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TRANSITIVE SIMPLE SUBGROUPS OF WREATH PRODUCTS IN PRODUCT ACTION

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Abstract

A transitive simple subgroup of a finite symmetric group is very rarely contained in a full wreath product in product action. All such simple permutation groups are determined in this paper. This remarkable conclusion is reached after a definition and detailed examination of 'Cartesian decompositions' of the permuted set, relating them to certain 'Cartesian systems of subgroups'. These concepts, and the bijective connections between them, are explored in greater generality, with specific future applications in mind.

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1. Introduction

The main result of this paper is that a transitive simple subgroup of a finite symmetric group is very rarely contained in a full wreath product in product action, so rarely that all such cases can be explicitly tabulated here. In other words, apart from a short list of exceptions, a simple subgroup of a finite wreath product in product action can never be transitive. A brief summary of the product action of wreath products is provided at the beginning of Section 2.

THEOREM 1.1. *Let Ω be a finite set, let $T < W < \operatorname{Sym}\Omega$ such that T is a finite simple group and W is permutationally isomorphic to a wreath product* $\operatorname{Sym}\Gamma \operatorname{wr} S_l$ *in product action. Then either T is intransitive or T, W, and $|\Omega|$ are as in Table* 1. *Moreover, if T is transitive, then $N_{\operatorname{Sym}\Omega}(T)$ is an almost simple group.*

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TABLE 1. Transitive simple subgroups of wreath products

	T	W	$\|\Omega\|$
1	A_6	$S_6 \operatorname{wr} S_2$	36
2	M_{12}	$S_{12} \operatorname{wr} S_2$	144
3	$P\Omega_8^+(q)$	$S_{(d/2)q^3(q^4-1)} \operatorname{wr} S_2$	$(d^2/4)q^6(q^4-1)^2$ $d=(4, q^4-1)$
4	$Sp_4(q),\ q \geqslant 4, q$ even	$S_{q^2(q^2-1)} \operatorname{wr} S_2$	$q^4(q^2-1)^2$

This classification is reached after observing that, in Theorem 1.1, the set Ω can be identified with the Cartesian product Γ^l such that the action of W is compatible with this identification. In order to make this idea precise, we introduce the concept of a 'Cartesian decomposition' of a set, and we also notice that W can be viewed as the full stabiliser in $\operatorname{Sym}\Omega$ of a Cartesian decomposition of Ω. Hence we reduce the problem of classifying the pairs T, W in Theorem 1.1 to the problem of classifying all Cartesian decompositions of finite sets that are invariant under the action of a transitive, simple group of permutations.

Let T be a finite simple group acting on a set Ω. In the classification of T-invariant Cartesian decompositions of Ω we use ideas that are familiar from the elementary theory of permutation groups. Namely, we investigate how the subgroup lattice of T might reflect the existence of a T-invariant Cartesian decomposition of Ω. In Definition 1.3 we define the concept of a 'Cartesian system of subgroups', and in Theorem 1.4 we establish a one-to-one correspondence between the set of T-invariant Cartesian decompositions of Ω and the set of Cartesian systems with respect to a fixed element of Ω.

The concepts of Cartesian decompositions and Cartesian systems, and the bijective connections between them, are explored in greater generality in Sections 2–4. Our motivation in doing so is to provide with a theoretical background for a future investigation of Cartesian decompositions that are invariant under a transitive permutation group.

Some of the concepts we use may be new to most of our readers. We define a permutation group to be *innately transitive* if it has a transitive minimal normal subgroup, and a transitive minimal normal subgroup of an innately transitive group is referred to as a *plinth*. Most of the results of this paper are expressed in the context of innately transitive groups. The structure of innately transitive groups is investigated in [4]. The problem of finding innately transitive subgroups of wreath products in product action is studied more extensively in [3]. Theorem 1.1 is equivalent to the following result, which is formulated in terms of innately transitive groups. Here, a permutation group is *quasiprimitive* if all of its minimal normal subgroups are transitive.

THEOREM 1.2. *Let Ω be a finite set, let $G < W < \mathrm{Sym}\,\Omega$ such that G is an innately transitive group with a simple plinth T, and W is permutationally isomorphic to a wreath product* $\mathrm{Sym}\,\Gamma \,\mathrm{wr}\, \mathrm{S}_l$ *in product action. Then T and W are as in Table* 1, *and G is an almost simple quasiprimitive group.*

Theorems 1.1 and 1.2 are easy consequences of Theorem 6.1 as explained at the end of Section 6.

A *Cartesian decomposition* of a finite set Ω is a collection $\mathscr{E}$ of partitions $\Gamma_1, \ldots, \Gamma_l$ of Ω such that $|\gamma_1 \cap \cdots \cap \gamma_l| = 1$ for all $\gamma_1 \in \Gamma_1, \ldots, \gamma_l \in \Gamma_l$. A Cartesian decomposition is said to be *homogeneous* if its elements have the same size and this common size is at least 2. The number of partitions in a Cartesian decomposition is called the *index*. A Cartesian decomposition is said to be *non-trivial* if it has index at least 2. In this paper, Cartesian decompositions are assumed to be non-trivial, unless it is explicitly stated otherwise. If $\mathscr{E}$ is a Cartesian decomposition of Ω, then Ω can be identified with the Cartesian product $\prod_{\Gamma\in\mathscr{E}} \Gamma$. More information on Cartesian decompositions is provided in [10], where a Cartesian decomposition $\mathscr{E}$ stabilised by a permutation group G such that the elements of $\mathscr{E}$ form a single G-orbit is said to be a system of product imprimitivity for G. A maximal subgroup W of $\mathrm{Sym}\,\Omega$ or $\mathrm{Alt}\,\Omega$ is said to be *of product action type*, or simply *PA type*, if W is the full stabiliser of a non-trivial, homogeneous Cartesian decomposition of Ω. If a permutation group G is contained in such a W, then we also say that W is a *maximal overgroup of G with product action type*, or simply *PA type*.

It is, in general, a difficult problem to describe maximal overgroups with PA type of a transitive permutation group. In the case where G itself is primitive, this question is answered by [13], but [2] leaves this problem open for a quasiprimitive G. Clearly our Theorem 1.2 gives a full classification of the maximal overgroups of product action type for an innately transitive permutation group G with a simple plinth. This is achieved by listing all non-trivial, homogeneous Cartesian decompositions stabilised by G. We found that such decompositions can be identified by information about the subgroups of the plinth. This motivates the following definition.

DEFINITION 1.3. Let M be a transitive permutation group on a set Ω and $\omega \in \Omega$. We say that a set $\{K_1, \ldots, K_l\}$ of subgroups of M is a *Cartesian system of subgroups* of M with respect to ω if

$$(1) \qquad \bigcap_{i=1}^{l} K_i = M_\omega \quad \text{and}$$

$$(2) \qquad K_i\left(\bigcap_{j\neq i} K_j\right) = M \quad \text{for all} \quad i \in \{1, \ldots, l\}.$$

A Cartesian system is said to be *homogeneous* if its elements are proper subgroups

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and they have the same size. A Cartesian system is *non-trivial* if it has at least two subgroups. If M is an abstract group and $\mathcal{K} = \{K_1, \ldots, K_l\}$ is a set of subgroups satisfying (2), then $\mathcal{K}$ is called a Cartesian system of M.

In this paper Cartesian systems are assumed to be non-trivial unless explicitly stated otherwise.

For a permutation group $G \leqslant \operatorname{Sym}\Omega$, let $\operatorname{CD}(G)$ denote the set of G-invariant Cartesian decompositions of Ω. Cartesian systems provide a way of identifying the set $\operatorname{CD}(G)$ from information internal to G.

