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CHAIN RECURRENT POINTS OF A TREE MAP

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We generalise a result of Hosaka and Kato by proving that if the set of periodic points of a continuous map of a tree is closed then each chain recurrent point is a periodic one. We also show that the topological entropy of a tree map is zero if and only if the ω−limit set of each chain recurrent point (which is not periodic) contains no periodic points.

1. INTRODUCTION

By a *tree* we mean a connected compact one-dimensional branched manifold containing no circle. The dynamics of a *tree map*, that is, a continuous map from a tree into itself, have been studied intensively in the recent years (see the references). In this paper we shall study the set of chain recurrent points of a tree map. It is known that if f is an interval map and the set of periodic points of f is closed, then each chain recurrent point is a periodic one [**6**, **13**]. Recently, Hosako and Kato [**5**] showed that if the set of non-wandering points of a continuous map of a tree is finite, then each non-wandering point is a periodic point. We shall generalise the result of [**5**] and prove some other results. To be more precise, we need some notation.

A *subtree* of T is a subset of T, which is itself a tree. For $x \in T$ the number of connected components of $T \setminus \{x\}$ is called the *valence* of x in T. A point of T of valence 1 is called an *end* of T, and a point of valence different from 2 is called a *vertex* of T. Let $V(T)$ be the set of vertices of T. The closure of each connected component of $T \setminus V(T)$ is called an *edge* of T. The set of ends of T and the number of ends of T will be denoted by $E(T)$ and $\mathrm{End}\,(T)$ respectively. Let $n \geqslant 2$. A tree is said to be an *n-star* if T has a point b of valence n and the closure of each connected component of $T \setminus b$ is an interval. Let $A \subset T$. We shall use $[A]$ to denote the smallest closed connected subset containing A. If $A = \{a, b\}$ then we use $[a, b]$ to denote $[A]$. We define $(a, b) = [a, b] \setminus \{a, b\}$ and we similarly define $(a, b]$ and $[a, b)$. For a subset A of T, we use $int(A)$, $\overline{A}$ and $b(A)$ to denote the interior, the closure and the boundary of A respectively.

Let f be a tree map. The set of periodic points of f, the set of almost periodic points of f, the set of recurrent points of f, the ω−limit set of x, the set of

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non-wandering points of f and the set of chain recurrent points of f will be denoted by $P(f)$, $AP(f)$, $R(f)$, $\omega(x,f)$, $\Omega(f)$ and $CR(f)$ respectively (see [3] for the definitions). It is known that $P(f) \subset AP(f) \subset R(f) \subset \bigcup_{x\in T} \omega(x,f) \subset \Omega(f) \subset CR(f)$. For the notion of *no division* for a periodic orbit of a tree map, the notion of *topological entropy* of f (denoted by $h(f)$) and the notion of *minimal set* see [**1, 11, 9, 12**]. It is known that if A is a minimal set of f, then $A \subset AP(f)$. For $x \in T$, let $\alpha(x,f) = \{y \in T :$ there are $n_i \longrightarrow \infty,\ y_i \in f^{-n_i}(x),$ with $y_i \longrightarrow y\}$.

Now we are in the position to state the main results of the paper.

THEOREM A. *Let T be a tree and $f : T \longrightarrow T$ be continuous. Then $CR(f) = P(f)$ if and only if $\overline{P(f)} = P(f)$.*

THEOREM B. *Let $f : T \longrightarrow T$ be a continuous map from a tree T into itself. Then f has zero topological entropy if and only if for each $x \in CR(f)\backslash P(f)$, $\omega(x,f)\cap P(f) = \emptyset$.*

REMARK. As there is a continuous map f from a graph G into itself such that $\Omega(f) = \{e,t\}$ with $f(e) = e$ and $t \notin P(f)$ [**2**], the conclusions of Theorem A,B do not hold for graph maps.

2. Proofs of the Main results

In this section we shall give the proofs of Theorem A and B. To do this we need the following known results.

