing the coefficients of x^n/n, we have, for $n = 1, 2, \ldots$,

$$\sum_{d|n} dp_d - \sum_{d|n} (-1)^{n/d}dq_d = (p + q)^n,$$

$$\sum_{d|n} dp_d - \sum_{d|n} dq_d = (p - q)^n.$$

Let $k > 0$ be an odd integer. Then, since

$$\sum_{d|2^\alpha k} (-1)^{2^\alpha k/d}dq_d = \sum_{d|2^{\alpha-1}k} dq_d - \sum_{d|k} 2^\alpha dq_{2^\alpha d},$$

$$\sum_{d|2^\alpha k} dp_d = \sum_{d|2^{\alpha-1}k} dp_d + \sum_{d|k} 2^\alpha dp_{2^\alpha d}$$

we obtain, from the above,

$$\sum_{d|k} 2^\alpha d(p_{2^\alpha d} + q_{2^\alpha d}) = (p+q)^{2^\alpha k} - (p-q)^{2^{\alpha-1}k}.$$

Then by the Möbius inversion formula, we have

$$p_{2^\alpha k} + q_{2^\alpha k} = \frac{1}{2^\alpha k} \sum_{d|k} \mu(d)((p+q)^{2^\alpha k/d} - (p-q)^{2^{\alpha-1}k/d}).$$

In case $\alpha = 0$, the above reduces (for odd k) to

$$p_k + q_k = \frac{1}{k} \sum_{d|k} \mu(d)(p+q)^{k/d}$$

Following Witt, we shall use the notations:

$$\psi(n) = \frac{1}{n} \sum_{d|n} \mu(d)(p+q)^{n/d};$$

$$\psi^*(n) = p_n + q_n.$$

Then the above can be summarized as

THEOREM 3.2. *The dimension $\psi^*(n)$ of the vector space spanned by all elements of the form*

$$[x_{i_1} x_{i_2} \ldots x_{i_n}]$$

is given, for odd k, by

$$\psi^*(k) = \psi(k);$$

$$\psi^*(2^\alpha k) = \psi(2^\alpha k) + \frac{1}{2^\alpha k} \sum_{d|k} \mu(d)((p+q)^{2^{\alpha-1}k/d} - (p-q)^{2^{\alpha-1}k/d}),$$

where p denotes the number of indices i such that $\lambda(ii) = \chi(\rho_i, \rho_i) = 1$ while q denotes the number of indices j such that $\lambda(jj) = -1$.

It should be remarked that the function $\psi^*(n)$ is completely determined by the values of $\lambda(ii)$, and independent of other values of $\lambda(ij)$. The Witt formula is obtained as the case $q = 0$. In case all $\lambda(ii) = -1$, we have $p = 0$, and we may deduce from the above that

$$\psi^*(n) = \begin{cases} \psi(n) & \text{for } n \equiv 0, 1, 3 \pmod 4, \\ \psi(n) + \psi(\tfrac{1}{2}n) & \text{for } n \equiv 2 \pmod 4. \end{cases}$$

4. An algebra associated with shuffles. We shall generalize the algebra defined in **(2)** to apply to generalized Lie elements. If r and s are positive integers, define a *shuffle of type* (r, s) to be a permutation σ of the numbers $1, 2, \ldots, r+s$ such that $1 \leqslant \sigma(\mu) < \sigma(\nu) \leqslant r$ or $r < \sigma(\mu) < \sigma(\nu) \leqslant r+s$ implies $\mu < \nu$. Take m^2 elements $\lambda(ij)$ in K arbitrarily, and define an algebra A over K as follows. A has the basis

$$\{1\} \cup \{a(i_1 \ldots i_n) | i_1, \ldots, i_n = 1, 2, \ldots, m;\ n = 1, 2, \ldots\}$$

with the multiplication table: 1 is a unity element;

$$a(i_1 \ldots i_r) a(i_{r+1} \ldots i_{r+s}) = \sum_\sigma \lambda(\sigma) a(i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(r+s)}),$$

where the sum ranges over all shuffles σ of type (r, s) while $\lambda(\sigma)$ denotes the product of all $\lambda(i_{\sigma(\mu)}, i_{\sigma(\nu)})$ such that $\mu < \nu$ and $\sigma(\mu) > \sigma(\nu)$. (We set $\lambda(\sigma) = 1$ if σ is the identity permutation.)
Thus, for example,

$$a(i)a(j) = a(ij) + \lambda(ji)a(ji);$$
$$a(i)a(jk) = a(ijk) + \lambda(ji)a(jik) + \lambda(ji)\lambda(ki)a(jki).$$