THEOREM 1.4. *Let G be an innately transitive permutation group on Ω with plinth M. Then, for a fixed $\omega \in \Omega$, there is a one-to-one correspondence between the set* $\operatorname{CD}(G)$ *and the set of G_ω-invariant Cartesian systems of M with respect to ω.*

Theorem 1.4 is an immediate consequence of Theorem 4.2 where an explicit one-to-one correspondence is constructed.

The major results of this paper are presented in Section 6. There we study innately transitive permutation groups with a non-abelian, simple plinth that preserve a Cartesian decomposition of the underlying set. The main result of Section 6 gives rise to a complete description of maximal overgroups with product action type for such an innately transitive group. Theorems 1.1–1.2 follow immediately from Theorem 6.1 (i), where we give a detailed description of G-invariant homogeneous Cartesian decompositions of Ω for innately transitive groups G with a simple plinth T. In particular, Table 3 contains the possibilities for G, T, W, $|\Omega|$, and the isomorphism types of the subgroups in the associated Cartesian system, as given by Theorem 1.4. Part (ii) of Theorem 6.1 gives a detailed description of Cartesian decompositions $\mathcal{E}$ of Ω with index at least 3 that are invariant under the action of an innately transitive group with a non-abelian, simple plinth. In Table 4, we list the possibilities for the plinth, $|\Omega|$, the full stabiliser of $\mathcal{E}$ in $\operatorname{Sym}\Omega$, and the isomorphism types of the elements in the corresponding Cartesian system. In the case where G is primitive, Theorem 6.1 reduces to [13, Proposition 6.1 (ii)]. Problems similar to ours were also addressed in [5].

Our notation concerning actions and permutation groups is standard. If G is a group acting on Ω and Δ is a subset of Ω, then G_Δ and $G_{(\Delta)}$ denote the setwise and the pointwise stabilisers of Δ, respectively. If $G_\Delta = G$ then G^Δ denotes the subgroup of $\operatorname{Sym}\Delta$ induced by G. If $\omega \in \Omega$, then ω^G denotes the G-orbit $\{\omega^g \mid g \in G\}$.

2. Cartesian decompositions

Let Γ be a finite set with at least two elements, $L \leqslant \operatorname{Sym}\Gamma$, $l \geqslant 2$ an integer, and $H \leqslant S_l$. The *wreath product* $L \operatorname{wr} H$ is the semidirect product $L^l \rtimes H$, where, for

$(x_1, \dots, x_l) \in L^l$ and $\sigma \in S_l$, $(x_1, \dots, x_l)^{\sigma^{-1}} = (x_{1^\sigma}, \dots, x_{l^\sigma})$. The product action of $L \operatorname{wr} H$ is the action of $L \operatorname{wr} H$ on Γ^l defined by

$$(\gamma_1, \dots, \gamma_l)^{(x_1, \dots, x_l)} = \left(\gamma_1^{x_1}, \dots, \gamma_l^{x_l}\right) \quad \text{and} \quad (\gamma_1, \dots, \gamma_l)^{\sigma^{-1}} = (\gamma_{1^\sigma}, \dots, \gamma_{l^\sigma})$$

for all $(\gamma_1, \dots, \gamma_l) \in \Gamma^l$, and $x_1, \dots, x_l \in L$ and $\sigma \in H$. The important properties of wreath products can be found in most textbooks on permutation group theory, see for instance Dixon and Mortimer [7].

The full stabiliser W in $\operatorname{Sym}\Omega$ of a homogeneous Cartesian decomposition $\mathcal{E}$ of Ω is isomorphic to $\operatorname{Sym}\Gamma \operatorname{wr} S_l$ acting in product action on Γ^l for $\Gamma \in \mathcal{E}$. Moreover, if $|\Gamma| \geqslant 3$ then W is primitive on Ω, and if $|\Gamma| \geqslant 5$ then W is a maximal subgroup of $\operatorname{Sym}\Omega$ or $\operatorname{Alt}\Omega$. As mentioned in Section 1, such maximal subgroups are usually referred to as *maximal subgroups of product action type*. They form one of several classes of primitive maximal subgroups of $\operatorname{Sym}\Omega$ and $\operatorname{Alt}\Omega$, identified by the O'Nan–Scott Theorem; see [12]. Thus an important part of classifying the primitive maximal subgroups of $\operatorname{Sym}\Omega$ or $\operatorname{Alt}\Omega$ containing a given (innately transitive) subgroup G is finding all homogeneous Cartesian decompositions of Ω that are stabilised by G. Our first result is that the plinth must leave invariant each partition in such a Cartesian decomposition.

PROPOSITION 2.1. *If G is an innately transitive group on a set Ω with plinth M and $\mathcal{E} \in \operatorname{CD}(G)$, then $M_{(\mathcal{E})} = M$.*

PROOF. We let $\Gamma \in \mathcal{E}$ and show that each element of the G-orbit Γ^G is stabilised by M. Suppose that $\{\Gamma_1, \dots, \Gamma_m\}$ is the G-orbit in $\mathcal{E}$ containing $\Gamma \in \mathcal{E}$. Set

$$\Sigma = \{\gamma_1 \cap \dots \cap \gamma_m \mid \gamma_1 \in \Gamma_1, \dots, \gamma_m \in \Gamma_m\}$$

and

$$\bar{\Gamma}_i = \{\{\sigma \in \Sigma \mid \sigma \subseteq \gamma\} \mid \gamma \in \Gamma_i\} \quad \text{for} \quad i = 1, \dots, m.$$

Then it is a routine calculation to check that Σ is a G-invariant partition of Ω, and that $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_m\}$ is a G-invariant Cartesian decomposition of Σ. Moreover, $|\bar{\Gamma}_i| = |\Gamma_i|$ for all i, and since $\Gamma_1, \dots, \Gamma_m$ form a G-orbit, $|\bar{\Gamma}_i| = |\bar{\Gamma}_j|$ for all i and j. It is also easy to see that if $g \in G_{(\bar{\Gamma}_i)}$ then $g \in G_{(\Gamma_i)}$. Since $G_{(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)}$ is a normal subgroup of G and M is a minimal normal subgroup of G, either $M \leqslant G_{(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)}$ or $M \cap G_{(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)} = 1$. Suppose that $M \cap G_{(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)} = 1$, so M acts on the set $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_m\}$ faithfully. Therefore M is isomorphic to a subgroup of S_m. Note that $|\Sigma| = |\bar{\Gamma}_1|^m$, and let p be a prime dividing $|\bar{\Gamma}_1|$. Then p^m divides $|\Sigma|$. Since M is transitive on Σ, $p^m \mid |M|$. However, M is isomorphic to a subgroup of S_m, and so p^m divides $m!$, which is a contradiction to [13, Lemma 4.2]. Hence $M \leqslant G_{(\bar{\Gamma}_1, \dots, \bar{\Gamma}_m)}$, that is, each $\bar{\Gamma}_i$ is stabilised by M, and so is each Γ_i. Thus M stabilises Γ, and, since Γ was chosen arbitrarily, this shows that every element of $\mathcal{E}$ is stabilised by M. □

LEMMA 2.2. *Let M be a transitive subgroup of* $\mathrm{Sym}\,\Omega$ *and let* $\mathcal{E} \in \mathrm{CD}(M)$ *such that* $M_{(\mathcal{E})} = M$. *Suppose that* $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_l\}$, *let* $\omega \in \Omega$ *be a fixed element, and for* $i = 1, \dots, l$ *let* $\gamma_i \in \Gamma_i$ *be such that* $\omega \in \gamma_i$. *Set* $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \dots, K_l\}$ *where* $K_i = M_{\gamma_i}$ *for* $i = 1, \dots, l$. *Then* $\mathcal{K}_\omega(\mathcal{E})$ *is a Cartesian system of subgroups of M with respect to ω. Moreover, if* $\omega^m = \omega'$ *for some* $m \in M$, *then* $\mathcal{K}_{\omega'}(\mathcal{E}) = \mathcal{K}_\omega(\mathcal{E})^m$.

