Bull. Aust. Math. Soc. **100** (2019), 297–303
doi:10.1017/S0004972719000248

EXTENDABLE TEMPERATURES

NEIL A. WATSON

(Received 8 November 2018; accepted 15 January 2019; first published online 27 February 2019)

Abstract

Let E and D be open subsets of $\mathbb{R}^{n+1}$ such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. We call a temperature u on D *extendable by* v if there is a supertemperature w on E such that $w = u$ on D and $w = v$ on $E\backslash\overline{D}$. Such a temperature need not be a thermic minorant of v on D. We show that either there is a unique temperature extendable by v, or there are infinitely many. Examples of temperatures extendable by v include the greatest thermic minorant GM_v^D of v on D, and the Perron–Wiener–Brelot solution of the Dirichlet problem S_v^D on D with boundary values the restriction of v to ∂D. In the case where these two examples are distinct, we give a formula for producing infinitely many more. Clearly GM_v^D is the greatest extendable thermic minorant, but we also prove that there is a least one, which is not necessarily equal to S_v^D.

2010 *Mathematics subject classification*: primary 35K05; secondary 31B05, 31B10, 31B20.

Keywords and phrases: temperature, supertemperature, thermic minorant.

1. Introduction

Given an open set E in $\mathbb{R}^{n+1}$, a function $u \in C^{2,1}(E)$ that satisfies the standard heat equation on E is called a *temperature*. If

$$W(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t) & \text{if } t > 0,\\ 0 & \text{if } t \le 0,\end{cases}$$

then W is a temperature on $\mathbb{R}^{n+1}\backslash\{0\}$. For any point $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and any positive number c, the set

$$\Omega(p_0; c) = \Omega(x_0, t_0; c) = \{(y, s) \in \mathbb{R}^{n+1} : W(x_0 - y, t_0 - s) > (4\pi c)^{-n/2}\}$$

is called the *heat ball* with *centre* (x_0, t_0) and *radius* c. The heat ball is of increasing importance and can now be found in several books, including [1, 3, 4, 6]. Temperatures can be characterised in terms of mean values over heat balls, in that a function $u \in C^{2,1}(E)$ is a temperature if and only if

$$u(x_0, t_0) = (4\pi c)^{-n/2} \iint_{\Omega(x_0,t_0;c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x, t)\, dx\, dt$$

whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$.

An extended real-valued function v on E is called a *supertemperature* on E if it satisfies the following four conditions:

(δ_1) $-\infty < v(p) \leq +\infty$ for all $p \in E$;
(δ_2) v is lower semicontinuous on E;
(δ_3) w is finite on a dense subset of E;
(δ_4) given any point $(x_0, t_0) \in E$ and positive number ϵ, there is a positive number $c < \epsilon$ such that the closed heat ball $\overline{\Omega}(x_0, t_0; c) \subseteq E$ and the inequality

$$v(x_0, t_0) \geq (4\pi c)^{-n/2} \iint_{\Omega(x_0,t_0;c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x, t)\, dx\, dt$$

holds.

If v is a supertemperature on E, D is an open subset of E, and u is a temperature such that $u \leq v$ on D, then u is called a *thermic minorant* of v on D.

Given an open set D such that $\overline{D}$ is a compact subset of E, and a supertemperature v on E, it is well known that the restriction of v to D can be replaced by a temperature in such a way that the resultant function, perhaps with some modification on ∂D, is a supertemperature on E. We shall study this phenomenon, using the following terminology.

Definition 1.1. Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. If u is a temperature on D such that the function w defined by

$$w = \begin{cases} u & \text{on } D, \\ v & \text{on } E \backslash \overline{D}, \end{cases}$$

can be extended to a supertemperature on E, we say that u is *extendable by v (to E)*.

Our starting point is a pair of known results. The first is the following corollary of [7, Theorem 2.5]. In this, a function f on ∂D is called *resolutive* if it has a Perron–Wiener–Brelot solution to the generalised Dirichlet problem, in the sense of [6]. That solution is denoted by S_f^D. Throughout this paper, our notation and terminology follow [6].

Lemma 1.2. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. Then the restriction of v to ∂D is resolutive for D, and the function w defined by*

$$w = \begin{cases} S_v^D & \text{on } D, \\ v & \text{on } E \backslash \overline{D}, \end{cases}$$

can be extended to a supertemperature $\overline{w}$ majorised by v on E.