THEOREM 4.1. *The algebra A is associative, and if $\lambda(ij)\lambda(ji) = 1$ for all i and j, then it satisfies the generalized commutativity:*

$$a(j_1 \ldots j_s)a(i_1 \ldots i_r) = \prod_{\mu=1}^{r} \prod_{\nu=1}^{s} \lambda(i_\mu j_\nu)a(i_1 \ldots i_r)a(j_1 \ldots j_s).$$

Proof. If

$$f = a(i_1 \ldots i_r), \; g = a(i_{r+1} \ldots i_{r+s}), \; h = a(i_{r+s+1} \ldots i_{r+s+t}).$$

then it is readily seen that both $(fg)h$ and $f(gh)$ are of the form

$$\sum \lambda(\sigma)\, a(i_{\sigma(1)}i_{\sigma(2)} \ldots i_{\sigma(r+s+t)}),$$

where σ runs over all permutations of $1, 2, \ldots, r + s + t$ such that any one of the three conditions

$$1 \leqslant \sigma(\mu) < \sigma(\nu) < r,$$
$$r < \sigma(\mu) < \sigma(\nu) \leqslant r + s,$$
$$r + s < \sigma(\mu) < \sigma(\nu) < r + s + t$$

implies $\mu < \nu$, and where $\lambda(\sigma)$ denotes the product of all $\lambda(i_{\sigma(\mu)}i_{\sigma(\nu)})$ such that $\mu < \nu$ and $\sigma(\mu) > \sigma(\nu)$. Hence $(fg)h = f(gh)$. The second half of the theorem may be verified easily.

In the rest of this section, we shall assume that $\lambda(ij)\lambda(ji) = 1$ for all i and j. Making the convention that $a(i_1 \ldots i_r)$ stands for 1 whenever the set $\{i_1, \ldots, i_r\}$ of indices is empty, we define the bilinear operation $\vee$ in A by

$$a(i_1 \ldots i_r) \vee a(j_1 \ldots j_s) = a(i_1 \ldots i_r\, j_1 \ldots j_s).$$

We also make the convention that the multiplication in A has priority over the operation $\vee$.

Define the elements $a[i_1i_2 \ldots i_n]$ in A inductively by $a[i] = a(i)$ and

$$a[i_1i_2 \ldots i_n] = a(i_1) \vee a[i_2 \ldots i_n] - \prod_{\nu=1}^{n-1} \lambda(i_n i_\nu)a(i_n) \vee a[i_1 \ldots i_{n-1}].$$

For the generalizations in the next section of some theorems on Lie elements, we need the following

THEOREM 4.2. *For $n > 0$, we have*

$$\sum_{s=1}^{n} a[i_1 \ldots i_s]a(i_{s+1} \ldots i_n) = na(i_1 \ldots i_n).$$

The above theorem may be proved in exactly the same way as in the case where all $\lambda(ij) = 1$ **(2)**, if we use the linear map $D: A \to A$ defined by $D(1) = 0$ and

$$Da(i_1 i_2 \ldots i_n) = \gamma(i_1)a(i_2 \ldots i_n),$$

where $\gamma(1), \ldots, \gamma(m)$ are m arbitrary elements in K. We omit the proof of (4.2). Incidentally, the map D becomes an anti-derivation of A if all $\lambda(ij) = -1$.