PROOF. Let us prove that $\bigcap_{i=1}^{l} K_i = M_\omega$. Since the Γ_i are M-invariant partitions of Ω, the stabiliser of a point stabilises the block in Γ_i that contains this point. Hence $M_\omega \leqslant K_i$ for all i, and so $M_\omega \leqslant \bigcap_i K_i$. Now suppose that $x \in \bigcap_i K_i$. Then x stabilises $\gamma_1, \dots, \gamma_l$. Since $\mathcal{E}$ is a Cartesian decomposition, $\gamma_1 \cap \dots \cap \gamma_l = \{\omega\}$, and so x stabilises ω. Thus $x \in M_\omega$, and so $\bigcap_i K_i = M_\omega$.

Now we prove that (2) also holds. We may suppose without loss of generality that $i = 1$. Let $x \in M$, $\delta_1 = \gamma_1^x, \dots, \delta_l = \gamma_l^x$, and $\{\xi\} = \delta_1 \cap \dots \cap \delta_l$. If $\{\zeta\} = \delta_1 \cap \gamma_2 \cap \dots \cap \gamma_l$ then the transitivity of M on Ω implies that there exists $z \in M$ with $\xi^z = \zeta$ and so $\delta_1^z = \delta_1, \delta_2^z = \gamma_2, \dots, \delta_l^z = \gamma_l$, whence $\gamma_j^{xz} = \gamma_j$ for $j = 2, \dots, l$ and $\gamma_1^{xzx^{-1}} = \gamma_1$, that is $xz \in \bigcap_{j=2}^{l} K_j$ and $xzx^{-1} \in K_1$. It follows that

$$x = (xzx^{-1})^{-1}(xzx^{-1}x) \in K_1 \left(\bigcap_{j=2}^{l} K_j\right),$$

and we deduce that the first factorisation of (2) holds. The other factorisations can be proved identically. Thus $\mathcal{K}_\omega(\mathcal{E})$ is a Cartesian system of M with respect to ω.

If $m \in M$ and $\omega' = \omega^m$ then $\{\omega'\} = \gamma_1^m \cap \dots \cap \gamma_l^m$ and $M_{\gamma_i^m} = M_{\gamma_i}^m$, which proves that $\mathcal{K}_{\omega'}(\mathcal{E}) = \mathcal{K}_\omega(\mathcal{E})^m$. □

If $M \leqslant \mathrm{Sym}\,\Omega$ and $\mathcal{E} \in \mathrm{CD}(M)$ such that $M_{(\mathcal{E})} = M$, then, for a fixed $\omega \in \Omega$, we define the Cartesian system $\mathcal{K}_\omega(\mathcal{E})$ with respect to ω as in Lemma 2.2. The last result of this section establishes one direction of the one-to-one correspondence in Theorem 1.4.

LEMMA 2.3. *Let G be an innately transitive group with plinth M acting on Ω, and let* $\omega \in \Omega$. *If* $\mathcal{E} \in \mathrm{CD}(G)$, *then* $M_{(\mathcal{E})} = M$. *Assume that* $\mathcal{K}_\omega(\mathcal{E})$ *is the Cartesian system of M with respect to ω. Then* $\mathcal{K}_\omega(\mathcal{E})$ *is invariant under conjugation by* G_ω, *and the* G_ω*-actions on* $\mathcal{K}_\omega(\mathcal{E})$ *and on* $\mathcal{E}$ *are equivalent.*

PROOF. It follows from Proposition 2.1 that $M_{(\mathcal{E})} = M$, and so we can use Lemma 2.2 to construct $\mathcal{K}_\omega(\mathcal{E})$ for ω. Suppose that $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_l\}$, and let $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \dots, K_l\}$ such that $K_i = M_{\gamma_i}$ where γ_i is the unique element of Γ_i containing ω. If $\Gamma_i, \Gamma_j \in \mathcal{E}$ and $g \in G_\omega$ such that $\Gamma_i^g = \Gamma_j$ then $\omega^g = \omega$, and so $\gamma_i^g = \gamma_j$. Hence

$$K_i^g = \left(M_{\gamma_i}\right)^g = M_{\gamma_i^g} = M_{\gamma_j} = K_j,$$

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and so $\mathcal{K}_\omega(\mathcal{E})$ is invariant under conjugation by G_ω. This argument also shows that the G_ω-actions on $\mathcal{E}$ and on $\mathcal{K}_\omega(\mathcal{E})$ are equivalent. □

3. Cartesian systems

In this section we summarise the most important properties of Cartesian systems of abstract groups. The following lemma is useful when working with Cartesian systems. If $\{K_1, \dots, K_l\}$ is a Cartesian system for a group M and $I \subseteq \{1, \dots, l\}$ then let K_I denote the subgroup $K_I = \bigcap_{i\in I} K_i$. We use the convention that if $I = \emptyset$ then $\bigcap_{i\in I} K_i = M$ for any collection $\{K_i\}_i$ of subgroups in M.

LEMMA 3.1. *Let $\{K_1, \dots, K_l\}$ be a (possibly trivial) Cartesian system for an abstract group M, and let I, J be subsets of $\{1, \dots, l\}$.*

(a) *If $x_1, \dots, x_l \in M$, then $\bigcap_{i\in I} K_i x_i$ is a coset modulo K_I.*
(b) $|M : K_I| = \prod_{i\in I} |M : K_i|$.
(c) $K_I K_J = K_{I\cap J}$.

PROOF. If an intersection of (right) cosets is nonempty then it is a (right) coset modulo the intersection of the relevant subgroups. The statement of (a) above and the simple proof below make use of this fact. We prove the lemma by induction on l. Notice that there is nothing to prove if $l = 1$. Our inductive hypothesis is that $l > 1$ and the lemma holds for all Cartesian systems for M which consist of fewer than l subgroups. Thus (a) and (b) only have to be proved for the case $I = \{1, \dots, l\}$. Put $L = \bigcap_{i>1} K_i$, and note that $\{K_1, L\}$ is also a Cartesian system for M (that is, $K_1 L = M$).

We also know from the inductive hypothesis that $\bigcap_{i>1} K_i x_i$ is a coset modulo L, so for (a) it is sufficient to show that $K_1 x_1 \cap Ly$ is never empty. In order to show this we choose $z \in L$ such that $K_1 z = K_1 x_1 y^{-1}$; this is possible, as $K_1 L = M$. Then $K_1 zy = K_1 x_1$, and so $zy \in K_1 x_1$, and also $zy \in Ly$. Hence $zy \in K_1 x_1 \cap Ly$, and consequently $K_1 x_1 \cap Ly$ is non-empty.

For (b), it is enough to show that $|M : K_I| = |M : K_1||M : L|$, but this follows from

$$|M| = |K_1 L| = |K_1||L|/|K_1 \cap L| = |K_1||L|/K_I|.$$

For an easy proof of (c) we first observe that

$$|K_I K_J| = |K_I||K_J|/|K_I \cap K_J|.$$

It is obvious that $K_I K_J \subseteq K_{I\cap J}$ and, as $K_I \cap K_J = K_{I\cup J}$, one can calculate from (b) and the last display that $|K_I K_J| = |K_{I\cap J}|$. This completes the proof of the lemma. □

Note that, in Lemma 3.1 (a), if we choose x to be any element of $\bigcap_{i\in I} K_i x_i$, then $K_i x_i = K_i x$ holds, for all $i \in I$.