LEMMA 2.1. [**10, 11, 1**] *Let T be a tree and $f : T \longrightarrow T$ continuous. Then*

1. $\overline{P(f)} = \overline{R(f)}$.
2. *f has a non-divisible periodic orbit if and only if there are some $x \in T$ and some $n \in \mathbb{N}$ with $(n,m) = 1$ for each $2 \leqslant m \leqslant \mathrm{End}\,(T)$ such that $x \in \big(f^n(x), f(x)\big)$.*
3. *$h(f) > 0$ if and only there is $n \in \mathbb{N}$ such that f^n has a non-divisible periodic orbit.*

LEMMA 2.2. [8] *Let $f : T \longrightarrow T$ be a continuous map of a tree T. Then $h(f) > 0$ if and only if there are some $n \in \mathbb{N}$ and two disjoint closed intervals J_1, J_2 contained in some edge of T such that $f^n(J_i) \supset J_1 \cup J_2$ for $i = 1, 2$.*

The following two lemmas will be used in the proof of Lemma 2.5.

LEMMA 2.3. *Let T be a compact metric space with metric d and $f : T \longrightarrow T$ be continuous. If A is an open subset of T such that $f\big(\overline{A}\big) \subset A$, then for each $x \in CR(f)\setminus A$ and each $n \in \mathbb{N}$ we have $f^n(x) \notin A$.*

PROOF: As $CR(f^n) = CR(f)$ for each $n \in \mathbb{N}$ we only need to show that $f(x) \notin A$ for each $x \in CR(f)\setminus A$. Assume the contrary. That is, there is $x \in CR(f)\setminus A$ such that $f(x) \in A$. Let

$$\varepsilon = \inf\Big\{d(y,z) : y \in T\setminus A,\ z \in f\big(\overline{A}\big)\Big\}.$$

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By our assumption, $\varepsilon > 0$. As $x \in CR(f)$ there are $x_0, x_1, \dots, x_n$ such that $x_0 = x_n = x$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for each $i = 0, \dots, n-1$. As $f(x_0) = f(x) \in A$ we have $x_1 \in A$ and inductively we have $x_i \in A$ for $1 \leqslant i \leqslant n$. That is, $x \in A$, a contradiction. ∎

LEMMA 2.4. *Let $n \in \mathbb{N}$ and $n \geqslant 2$. Then there is $l(n) \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there is $m' \in \{m+1, \dots, m+l(n)\}$ such that $(m', i) = 1$ for each $2 \leqslant i \leqslant n$.*

PROOF: The lemma can be checked by taking $l(n) = n!$. ∎

The key lemma for the proofs of the main theorems will be the following. Note that we use $F(f)$ to denote the set of fixed points of f.

LEMMA 2.5. *Let T be a tree and $f : T \longrightarrow T$ be continuous. If there is an $x \in CR(f) \setminus P(f)$ such that $\omega(x, f) \cap F(f) \neq \emptyset$, then $h(f) > 0$.*

PROOF: Let $x \in CR(f) \setminus P(f)$ and $e \in \omega(x, f) \cap F(f)$. ∎

(A) if there are $x_1 \in (x, e)$ and some $n \in \mathbb{N}$ such that $f^n(x_1) = x$ then $h(f) > 0$.

As $f^n[x_1, e] \supset [x, e]$, there is $x_2 \in (x_1, e)$ with $f^n(x_2) = x_1$. Inductively, for each $i \geqslant 3$ there is $x_i \in (x_{i-1}, e)$ with $f^n(x_i) = x_{i-1}$. Since $e \in \omega(x, f)$ we have that $e \in \omega(x, f^n)$. Let S be the component of $T \setminus \{x_{l(n)}\}$ which contains x. Then there is $m \in \mathbb{N}$ such that $f^{mn}(x) \notin S$. We have:

$$x_i \in \left(f^n(x_i), f^{(m+i)n}(x_i)\right), \qquad i = 1, 2, \dots, l(n).$$

By Lemma 2.4 we know that there is $1 \leqslant i_0 \leqslant l(n)$ such that $(m + i_0, j) = 1$ for each $j = 2, \dots, \mathrm{End}\,(T)$. By Lemma 2.1, $h(f^n) > 0$, and hence $h(f) > 0$.

Let T_1 be the component of $T \setminus \{x\}$ containing e and assume that for each $n \in \mathbb{N}$ there is no $y \in (x, e)$ with $f^n(y) = x$.

(B) There is $y \in T_1$ such that $f(y) = x$.

Assume the contrary. That is, there is no $y \in T_1$ such that $f(y) = x$. Then $f\left(\overline{T_1}\right) \subset T_1$, contradicting Lemma 2.3 as $e \in T_1$ and $e \in \omega(x, f)$.