THEOREM 4.3. *If the linear map* $\phi: A \to A$ *is defined by* $\phi(1) = 0$ *and* $\phi(a(i_1 i_2 \ldots i_n)) = a[i_1 i_2 \ldots i_n]$, *then* $\phi(a(i_1 \ldots i_r)a(i_{r+1} \ldots i_{r+s})) = 0$ *for all* $i_1, i_2, \ldots, i_{r+s} = 1, 2, \ldots, m$; $r > 0$, $s > 0$.

Proof. We shall proceed by induction on $n = r + s$. If $n = 2$, then the theorem can be verified easily. Assume $n > 2$ and that the theorem is proved for smaller values of n. By the definition of the multiplication in A, we have

$$\begin{aligned} \phi(a(i_1 \ldots i_r)a(i_{r+1} \ldots i_n)) \\ &= \sum \lambda(\sigma)a(i_{\sigma(1)}) \vee a[i_{\sigma(2)} \ldots i_{\sigma(n)}] \\ &\quad - \sum \lambda(\sigma) \prod_{\nu=1}^{n-1} \lambda(i_{\sigma(n)} i_{\sigma(\nu)})a(i_{\sigma(n)}) \vee a[i_{\sigma(1)} \ldots i_{\sigma(n-1)}], \end{aligned}$$

where the sums run over all shuffles of type (r, s), $r + s = n$. Since $\sigma(1) = 1$ or $r + 1$, and $\sigma(n) = r$ or n, the right-hand side of the above equation can be written

$$\begin{aligned} &\sum_{\sigma(1)=1} \lambda(\sigma)a(i_1) \vee a[i_{\sigma(2)} \ldots i_{\sigma(n)}] \\ &\qquad + \sum_{\sigma(1)=r+1} \lambda(\sigma)a(i_{r+1}) \vee a[i_{\sigma(2)} \ldots i_{\sigma(n)}] \\ &\qquad - \sum_{\sigma(n)=r} \lambda(\sigma) \prod_{\nu=1}^{n-1} \lambda(i_r i_{\sigma(\nu)})a(i_r) \vee a[i_{\sigma(1)} \ldots i_{\sigma(n-1)}] \\ &\qquad - \sum_{\sigma(n)=n} \lambda(\sigma) \prod_{\nu=1}^{n-1} \lambda(i_n i_{\sigma(\nu)})a(i_n) \vee a[i_{\sigma(1)} \ldots i_{\sigma(n-1)}] \\ &= a(i_1) \vee \phi(a(i_2 \ldots i_r)a(i_{r+1} \ldots i_n)) \\ &\qquad + \prod_{\nu=1}^{r} \lambda(i_{r+1} i_\nu)a(i_{r+1}) \vee \phi(a(i_1 \ldots i_r)a(i_{r+2} \ldots i_n)) \\ &\qquad - \prod_{\mu=r+1}^{n} \lambda(i_\mu i_r) \prod_{\nu=1, \neq r}^{n} \lambda(i_r i_\nu)a(i_r) \vee \phi(a(i_1 \ldots i_{r-1})a(i_{r+1} \ldots i_n)) \\ &\qquad - \prod_{\nu=1}^{r-1} \lambda(i_n i_\nu)a(i_n) \vee \phi(a(i_1 \ldots i_r)a(i_{r+1} \ldots i_{n-1})) \\ &= 0 \end{aligned}$$

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because of the induction assumption and the fact that, for $r = 1$,

$$\prod_{\mu=r+1}^{n} \lambda(i_\mu i_r) \prod_{\nu=1,\neq r}^{n} \lambda(i_r i_\nu) = 1.$$

COROLLARY 4.3. *If* $0 < r < n$, *then*

$$na(i_1, \ldots, i_r)a(i_{r+1} \ldots i_n) = \sum \lambda(\sigma)(na(i_{\sigma(1)} \ldots i_{\sigma(n)}) - a[i_{\sigma(1)} \ldots i_{\sigma(n)}]),$$

where the sum ranges over all shuffles of type $(r, n - r)$.