4. Cartesian systems and Cartesian decompositions

In a transitive group $M \leqslant \mathrm{Sym}\,\Omega$, a subgroup K satisfying $M_\omega \leqslant K \leqslant M$ for some $\omega \in \Omega$ determines an M-invariant partition of Ω comprising the M-translates of the K-orbit ω^K.

LEMMA 4.1. *Let G be an innately transitive group on Ω with plinth M, and let ω be a fixed element of Ω. Suppose that $\mathcal{K} = \{K_1, \dots, K_l\}$ is a G_ω-invariant Cartesian system of subgroups of M with respect to ω, and let $\Gamma_1, \dots, \Gamma_l$ be the M-invariant partitions of Ω determined by $K_1, \dots, K_l$, respectively. Then $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_l\}$ is a G-invariant Cartesian decomposition of Ω, such that $\mathcal{K}_\omega(\mathcal{E}) = \mathcal{K}$. Moreover, if M is non-abelian and the Cartesian system $\{K_1, \dots, K_l\}$ is homogeneous, then the stabiliser W in* $\mathrm{Sym}\,\Omega$ *of $\mathcal{E}$ is a maximal subgroup of* $\mathrm{Sym}\,\Omega$ *or* $\mathrm{Alt}\,\Omega$ *such that $G \leqslant W$.*

PROOF. As $M_\omega \leqslant K_i \leqslant M$, each Γ_i is an M-invariant partition of Ω. For $i = 1, \dots, l$ let γ_i be the unique element of Γ_i containing ω. In order to prove that $\mathcal{E}$ is a Cartesian decomposition, we only have to show that

$$\left|\bigcap_{i=1}^{l} \delta_i\right| = 1 \quad \text{whenever} \quad \delta_1 \in \Gamma_1, \dots, \delta_l \in \Gamma_l.$$

To see this, choose $\delta_1 \in \Gamma_1, \dots, \delta_l \in \Gamma_l$. Now $\delta_i = \gamma_i^{x_i}$ for some $x_i \in M$, and by Lemma 3.1 (a), there exists some $x \in M$ such that $K_i x_i = K_i x$ for $i = 1, \dots, l$. Then

$$\begin{aligned}\delta_i = \gamma_i^{x_i} &= \{\omega^k \mid k \in K_i\}^{x_i} = \{\omega^{k'} \mid k' \in K_i x_i\}\\ &= \{\omega^{k'} \mid k' \in K_i x\} = \{\omega^k \mid k \in K_i\}^x = \gamma_i^x.\end{aligned}$$

Thus

$$\bigcap_{i=1}^{l} \delta_i = \bigcap_{i=1}^{l} \gamma_i^x = \left(\bigcap_{i=1}^{l} \gamma_i\right)^x,$$

and therefore we only have to prove that $\left|\bigcap_{i=1}^{l} \gamma_i\right| = 1$. Note that $\omega \in \gamma_i$ for $i = 1, \dots, l$. Suppose that $\omega' \in \gamma_1 \cap \dots \cap \gamma_l$ for some $\omega' \in \Omega$. Then there is some $x \in M$ such that $\omega^x = \omega'$. Then x must stabilise $\gamma_1, \dots, \gamma_l$, and hence $x \in K_i$ for all $i = 1, \dots, l$. Since $\bigcap_{i=1}^{l} K_i = M_\omega$, it follows that $x \in M_\omega$, and so $\omega^x = \omega$. Thus $\bigcap_{i=1}^{l} \gamma_i = \{\omega\}$, and $\mathcal{E}$ is a Cartesian decomposition.

Since each Γ_i is an M-invariant partition of Ω, $\mathcal{E}$ is invariant under M. Since $\{K_1, \ldots, K_l\}$ is G_ω-invariant, $\mathcal{E}$ is also G_ω-invariant, and so $\mathcal{E}$ is MG_ω-invariant. Since M is transitive, $MG_\omega = G$. Therefore $\mathcal{E}$ is G-invariant. Note that

$$\mathcal{K} = \{M_{\gamma_1}, \ldots, M_{\gamma_l}\} \quad \text{and} \quad K_\omega(\mathcal{E}) = \{M_{\gamma_1}, \ldots, M_{\gamma_l}\}.$$

Thus $\mathcal{K} = \mathcal{K}_\omega(\mathcal{E})$, as required.

Since M is non-abelian, M is a direct product of isomorphic non-abelian, simple groups. Hence for $i = 1, \ldots, l$, the group M^{Γ_i} is also isomorphic to a direct product of non-abelian simple groups. Moreover, M^{Γ_i} is transitive and faithful on Γ_i, and so $|\Gamma_i| \geqslant 5$ for all i. As $\{K_1, \ldots, K_l\}$ is homogeneous, $\mathcal{E}$ is also homogeneous and W is permutationally isomorphic to $\operatorname{Sym}\Gamma \operatorname{wr} S_l$ in product action for some set Γ and $l \geqslant 2$. Hence the results of [12] show that W is a maximal subgroup of $\operatorname{Sym}\Omega$ if $W \not\leqslant \operatorname{Alt}\Omega$, and W is a maximal subgroup of $\operatorname{Alt}\Omega$ otherwise. Since $\mathcal{E}$ is G-invariant, clearly $G \leqslant W$. □

THEOREM 4.2. *Let G be an innately transitive group on Ω with plinth M. For a fixed $\omega \in \Omega$ the map $\mathcal{E} \mapsto \mathcal{K}_\omega(\mathcal{E})$ is a bijection between the set* $\operatorname{CD}(G)$ *and the set of G_ω-invariant Cartesian systems of subgroups of M with respect to ω.*

PROOF. Let $\mathcal{C}$ denote the set of G_ω-invariant Cartesian systems of subgroups of M with respect to ω. In Lemma 2.2, we explicitly constructed a map $\Psi : \operatorname{CD}(G) \to \mathcal{C}$ for which $\Psi(\mathcal{E}) = \mathcal{K}_\omega(\mathcal{E})$. We claim that Ψ is a bijection. Let $\mathcal{K} \in \mathcal{C}$, let $\Gamma_1, \ldots, \Gamma_l$ be the M-invariant partitions determined by the elements $K_1, \ldots, K_l$ of $\mathcal{K}$, and let $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_l\}$. We proved in Lemma 4.1 that $\mathcal{E}$ is a G-invariant Cartesian decomposition of Ω such that $\mathcal{K}_\omega(\mathcal{E}) = \mathcal{K}$. Hence Ψ is surjective.

Suppose now that $\mathcal{E}_1, \mathcal{E}_2 \in \operatorname{CD}(G)$ is such that $\Psi(\mathcal{E}_1) = \Psi(\mathcal{E}_2)$ and let $\mathcal{K}$ denote this common Cartesian system. Let $\mathcal{E}$ be the set of M-invariant partitions determined by the elements of $\mathcal{K}$. Then, by the definition of $\Psi(\mathcal{E}_i)$ in Lemma 2.2, $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \mathcal{E}$. Thus Ψ is injective, and so Ψ is a bijection. □

Theorem 1.4 is an immediate consequence of the previous result.

5. Some factorisations of finite simple groups

To prove Theorem 1.1 we need first to prove some results about factorisations of certain finite simple groups. If G is a group and $A, B \leqslant G$ such that $G = AB$, then we say that the expression $G = AB$ or the set $\{A, B\}$ is a *factorisation* of G. In [1] full factorisations of almost simple groups were classified up to the following equivalence relation. The factorisations $G = A_1B_1$ and $G = A_2B_2$ of a group G are said to be

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equivalent if there are $\alpha \in \mathrm{Aut}(G)$, and $x,\ y \in G$ such that $\{A_1, B_1\} = \{A_2^{\alpha x}, B_2^{\alpha y}\}$. The following lemma shows that this equivalence relation can be expressed in a simpler way.