The second result is not quite [7, Corollary 3.3] but can be proved by an almost identical method.

https://doi.org/10.1017/S0004972719000248 Published online by Cambridge University Press

LEMMA 1.3. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, let v be a supertemperature on E, and let GM_v^D be the greatest thermic minorant of v on D. Then the function w defined by*

$$w = \begin{cases} GM_v^D & \text{on } D, \\ v & \text{on } E\backslash\overline{D}, \end{cases}$$

can be extended to a supertemperature $\overline{w}$ majorised by v on E.

Thus each of S_v^D and GM_v^D is extendable by v to E.

We note that the temperature u, in the definition of 'extendable', is not necessarily a thermic minorant of v on D. For if $S_v^D \neq GM_v^D$ and we use the function $\overline{w}$ of Lemma 1.2 in place of the original supertemperature v, then GM_v^D is extendable by $\overline{w}$, by Lemma 1.3, but is not a thermic minorant of $\overline{w}$ on D.

It is known that, for some open subsets D such as heat balls, there is only one temperature u on D that is extendable by a given supertemperature v. See [5, Theorem 6] for details. On the other hand, if there are two distinct temperatures u_1 and u_2 on D that are extendable by v, then there are infinitely many. This is because, whenever $0 < \alpha < 1$, the temperature αu_1 is extendable by αv, and $(1-\alpha)u_2$ is extendable by $(1-\alpha)v$, so that $\alpha u_1 + (1-\alpha)u_2$ is extendable by $\alpha v + (1-\alpha)v = v$.

Even if $S_v^D = GM_v^D$, there may still be infinitely many temperatures on D that are extendable by v, as the following example shows.

EXAMPLE 1.4. We take $E = \mathbb{R}^{n+1}$ and v the characteristic function of $\mathbb{R}^n \times]0, \infty[$. Given any ball B in $\mathbb{R}^n$, we take $D = B \times (]-1, 1[\cup]1, 2[)$. Since the set $B \times \{-1, 1\}$ is a null set for the Riesz measure associated with v, [8, Theorem 12] implies that $S_v^D = GM_v^D$. By Lemma 1.2, the function w_1 defined by

$$w_1 = \begin{cases} S_v^D & \text{on } D, \\ v & \text{on } E\backslash\overline{D}, \end{cases}$$

can be extended to a supertemperature on E. We now define $D_+ = B \times]1, 2[$ and $D_- = B \times]-1, 1[$. Then $S_v^D = S_v^{D_-}$ on D_- and $S_v^D = S_v^{D_+}$ on D_+. If $C = B \times]-1, 2[$, then $\overline{C} = \overline{D}$ and the function

$$w_2 = \begin{cases} S_v^C & \text{on } C, \\ v & \text{on } E\backslash\overline{C} = E\backslash\overline{D}, \end{cases}$$

can be extended to a supertemperature on E. Furthermore $S_v^C = S_v^{D_-}$ on D_-, but $S_v^C < S_v^{D_+}$ on D_+ because $S_v^C < 1 = v$ on $B \times \{1\}$. Therefore $w_2 < w_1$ on D_+. Thus S_v^D and S_v^C are both temperatures on D that are extendable by v, but they are not equal. Hence there are infinitely many such temperatures.

https://doi.org/10.1017/S0004972719000248 Published online by Cambridge University Press

2. The least extendable temperature

Example 1.4 also shows that S_v^D may not be the least temperature on D that is extendable by v. However this is because of the choice of v and a different choice can satisfy this equation. There is a smallest supertemperature v_0 on E such that $v_0 = v$ on $E \backslash \overline{D}$, and $S_{v_0}^D = v_0$ on D, as we now show.

THEOREM 2.1. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let v be a supertemperature on E. Then the class $\mathcal{F}$ of all supertemperatures w on E such that $w = v$ on $E \backslash \overline{D}$ has a minimal element v_0 for which $S_{v_0}^D = v_0$ on D.*

PROOF. If B is an open superset of $\overline{D}$ such that $\overline{B}$ is a compact subset of E, then v has a lower bound K on ∂B. Now the minimum principle shows that the restrictions to B of all members of $\mathcal{F}$ are lower bounded by K. Thus the function $u = \inf \mathcal{F}$ is locally lower bounded on E, so that its lower semicontinuous smoothing $\widehat{u}$ is a supertemperature on E which equals u wherever the latter is lower semicontinuous, by [6, Theorem 7.13]. Therefore $\widehat{u} = v$ on $E \backslash \overline{D}$, so that $\widehat{u} \in \mathcal{F}$ and $\mathcal{F}$ has a minimal element. Applying Lemma 1.2, we see that the function w_0, defined by

$$w_0 = \begin{cases} S_{\widehat{u}}^D & \text{on } D, \\ \widehat{u} = v & \text{on } E \backslash \overline{D}, \end{cases}$$

can be extended to a supertemperature majorised by $\widehat{u}$ on E. That extension belongs to $\mathcal{F}$, so that it also majorises $\widehat{u}$ and hence the two are equal. □

3. A formula for extendable temperatures

For the case where S_v^D and GM_v^D are distinct, we can give a formula for an infinity of temperatures on D that are extendable by v. The key to this is [8, Theorem 7(b)], which we state below as Lemma 3.1.