Let $n \in \mathbb{N}$ and $W_n = T_1 \cap \left(\bigcup_{j=1}^{n} f^{-j}(x)\right)$. Let V_n be the component of $T_1 \setminus W_n$ containing e. As for each $n \in \mathbb{N}$ there is no $y \in (x, e)$ with $f^n(y) = x$, we have that $x \in \overline{V_n}$. Moreover, $E\left(\overline{V_n}\right) \subset E(T_1) \cup \bigcup_{j=1}^{n} f^{-j}(x)$.

(C) For each $n \in \mathbb{N}$ there is $y_n \in V_n$ such that $y_n \in f^{-(n+1)}(x)$.

Note that V_n is an open subset of T as $\bigcup_{j=1}^{n} f^{-j}(x)$ is closed. If there is no $y \in V_n$ such that $y \in f^{-(n+1)}(x)$ then we have $f\left(\overline{V_n}\right) \subset V_n$, as $e \in V_n$ and T is uniquely arc-wise connected. This contradicts Lemma 2.3 since $e \in \omega(x, f)$ and $e \in V_n$.

Let $W = T_1 \cap \left(\bigcup_{j=1}^{\infty} f^{-j}(x)\right)$ and V be the component of $T_1 \setminus W$ containing e. It is easy to see that V contains a degenerate interval. Let $P = \overline{V} \cap \alpha(x, f)$. Then $P \subset E\left(\overline{V}\right) \cup V(T_1)$ is finite. We claim:

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(D) P is not empty and $f(P) \subset P$.

If $P = \emptyset$, then $V = V_n$ for some $n \in \mathbb{N}$. By (C), there is $y_n \in V$ such that $y_n \in f^{-n+1}(x)$, a contradiction.

Let $y \in P$. Then $f(y) \in \alpha(x,f)$ and there are $y_{n_i} \in f^{-n_i}(x)$ such that $y_{n_i} \longrightarrow y$ and $y_{n_i} \notin V$. Assume that $f(y) \notin \overline{V}$. Then there is $p \in P$ such that $p \in \big(e, f(y)\big)$. If $p \in \bigcup_{j=1}^{\infty} f^{-j}(x)$, then there is $z \in (y,e)$ such that $f(z) = p$. This implies that $z \in \bigcup_{j=1}^{\infty} f^{-j}(x)$, and hence $y \notin \overline{V}$, a contradiction. Thus we have $p \notin \bigcup_{j=1}^{\infty} f^{-j}(x)$. If $\big(p, f(y)\big) \cap \left(\bigcup_{j=1}^{\infty} f^{-j}(x)\right) \neq \emptyset$, then there is $z \in (y,e)$ such that $f(z) = p$. This implies that $y \notin \overline{V}$, a contradiction. We must have $\big(p, f(y)\big) \cap \left(\bigcup_{j=1}^{\infty} f^{-j}(x)\right) = \emptyset$. Thus $f(y) \in \overline{V}$, a contradiction. Hence $f(y) \in \overline{V} \cap \alpha(x,f)$. That is, P is invariant under f.

As P is a finite invariant subset, there is n such that $f^n(y) = y$ for each $y \in P$. Let $g = f^n$ and $y_1 \in P$. Then there are $y_{n_i} \in f^{-n_i}(x)$ with $y_{n_i} \longrightarrow y_1$. It is easy to see that there is $1 \leqslant i_0 \leqslant n$ such that $f^i(y_1) \in f^{-n_i+i_0}(x)$ and $-n_i + i_0 | n$. Let $y = f^i(y_1)$. Then $y \in \alpha(x,g)$ and $g(y) = y$. We have:

(E) There are $z \in T$, $m \geqslant 2$ with $(m,i) = 1$ for each $2 \leqslant i \leqslant \mathrm{End}\,(T)$, $1 \leqslant t \leqslant \mathrm{Val}(y)$ such that $z \in \big(g^t(z), g^{tm}(z)\big)$.