LEMMA 5.1. *Let G be a group.*

(i) *If $G = AB$ for some $A,\ B \leqslant G$, then the conjugation action of A is transitive on the conjugacy class B^G, and B is transitive on A^G.*

(ii) *The factorisations $G = A_1B_1$ and $G = A_2B_2$ of G are equivalent if and only if there is $\beta \in \mathrm{Aut}(G)$ such that $\{A_1, B_1\} = \{A_2^\beta, B_2^\beta\}$.*

PROOF. (i) As $AB = G$, we also have $A\mathrm{N}_G(B) = G$. Since $\mathrm{N}_G(B)$ is a point stabiliser for the conjugation action of G on the conjugacy class B^G, we obtain that A is a transitive subgroup of G with respect to this action. Similar argument shows that B is transitive by conjugation on A^G.

(ii) It is clear that if there is $\beta \in \mathrm{Aut}(G)$ such that $\{A_1, B_1\} = \{A_2^\beta, B_2^\beta\}$ then the two factorisations in the lemma are equivalent. Suppose that $G = A_1B_1$ and $G = A_2B_2$ are equivalent factorisations. By assumption, there is $\alpha \in \mathrm{Aut}(G)$ and $x,\ y \in G$ such that $\{A_1, B_1\} = \{A_2^{\alpha x}, B_2^{\alpha y}\}$. Then we have $A_1^G = (A_2^\alpha)^G$ and $B_1^G = (B_2^\alpha)^G$, or $A_1^G = (B_2^\alpha)^G$ and $B_1^G = (A_2^\alpha)^G$. Suppose without loss of generality that $A_1^G = (A_2^\alpha)^G$ and $B_1^G = (B_2^\alpha)^G$. Since A_1 and A_2^α are conjugate, there is some $g \in G$ such that $A_1^g = A_2^\alpha$, and B_1^g is conjugate to B_2^α. As $G = (A_1B_1)^g = A_1^gB_1^g$, we have that A_1^g is transitive by conjugation on $(B_1^g)^G = B_1^G$. Hence there is some $a \in A_1^g$ such that $A_1^{ga} = A_1^g = A_2^\alpha$, and $B_1^{ga} = B_2^\alpha$. Hence $A_1 = A_2^{\alpha a^{-1}g^{-1}}$ and $B_1 = B_2^{\alpha a^{-1}g^{-1}}$. Thus we may take β as α followed by the inner automorphism corresponding to $a^{-1}g^{-1}$. □

If G is a group and A and B are subgroups then let

$$\mathrm{N}_G(\{A, B\}) = \{g \in G \mid \{A^g, B^g\} = \{A, B\}\}.$$

In the proof of the following result we use the following simple fact, called Dedekind's modular law. If K, L, H are subgroups of a group G such that $K \leqslant L$, then

$$(HK) \cap L = (H \cap L)K. \tag{3}$$

LEMMA 5.2. *Let T be a finite simple group and $A,\ B$ proper subgroups of T such that $|A| = |B|$ and $T = AB$. Then the following hold.*

(i) *The isomorphism types of T, A, and B are as in Table 2, and A, B are maximal subgroups of T.*

(ii) *There is an automorphism $\vartheta \in \mathrm{Aut}(T)$ such that ϑ interchanges A and B.*

(iii) *The group $A \cap B$ is self-normalising in T.*

TABLE 2. Factorisations of finite simple groups in Lemma 5.2

	T	A, B
1	A_6	A_5
2	M_{12}	M_{11}
3	$P\Omega_8^+(q)$	$\Omega_7(q)$
4	$Sp_4(q)$, $q \geqslant 4$ even	$Sp_2(q^2).2$

(iv) *If T is as in row* 1, 2, *or* 4 *of Table* 2, *then*

$$N_{Aut(T)}(A \cap B) = N_{Aut(T)}(\{A, B\}) = N,$$

say, and moreover $TN = Aut(T)$.

PROOF. (i) Note that, since $|A| = |B|$, the factorisation $T = AB$ is a full factorisation of T, that is, the sets of primes dividing $|T|$, $|A|$, and $|B|$ are the same. It was proved in [1], that T, A, and B are as in [1, Table I]. It is easy to see that the only possibilities where $|A| = |B|$ are those in Table 2, and it follows that in these cases A and B are maximal subgroups of T.

(ii) In each line of Table 2, the groups A and B are not conjugate, but there is an outer automorphism $\sigma \in Aut(T)$ which swaps the conjugacy classes A^T and B^T (see the Atlas [6] for $T \cong A_6$, M_{12}, [9] for $T \cong P\Omega_8^+(q)$, and [1, page 155] for $T \cong Sp_4(q)$). By Lemma 5.1 (i), the group A is transitive in its conjugation action on the conjugacy class B^T and B is transitive on A^T. Thus there is an element $a \in A$ such that $A^{\sigma a} = B$ and $B^{\sigma a}$ is conjugate in T to A. Since B is transitive on A^T, there is an element $b \in B$ such that $A^{\sigma ab} = B$ and $B^{\sigma ab} = A$. Therefore we can take ϑ as σ followed by the inner automorphism induced by the element ab.

(iii) Set $C = A \cap B$. First we prove that C is self-normalising in T. If T is isomorphic to A_6 or M_{12} then the information given in the Atlas [6] shows that if N is a proper subgroup of T properly containing C, then N is isomorphic to A or B. In all cases A and B are simple, and so $N_T(C) = C$. If $T \cong P\Omega_8^+(q)$ then we obtain from [9, 3.1.1 (vi)] that $C \cong G_2(q)$ and [9, 3.1.1 (iii)] yields that $N_T(C) = C$.

Now let $T \cong Sp_4(q)$ for $q \geqslant 4$, q even. In this case $A \cong B \cong Sp_2(q^2) \cdot 2$. Consider the fields $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ as subfields of the field $\mathbb{F}_{q^4}$ and consider the field $\mathbb{F}_{q^4}$ as a 4-dimensional vector space V over $\mathbb{F}_q$. Let $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}} : \mathbb{F}_{q^4} \to \mathbb{F}_{q^2}$ and $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2} \to \mathbb{F}_q$ denote the norm and the trace map, respectively. For the basic properties of these maps see [11, 2.3]. Using the fact that $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x) = x^{q^2+1}$ for all $x \in \mathbb{F}_{q^4}$, we obtain that $x \mapsto \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)$ is an $\mathbb{F}_{q^2}$-quadratic form on V, such that

$$(x, y) \mapsto \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x + y) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(y)$$

is a non-degenerate, symmetric, $\mathbb{F}_{q^2}$-bilinear form with Witt defect 1 (we recall that q is a 2-power). Hence $Q = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \circ \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}$ is an $\mathbb{F}_q$-quadratic form $V \to \mathbb{F}_q$, and $f(x, y) = Q(x+y) + Q(x) + Q(y)$, is a non-degenerate, symmetric $\mathbb{F}_q$-bilinear form on V with Witt defect 1. Then without loss of generality we may assume that T is the stabiliser of f in $\mathrm{GL}_4(q)$, A consists of elements of T that are $\mathbb{F}_{q^2}$-semilinear, and B is the stabiliser of Q.