The above corollary, together with (4.2), shows that the $(n - 1)m^n$ elements $a(i_1 \ldots i_r)a(i_{r+1} \ldots i_n)$, $i_1, \ldots, i_n = 1, 2, \ldots, m$; $0 < r < n$, and the m^n elements $na(i_1 \ldots i_n) - a[i_1 \ldots i_n]$ span the same vector space over K. Also from (4.2) we obtain

COROLLARY 4.4. *The linear map* $\phi_0: A \to A$ *defined by* $\phi_0(1) = 0$ *and*

$$\phi_0(a(i_1 i_2 \ldots i_n)) = n^{-1}a[i_1 i_2 \ldots i_n],$$

for $n > 0$, *is a projection, that is,* $\phi_0^2 = \phi_0$.

The following theorem is essentially a generalization of Theorem 2.6 of **(2)**, and may be proved by using the map D introduced in the above.

THEOREM 4.5. *For* $n > 0$, *we have*

$$\sum_{s=0}^{n} (-1)^s \prod_{s<\mu<\nu<n} \lambda(i_\nu i_\mu)a(i_1 \ldots i_s)a(i_n i_{n-1} \ldots i_{s+1}) = 0.$$

5. Generalization of a theorem of Friedrichs. Let L be a free χ-algebra of rank m, and $\eta: L \to U$ the linearization of L into its universal enveloping algebra. Let U' be a replica of U with the grade-respecting isomorphism $\iota: U \to U'$ and $\eta' = \iota \circ \eta$. Let $U \otimes U'$ be the tensor product of U and U' with respect to χ. In the course of the proof of (3.1) we have seen that the map $\bar{\eta}: L \to U \otimes U'$ defined by

$$\bar{\eta}(f) = \eta(f) \otimes 1 + 1 \otimes \eta'(f)$$

is a linearization and that there exists a homomorphism $\xi: U \to U \otimes U'$ such that $\xi \circ \eta = \bar{\eta}$. Now the following theorem generalizes a theorem of Friedrichs **(2)**.

THEOREM 5.1. *Let* η, ι, *and* ξ *be as above. Then an element* u *in* U *belongs to the image* $\eta(L)$ *of* L *under* η *if and only if*

$$\xi(u) = u \otimes 1 + 1 \otimes \iota(u).$$

Proof. The "only if" part follows from the fact that $\bar{\eta} = \xi \circ \eta$. In order to prove the "if" part, let $x_1, x_2, \ldots, x_m$ be free generators of U and write, for simplicity, x_i and x_i' for $x_i \otimes 1$ and $1 \otimes \iota(x_i)$, respectively. If

$$u = \sum \alpha_{i_1 \ldots i_n} x_{i_1} \ldots x_{i_n}$$

with coefficients in K, then

$$\xi(u) = \sum \alpha_{i_1\dots i_n}(x_{i_1} + x'_{i_1}) \dots (x_{i_n} + x'_{i_n})$$

$$= \sum \sum_{s=0}^{n} \phi(a(i_1 \dots i_s)a(i_{s+1} \dots i_n))x_{i_1} \dots x_{i_s}x'_{i_{s+1}} \dots x'_{i_n},$$

where ϕ is a linear map: $A_n \to K$ defined by

$$\phi(a(i_1 \dots i_n)) = \alpha_{i_1 \dots i_n}.$$

Hence the condition given in (5.1) is equivalent to

$$\phi(a(i_1 \dots i_s)a(i_{s+1} \dots i_n)) = 0 \qquad (0 < s < n).$$

The rest of the proof is exactly the same as in the case $\lambda(ij) = 1$ (**2**, p. 214), and may be omitted. Here we have to use

$$\sum a[i_1 \dots i_n]x_{i_1} \dots x_{i_n} = \sum a(i_1 \dots i_n)[x_{i_1} \dots x_{i_n}],$$

but this, too, can be proved as in (**2**, p. 213).

Similarly we may prove the following

THEOREM 5.2. *A homogeneous element*

$$f = \sum \alpha_{i_1\dots i_n}x_{i_1} \dots x_{i_n}$$

in U of degree $n > 0$ is a generalized Lie element if and only if

$$nf = \sum \alpha_{i_1\dots i_n}[x_{i_1} \dots x_{i_n}].$$

This generalizes a theorem of Dynkin-Specht-Wever (**2**, p. 214).

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