We use the following notations. If ν is a nonnegative Borel measure on an open set E and A is a ν-measurable set, we denote the restriction of ν to A by ν_A. If D is an open subset of E, we denote the extended Green function of D by $G_D^{\equiv}$ (see [2, 8] for details), and define the function $G_D^{\equiv}\nu$ by

$$G_D^{\equiv}\nu(p) = \int_E G_D^{\equiv}(p; q)\, d\nu(q)$$

for all $p \in D$.

We require a classification of the boundary points of E in which we use the following notations for upper and lower half-balls. Given any point $p_0 = (x_0, t_0)$ in $\mathbb{R}^{n+1}$ and $r > 0$, we put $H(p_0, r) = \{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2,\ t < t_0\}$ and $H^*(p_0, r) = \{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2,\ t > t_0\}$. Let q be a boundary point of the bounded open set D. In our classification of boundary points, we always suppose that the boundary of D does not contain any polar set whose union with D would give another open set. We call q a *normal* boundary point if every lower half-ball centred at

q meets the complement of D. If this condition fails, and also for every $r > 0$ we have $H^*(q, r) \cap D \neq \emptyset$, then q is called a *semisingular* boundary point. The set of all normal boundary points of D is denoted by $\partial_n D$ and that of all semisingular points by $\partial_{ss} D$. A similar classification is made relative to the adjoint equation, by interchanging H and H^* throughout. This leads, in particular, to the idea of a point $q \in \partial D$ being a *cothermal* normal boundary point if every upper half-ball centred at q meets the complement of D; the set of all such points is denoted by $\partial_n^* D$. A point $q \in \partial_n E$ is called *regular* if $\lim_{p \to q} S_f^E(p) = f(q)$ for all $f \in C(\partial E)$; a point $q \in \partial_n^* E$ is called *coregular* if this holds relative to the adjoint equation. A point $q = (y, s) \in \partial_{ss} E$ is called *regular* if $\lim_{(x,t)\to(y,s+)} S_f^E(x, t) = f(y, s)$ for all $f \in C(\partial E)$.

Lemma 3.1. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let Y be the complement in ∂D of the set of coregular points of $\partial_n^* D$. Let v be a supertemperature on E, and let ν be its associated Riesz measure. Then we have $GM_v^D = S_v^D + G_{\overline{D}} \nu_Y$ on D.*

We can now give our formula for temperatures on D that are extendable by v, which gives an infinity of such functions in the case where $S_v^D \neq GM_v^D$.

Theorem 3.2. *Let E and D be open sets such that $\overline{D}$ is a compact subset of E, and let Y be the complement in ∂D of the set of coregular points of $\partial_n^* D$. Let v be a supertemperature on E and let ν be its associated Riesz measure. If μ is a measure on E such that $0 \leq \mu \leq \nu_Y$, then the function w defined on $E \backslash \partial D$ by*

$$w = \begin{cases} S_v^D + G_{\overline{D}} \mu & \text{on } D, \\ v & \text{on } E \backslash \overline{D}, \end{cases}$$

is a temperature on D and can be extended to a supertemperature majorised by v on E. Moreover,

$$\lim_{p \to q,\, p \in D} G_{\overline{D}} \mu(p) = 0$$

whenever q is a regular point of $\partial_n D \backslash \overline{Y}$, and

$$\lim_{p \to q+,\, p \in D} G_{\overline{D}} \mu(p) = 0$$

whenever q is a regular point of $\partial_{ss} D \backslash \overline{Y}$.