For $a \in V \setminus \{0\} = \mathbb{F}_{q^4}^*$, define the map $s_a : x \mapsto xa$. Then it is well-known that $S = \{s_a \mid a \in \mathbb{F}_{q^4}^*\}$ is a cyclic subgroup of $\mathrm{GL}_4(q)$. A generator of S is called a Singer cycle; see Satz II.7.3 in Huppert [8]. Let Z denote the subgroup $\{s_a \mid \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a) = 1\}$ of S. Since the restriction of $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}$ to $\mathbb{F}_{q^4}^*$ is an epimorphism $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}} : \mathbb{F}_{q^4}^* \to \mathbb{F}_{q^2}^*$, and Z is the kernel of this epimorphism, we have that $|Z| = q^2 + 1$. If σ is the Frobenius automorphism $x \mapsto x^q$ of $\mathbb{F}_{q^4}$ then $(s_a)^\sigma = s_{a^\sigma}$ for all $s_a \in S$. Therefore σ normalises S, and, since S is cyclic, σ also normalises Z. We claim that $C = Z\langle\sigma\rangle$. Since $T = AB$, $|C| = 4(q^2+1)$, and hence it suffices to prove that $Z\langle\sigma\rangle \leqslant C$. It is clear that σ is $\mathbb{F}_{q^2}$-semilinear, and so $\sigma \in A$. Also

$$Q(\sigma(x)) = Q(x^q) = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x^q)\right) = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)^q\right)$$
$$= \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)\right) = Q(x).$$

Therefore $\sigma \in B$, and so $\sigma \in C$. Let $a \in \mathbb{F}_{q^4}$ such that $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a) = 1$. Then

$$Q(s_a(x)) = Q(xa) = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(xa)\right)$$
$$= \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a)\right) = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)\right) = Q(x).$$

Thus $s_a \in B$. Since s_a is also $\mathbb{F}_{q^2}$-linear, we obtain $s_a \in C$. Hence $C = Z\langle\sigma\rangle$.

We will now prove that C is self-normalising in T. First notice that q^2+1 is divisible by an odd prime r such that $r \nmid (q^2 - 1)$. Hence there is a unique subgroup R in Z with order r. Since Z is the commutator subgroup of C, it is a characteristic subgroup of C. Also R is the unique subgroup of Z with order r, and so R is characteristic in Z. Thus R is characteristic in C and $\mathrm{N}_T(C)$ must normalise R. By [8, Satz II.7.3], $\mathrm{N}_{\mathrm{GL}_4(q)}(R) = SC = S\langle\sigma\rangle$.

Let us now determine how much of $S\langle\sigma\rangle$ is contained in T. Since $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is additive,

$$f(x^q, y^q) = \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x^q + y^q) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x^q) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(y^q)\right)$$
$$= \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x+y) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x) + \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(y)\right)^q\right) = f(x, y),$$

and hence the cyclic subgroup $\langle\sigma\rangle$ is in T. Using (3), we have $(S\langle\sigma\rangle) \cap T = (S \cap T)\langle\sigma\rangle$. Thus we need to compute $S \cap T$. If $x \in \mathbb{F}_{q^4}^*$ such that $f(xa, xb) = f(a, b)$ for all

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$a,\ b \in V$ then

$$(4)\quad \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a+b)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(b)\right)$$
$$=\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(xa+xb)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(xa)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(xb)\right)$$
$$=\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)\left(\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a+b)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(a)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(b)\right)\right).$$

As observed above,

$$(u,v)\mapsto \mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(u+v)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(u)+\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(v)$$

is a non-degenerate, symmetric, $\mathbb{F}_{q^2}$-bilinear form, and so it maps $V\times V$ onto $\mathbb{F}_{q^2}$. Hence (4) shows that $y=\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)$ has the property that $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(yu)=\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(u)$ for all $u\in\mathbb{F}_{q^2}$, that is, $yu+y^qu^q=u+u^q$, for all $u\in\mathbb{F}_{q^2}$. Thus $(yu+u)^q=yu+u$. Hence $u(y+1)\in\mathbb{F}_q$ for all $u\in\mathbb{F}_{q^2}$, and consequently $y=1$. Thus if the map s_x preserves f then $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)=1$. On the other hand from (4) it is clear that if $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}(x)=1$ then multiplication by x preserves f. Since the norm is a group epimorphism $\mathbb{N}_{\mathbb{F}_{q^4}/\mathbb{F}_{q^2}}:\mathbb{F}^*_{q^4}\to\mathbb{F}^*_{q^2}$ it follows that the elements of norm 1 form a cyclic group of order q^2+1. Hence $S\cap T=Z$ and $\mathrm{N}_{\mathrm{GL}_4(q)}(C)\cap T=C$, that is, C is self-normalising in T.

(iv) Finally we assume that T is as in row 1, 2, or 4 of Table 2, and we prove the assertion that $N_1=N_2$, where $N_1=\mathrm{N}_{\mathrm{Aut}(T)}(C)$ and $N_2=\mathrm{N}_{\mathrm{Aut}(T)}(\{A,B\})$. It is clear that $N_2\leqslant N_1$, and so we only have to prove $|N_1|\leqslant|N_2|$. Since A and B are not conjugate in T, we have that $N_2\cap T=\mathrm{N}_T(A)\cap\mathrm{N}_T(B)=A\cap B=C$, and, since C is self-normalising in T, we also have $N_1\cap T=C$. Thus it suffices to prove that $TN_1\leqslant TN_2$, which follows immediately once we show that $TN_2=\mathrm{Aut}(T)$. Since N_2 interchanges A and B, we have that $N_2=(\mathrm{N}_{\mathrm{Aut}(T)}(A)\cap\mathrm{N}_{\mathrm{Aut}(T)}(B))\langle\vartheta\rangle$ where $\vartheta\in\mathrm{Aut}(T)$ is as in (ii). If $T\cong\mathrm{A}_6$ then $|\mathrm{N}_{\mathrm{Aut}(T)}(A):\mathrm{N}_T(A)|=|\mathrm{N}_{\mathrm{Aut}(T)}(B):\mathrm{N}_T(B)|=2$, and so $TN_2=\mathrm{Aut}(T)$ (see [6]). If $T\cong\mathrm{M}_{12}$ then $\mathrm{N}_{\mathrm{Aut}(T)}(A)=\mathrm{N}_T(A)$ and $\mathrm{N}_{\mathrm{Aut}(T)}(B)=\mathrm{N}_T(B)$, and so $TN_2=\mathrm{Aut}(T)$ (see [6]). If $T\cong\mathrm{Sp}_4(q)$ then the field automorphism group Φ normalises A and B. If $\vartheta\in\mathrm{Aut}(T)$ is as in (ii), then $\mathrm{Aut}(T)=T\Phi\langle\vartheta\rangle$, and so we obtain that $TN_2=\mathrm{Aut}(T)$. Hence if T is as in row 1, 2, or 4 of Table 2, then $TN_2=\mathrm{Aut}(T)$, and $TN_1\leqslant TN_2$ clearly holds. Thus $N_1=N_2$ follows. □

We recall a couple of facts about automorphisms of $\mathrm{P}\Omega_8^+(q)$. Let $T=\mathrm{P}\Omega_8^+(q)$. Then, as shown in [9, pp. 181–182], $\mathrm{Aut}(T)=\Theta\rtimes\Phi$, where Φ is the group of field automorphisms of T, and Θ is a certain subgroup of $\mathrm{Aut}(T)$ containing the commutator subgroup $\mathrm{Aut}(T)'$. We also have $\mathrm{Out}(T)=\mathrm{Aut}(T)/T=\Theta/T\times\Phi T/T$, and $\Theta/T\cong\mathrm{S}_m$ where $m=3$ for even q, and $m=4$ for odd q. Let $\pi:\Theta\to\mathrm{S}_m$ denote the natural epimorphism. The following lemma derives the information about $\mathrm{P}\Omega_8^+(q)$ similar to that in Lemma 5.2 (iv).

LEMMA 5.3. *Let* $T = \mathrm{P}\Omega_8^+(q)$, *let* A, B *be subgroups of* T *such that* A, $B \cong \Omega_7(q)$ *and* $AB = T$, *and set* $C = A \cap B$. *Then the following hold.*

(i) *We have* $\Phi \leqslant \mathrm{N}_{\mathrm{Aut}(T)}(A) \cap \mathrm{N}_{\mathrm{Aut}(T)}(B)$.

(ii) *The groups* $(\mathrm{N}_{\mathrm{Aut}(T)}(A) \cap \Theta)T/T$ *and* $(\mathrm{N}_{\mathrm{Aut}(T)}(B) \cap \Theta)T/T$ *are conjugate to the subgroup in column* X *of* [9, Results Matrix], *so that*

$$\pi(\mathrm{N}_{\mathrm{Aut}(T)}(A) \cap \Theta) \cong \pi(\mathrm{N}_{\mathrm{Aut}(T)}(B) \cap \Theta) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(iii) *We have* $\Phi \leqslant \mathrm{N}_{\mathrm{Aut}(T)}(C)$ *and* $(\mathrm{N}_{\mathrm{Aut}(T)}(C) \cap \Theta)T/T$ *is conjugate to the subgroup in column* VII *of* [9, Results Matrix], *so that* $\pi(\mathrm{N}_{\mathrm{Aut}(T)}(C) \cap \Theta) \cong \mathrm{S}_3$ *and*

$$\left|\mathrm{N}_{\mathrm{Aut}(T)}(C) : \mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\})\right| = 3. \tag{5}$$

(iv) *We have* $T\mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\}) = T\Phi\langle\vartheta\rangle$, *where* ϑ *is as in Lemma* 5.2 (ii), *so that* $\pi(\mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\}) \cap \Theta)/T \cong \mathbb{Z}_2$.

PROOF. Claims (i)–(ii) can easily be verified by inspection of [9, Results Matrix]. In (iii) we only need to prove (5). Let $N_1 = \mathrm{N}_{\mathrm{Aut}(T)}(C)$ and $N_2 = \mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\})$. Clearly $N_2 \leqslant N_1$. By [9, Proposition 3.1.1 (vi)], $C \cong \mathrm{G}_2(q)$, and [9, Proposition 3.1.1 (iii)] shows that $\pi(N_1 \cap \Theta) \cong \mathrm{S}_3$. From [9, Results Matrix] we obtain $\pi(N_2 \cap \Theta) = \mathbb{Z}_2$. As in the proof of Lemma 5.2, we have $N_1 \cap T = N_2 \cap T = C$. As $T = \ker \pi$ this implies $N_1 \cap \ker \pi = N_2 \cap \ker \pi$, and so $|N_1 \cap \Theta| = 3 \cdot |N_2 \cap \Theta|$. Since $\Phi \leqslant N_1 \cap N_2$ we have $N_1\Theta = N_2\Theta = \mathrm{Aut}(T)$, and so $|N_1| = 3 \cdot |N_2|$, as required. In (iv) we notice that $T\Phi\langle\vartheta\rangle \leqslant T\mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\})$. On the other hand, (iii) implies that $|T\Phi\langle\vartheta\rangle| = |T\mathrm{N}_{\mathrm{Aut}(T)}(\{A, B\})|$, hence equality follows. □

6. Innately transitive groups with a non-abelian, simple plinth

In this section we prove our second main theorem, namely Theorem 1.1, which is a consequence of the following result.

THEOREM 6.1. *Let* G *be an innately transitive permutation group on* Ω *with a non-abelian, simple plinth* T, *let* $\omega \in \Omega$, $\mathcal{E} \in \mathrm{CD}(G)$, *and let* W *be the stabiliser of* $\mathcal{E}$ *in* $\mathrm{Sym}\,\Omega$. *Then* $|\mathcal{E}| \leqslant 3$ *and the following hold.*

(i) *Suppose that* $\mathcal{E}$ *is homogeneous. Then* $|\mathcal{E}| = 2$, W *is a maximal subgroup of* $\mathrm{Sym}\,\Omega$ *or* $\mathrm{Alt}\,\Omega$, *and* G, T, W, *the subgroups* $K \in \mathcal{K}_\omega(\mathcal{E})$, *and* $|\Omega|$ *are as in Table* 3. *In particular, the set* $\mathcal{K}_\omega(\mathcal{E})$ *contains two isomorphic subgroups. Moreover, the group* G *is quasiprimitive and* T *is the unique minimal normal subgroup of* G. *Moreover exactly one of the following holds*:

(a) $|\mathrm{CD}(G)| = 1$;

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TABLE 3. Homogeneous Cartesian decompositions preserved by almost simple groups

	G	T	W	K	$\|\Omega\|$
1	$A_6 \leqslant G \leqslant P\Gamma L_2(9)$	A_6	$S_6 \operatorname{wr} S_2$	A_5	36
2	$M_{12} \leqslant G \leqslant \mathrm{Aut}(M_{12})$	M_{12}	$S_{12} \operatorname{wr} S_2$	M_{11}	144
3	$P\Omega_8^+(q) \leqslant G \leqslant P\Omega_8^+(q)\Phi\langle\vartheta\rangle$ Φ: field automorphisms ϑ is as in Lemma 5.2 (ii)	$P\Omega_8^+(q)$	$S_{(d/2)q^3(q^4-1)} \operatorname{wr} S_2$ $d = (4, q^4 - 1)$	$\Omega_7(q)$	$\left(\dfrac{d^2}{4}\right) q^6(q^4-1)^2$
4	$Sp_4(q) \leqslant G \leqslant \mathrm{Aut}(Sp_4(q))$	$Sp_4(q)$ $q \geqslant 4$ even	$S_{q^2(q^2-1)} \operatorname{wr} S_2$	$Sp_2(q^2).2$	$q^4(q^2-1)^2$

TABLE 4. Cartesian decompositions with index 3 preserved by almost simple groups

	T	W	$\mathcal{K}_\omega(\mathcal{E})$	$\|\Omega\|$
1	$Sp_{4a}(2)$ $a \geqslant 2$	$S_{n_1} \times S_{n_2} \times S_{n_3}$ $n_1 = \|Sp_{4a}(2) : Sp_{2a}(4) \cdot 2\|$ $n_2 = \|Sp_{4a}(2) : O_{4a}^-(2)\|$ $n_3 = \|Sp_{4a}(2) : O_{4a}^+(2)\|$	$Sp_{2a}(4) \cdot 2,\ O_{4a}^-(2),\ O_{4a}^+(2)$	$n_1 \cdot n_2 \cdot n_3$
2	$P\Omega_8^+(3)$	$S_{1080} \times S_{1120} \times S_{28431}$	$\Omega_7(3),\ \mathbb{Z}_3^6 \rtimes PSL_4(3),\ P\Omega_8^+(2)$	34, 390, 137, 600
3	$Sp_6(2)$	$S_{120} \times S_{28} \times S_{36}$ $S_{240} \times S_{28} \times S_{36}$ $S_{120} \times S_{56} \times S_{36}$ $S_{120} \times S_{28} \times S_{72}$	$G_2(2),\ O_6^-(2),\ O_6^+(2)$ $G_2(2)',\ O_6^-(2),\ O_6^+(2)$ $G_2(2),\ O_6^-(2)',\ O_6^+(2)$ $G_2(2),\ O_6^-(2),\ O_6^+(2)'$	120, 960 241, 920 241, 920 241, 920

(b) $|\mathrm{CD}(G)| = 3$, *T is as in row* 3 *of Table* 3, $G \leqslant T\Phi$ *where* Φ *is the group of field automorphisms of* T.