Proof. To prove the extendability, we first suppose that $v \geq 0$, so that we can write $v = G_E \nu + GM_v^E$ on E. Given a measure μ on E such that $0 \leq \mu \leq \nu_Y$, we put $v_1 = G_E(\nu - \mu)$ and $v_2 = G_E \mu + GM_v^E$ on E. Then v_1 and v_2 are supertemperatures on E and $v = v_1 + v_2$. By [8, Lemma 6], $G_{\overline{D}} \mu$ is a temperature on D because μ is supported in ∂D. By Lemma 1.2, the function w_1 defined by

$$w_1 = \begin{cases} S_{v_1}^D & \text{on } D, \\ v_1 & \text{on } E \backslash \overline{D}, \end{cases}$$

https://doi.org/10.1017/S0004972719000248 Published online by Cambridge University Press

can be extended to a supertemperature majorised by v_1 on E. Moreover, by Lemma 1.3, the function w_2 defined by

$$w_2 = \begin{cases} GM_{\nu_2}^D & \text{on } D, \\ v_2 & \text{on } E\backslash\overline{D}, \end{cases}$$

can be extended to a supertemperature majorised by v_2 on E. Furthermore, Lemma 3.1 shows that $GM_{\nu_2}^D = S_{\nu_2}^D + G_D^{=}\mu$ on D, so that $w_1 + w_2 = S_\nu^D + G_D^{=}\mu = w$ on D, and hence $w_1 + w_2 = w$ on $E\backslash\partial D$. It follows that w can be extended to a supertemperature majorised by $v_1 + v_2 = v$ on E.

In the general case, we choose an open superset C of $\overline{D}$ such that $\overline{C}$ is a compact subset of E. Then v is lower bounded on C, so that we can apply the case just proved to $v - m$ on C, where $m = \inf_C v$, noting that ν_C is the Riesz measure associated with $v - m$ on C. Thus, if μ is a measure on E such that $0 < \mu \leq \nu_Y$, then the function w_3 defined by

$$w_3 = \begin{cases} S_{v-m}^D + G_D^{=}\mu_C & \text{on } D, \\ v - m & \text{on } C\backslash\overline{D}, \end{cases}$$

is a temperature on D and can be extended to a supertemperature majorised by $v - m$ on C. The addition of m to w_3 gives the restriction of w to $C\backslash\partial D$ and the extendability follows.

To prove the last part, we first put $v_\mu = G_E\mu$. Then, by [8, Theorem 3],

$$\begin{aligned} G_D^{=}\mu(p) &= \int_E \left(G_E(p;q) - \int_{\partial D} G_E(\cdot;q)\, d\omega_p^D \right) d\mu(q) \\ &= G_E\mu(p) - \int_{\partial D} G_E\mu\, d\omega_p^D \\ &= v_\mu(p) - S_{v_\mu}^D(p) \end{aligned}$$

for all $p \in D$. The function v_μ is a temperature on $E\backslash\overline{Y}$, so that it is continuous on $\partial D\backslash\overline{Y}$. Therefore

$$\lim_{p\to q,\, p\in D} S_{v_\mu}^D(p) = v_\mu(q)$$

whenever q is a regular point of $\partial_n D\backslash\overline{Y}$, and

$$\lim_{p\to q+,\, p\in D} S_{v_\mu}^D(p) = v_\mu(q)$$

whenever q is a regular point of $\partial_{ss} D\backslash\overline{Y}$. The result follows. □

References

[1] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knoph, P. Lu, F. Luo and L. Ni, *The Ricci Flow: Techniques and Applications: Part III: Geometric-Analytic Aspects*, Mathematical Surveys and Monographs, 163 (American Mathematical Society, Providence, RI, 2010).

https://doi.org/10.1017/S0004972719000248 Published online by Cambridge University Press

[2] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Grundlehren der mathematischen Wissenschaften, 262 (Springer, New York, 1984).
[3] K. Ecker, *Regularity Theory for Mean Curvature Flow*, Progress in Nonlinear Differential Equations and their Applications, 57 (Birkhäuser, Basel, 2004).
[4] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 19 (American Mathematical Society, Providence, RI, 1998).
[5] N. A. Watson, 'Mean values of subtemperatures over level surfaces of Green functions', *Ark. Mat.* **30** (1992), 165–185.
[6] N. A. Watson, *Introduction to Heat Potential Theory*, Mathematical Surveys and Monographs, 182 (American Mathematical Society, Providence, RI, 2012).
[7] N. A. Watson, 'Thermic minorants and reductions of supertemperatures', *J. Aust. Math. Soc.* **99** (2015), 128–144.
[8] N. A. Watson, 'Extensions of Green functions and the representation of greatest thermic minorants', *New Zealand J. Math.* **47** (2017), 97–110.

NEIL A. WATSON, School of Mathematics and Statistics,
University of Canterbury, Private Bag, Christchurch,
New Zealand
e-mail: n.watson@math.canterbury.ac.nz