(ii) *Suppose that* $|\mathcal{E}| = 3$. *If* W *is the stabiliser in* $\mathrm{Sym}\,\Omega$ *of* $\mathcal{E}$, *then* T, W, *the elements of* $\mathcal{K}_\omega(\mathcal{E})$, *and* $|\Omega|$ *are as in Table* 4.

PROOF. Suppose that $\mathcal{E} \in \mathrm{CD}(G)$. Then Proposition 2.1 implies that $T_{(\mathcal{E})} = T$. Let l be the index of $\mathcal{E}$, and let $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \dots, K_l\}$ be the corresponding Cartesian system for T. Then the definition of $\mathcal{K}_\omega(\mathcal{E})$ implies that if $l \geqslant 3$ then $\{K_1, \dots, K_l\}$ is a strong multiple factorisation of the finite simple group T. Strong multiple factorisations of finite simple groups are defined and classified in [1]; in particular it is proved that $l \leqslant 3$.

(a) If $l = 3$ then [1, Table V] shows that K_1, K_2, K_3 have different sizes. Thus if $\mathcal{E}$ is homogeneous then $l = 2$ and the factorisation $T = K_1K_2$ is as in Lemma 5.2. Hence T, K_1, K_2, and $|\Omega|$ are as in Table 3. The maximality of W follows from Lemma 4.1.

Let us now prove that G is quasiprimitive. As T is transitive on Ω, we have $C_{\mathrm{Sym}\,\Omega}(T) \cong N_T(T_\omega)/T_\omega$; see [7, Theorem 4.2A]. On the other hand, $T_\omega = K_1 \cap K_2$, and Lemma 5.2 shows that $N_T(K_1 \cap K_2) = K_1 \cap K_2 = T_\omega$. Hence $C_{\mathrm{Sym}\,\Omega}(T) = 1$, and so T is the unique minimal normal subgroup of G. Hence G is an almost simple quasiprimitive group acting on Ω.

Now we prove that the information given in the G-column of Table 3 is correct. Since T is the unique minimal normal subgroup of G, we have that G is an almost simple group and $T \leqslant G \leqslant \mathrm{Aut}(T)$. Let $N = \mathrm{N}_{\mathrm{Aut}(T)}(\{K_1, K_2\})$. Note that $G = TG_\omega$ and $G_\omega \leqslant N$. On the other hand, N has the property that, since A and B are not conjugate in T,

$$T \cap N = \mathrm{N}_T(K_1) \cap \mathrm{N}_T(K_2) = K_1 \cap K_2 = T_\omega,$$

and so the T-action on Ω can be extended to TN with point stabiliser N. Thus $G \leqslant TN$. By Lemmas 5.2 (iv) and 5.3 (iv), for $T \cong \mathrm{A}_6$, M_{12}, $\mathrm{P}\Omega_8^+(q)$, and $\mathrm{Sp}_4(q)$, we have $TN = \mathrm{P}\Gamma\mathrm{L}_2(9)$, $\mathrm{Aut}(\mathrm{M}_{12})$, $\mathrm{P}\Omega_8^+(q)\Phi\langle\vartheta\rangle$ (where Φ is the group of field automorphisms and ϑ is as in Lemma 5.2 (ii)), and $\mathrm{Aut}(\mathrm{Sp}_4(q))$, respectively. Hence the assertion follows.

Finally we prove the claim concerning $|\mathrm{CD}(G)|$. Suppose that $L_1, L_2 \leqslant T$ is such that $|L_1| = |L_2|$, $L_1L_2 = T$ and $L_1 \cap L_2 = T_\omega$. By [1], the full factorisation $T = K_1K_2$ is unique up to equivalence, Lemma 5.1 (ii) shows that there is an element $\alpha \in \mathrm{Aut}(T)$ such that $\{K_1, K_2\}^\alpha = \{L_1, L_2\}$, and so $\alpha \in \mathrm{N}_{\mathrm{Aut}(T)}(T_\omega) = \mathrm{N}_{\mathrm{Aut}(T)}(K_1 \cap K_2)$. Lemma 5.2 (iii) implies that if T is as in row 1, 2, or 4 of Table 3 then $\mathrm{N}_{\mathrm{Aut}(T)}(\{K_1, K_2\}) = \mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$ and so $\{L_1, L_2\} = \{K_1, K_2\}^\alpha = \{K_1, K_2\}$. Thus $|\mathrm{CD}(G)| = 1$ in these cases, as asserted.

Suppose now that $T \cong \mathrm{P}\Omega_8^+(q)$ for some q. Then we obtain from Lemma 5.3 (iii) that $\left|\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega) : \mathrm{N}_{\mathrm{Aut}(T)}(\{K_1, K_2\})\right| = 3$, and so the $\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$-orbit containing $\{K_1, K_2\}$ has 3 elements, which gives rise to 3 different choices of Cartesian systems with respect to ω. Let $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ denote the corresponding Cartesian decompositions of Ω, such that $\mathcal{E} = \mathcal{E}_1$. We computed above that $\mathrm{C}_{\mathrm{Sym}\,\Omega}(T) = 1$, and this implies that $\mathrm{N}_{\mathrm{Sym}\,\Omega}(T) = \mathrm{Aut}(T) \cap \mathrm{Sym}\,\Omega$. In other words, $N = \mathrm{N}_{\mathrm{Sym}\,\Omega}(T)$ is the largest subgroup of $\mathrm{Aut}(T)$ that extends the T-action on Ω. Since T is a transitive subgroup of N, we have $N = TN_\omega$. As T_ω is a normal subgroup of N_ω, it follows that $N \leqslant T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$. On the other hand,

$$|T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega) : \mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)| = |T : T \cap \mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)| = |T : \mathrm{N}_T(T_\omega)| = |T : T_\omega|,$$

by Lemma 5.2 (iii). This shows that the T-action on Ω can be extended to $T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$ with point stabiliser $\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$. In other words, $T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$ is the largest subgroup of $\mathrm{Aut}(T)$ that extends the T-action on Ω. The stabiliser of $\mathcal{E}_1$ in $T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$ is $T\mathrm{N}_{\mathrm{Aut}(T)}(\{K_1, K_2\})$. Hence if $G \leqslant \mathrm{Aut}(T)$ is such that $T \leqslant G$ and G leaves the Cartesian decomposition $\mathcal{E}_1$ invariant, then $G \leqslant T\mathrm{N}_{\mathrm{Aut}(T)}(\{K_1, K_2\}) = T\Phi\langle\vartheta\rangle$, by Lemma 5.3 (iii). If $\mathrm{CD}(G) \neq \{\mathcal{E}\}$ then, G leaves $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ invariant. Therefore G lies in the kernel of the action of $T\mathrm{N}_{\mathrm{Aut}(T)}(T_\omega)$ on $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. Hence $G \leqslant T\Phi$, as required.

(b) Suppose that $|\mathcal{E}| = 3$. Then $\{K_1, K_2, K_3\}$ is a strong multiple factorisation of T. Therefore using [1, Table V] we obtain that T, K_1, K_2, K_3, and the degree $|\Omega| = |T : K_1 \cap K_2 \cap K_3|$ of G are as in Table 4. □

The proof of Theorem 1.1 is now easy, as Theorem 6.1 implies that $C_{\mathrm{Sym}\,\Omega}(T) = 1$, and so $N_{\mathrm{Sym}\,\Omega}(T)$ is an almost simple group with socle T. For the proof of Theorem 1.2, notice that W is the full stabiliser of a Cartesian decomposition $\mathcal{E}$ of Ω. As $G \leqslant W$, the Cartesian decomposition $\mathcal{E}$ is also G-invariant. Hence Theorem 6.1 implies the required result.

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