Let U be a small connected neighbourhood of y such that $\overline{U}$ is homeomorphic to some n−star with $n = Val(y)$ and $x \notin U$. Let $b_1, \dots, b_n$ be the connected components of $U \setminus \{y\}$. As y is a fixed point of g, there is a small connected neighbourhood V of y such that $g^i(V) \subset U$ for $i = 0, 1, \dots, n+1$. Since $y \in \alpha(x,g)$ there is n_i such that $y_{n_i} \in g^{-n_i}(x) \cap V$. Then there is $q \in V$ such that $q, g(q), \dots, g^t(q) \in U$ with $q \in \big(y, g^t(q)\big) \subset b_{n_0}$ and $g^t(q) \in Orb(y_{n_i}, g)$ for some $1 \leqslant t \leqslant n$, $1 \leqslant n_0 \leqslant n$. Then we have $q_{i+1} \in b_{n_0}$ such that $q_{i+1} \in (y, q_i)$ and $g^t(q_{i+1}) = q_i$ for each $i \in \mathbb{N}$. (Set $q = q_1$.) By using the same idea as in the proof of (A) and the fact that $e \in \omega(x,g)$ we get the conclusion of (E).

By Lemma 2.2 we have $h(g) = \big(h(g^t)\big)/t > 0$, and a consequently $h(f) > 0$.

Corollary 2.6. *Let $f : T \longrightarrow T$ be a continuous map from a tree T into itself. If $CR(f) \neq P(f)$, then $AP(f) \neq P(f)$ and consequently, $P(f)$ is not closed.*

Proof: If $h(f) > 0$, then by Lemma 2.2 there are two disjoint closed intervals J_1, J_2 contained in some edge E of T and $n \in \mathbb{N}$ such that $f^n(J_1) \cap f^n(J_2) \supset J_1 \cup J_2$. Hence there are a closed invariant (under f^n) subset X of E and a continuous surjective map $\phi : X \longrightarrow \Sigma_2$ such that $\phi \circ f^n = f^n \circ \sigma$, where (Σ_2, σ) is the one-sided shift with two symbols. Moreover, ϕ is one-to-one except on a countable subset of X (see **[3]**). Then using **[12]** we get a non-trivial minimal set of f^n. That is, $AP(f^n) \neq P(f^n)$ and hence $AP(f) \neq P(f)$. Now we assume that $h(f) = 0$ and $x \in CR(f) \setminus P(f)$. If $\omega(x,f) \cap P(f) \neq \emptyset$, then there is $y \in \omega(x,f) \cap P(f)$ with period n for some $n \in \mathbb{N}$. Thus there is $i \in \mathbb{N}$ such that

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$f^i(y) \in \omega(x, f^n)$ as $\omega(x,f) = \bigcup_{i=1}^{n} \omega\big(f^i(x), f^n\big)$ and $f\Big(\omega\big(f^j(x), f^n\big)\Big) = \omega\big(f^{j+1}(x), f^n\big)$ for each $j \in \mathbb{N}$. Note that $f^i(y)$ is a fixed point of f^n. By Lemma 2.5, we have $h(f^n) > 0$, a contradiction. Hence we have $\omega(x,f) \cap P(f) = \emptyset$. Let A be a minimal set contained in $\omega(x,f)$. Then A is not trivial. That is, $AP(f) \neq P(f)$.

As $P(f) \subset AP(f) \subset \overline{R(f)} = \overline{P(f)}$, we have that $P(f)$ is not closed. ∎

Proof of Theorem A: It is clear $CR(f) = P(f)$ implies $P(f)$ is closed. Now assume that $P(f)$ is closed. By Corollary 2.6, we have $CR(f) = P(f)$. This ends the proof. ∎

To prove Theorem B we need the following lemma.

Lemma 2.7. *Let $f : T \longrightarrow T$ be a continuous map from a tree T into itself. If $h(f) > 0$, then there is $x \in CR(f) \setminus P(f)$ such that $f^n(x) \in P(f)$ for some $n \in \mathbb{N}$.*

Proof: By Lemma 2.2 there are two disjoint closed intervals J_1, J_2 contained in some edge E of T and $n \in \mathbb{N}$ such that $f^n(J_1) \cap f^n(J_2) \supset J_1 \cup J_2$. Let $g = f^n$, $J_i = [a_i, b_i]$ and give an orientation of E such that for each $x_i \in J_i$, $x_1 < x_2$. Then there are a fixed point $e \in J_1$ of g and $z \in J_1$ such that $g(z) = b_2$. Without loss of generality we assume that $e < z$ and $(e,z) \cap F(g) = \emptyset$. Take a point $z < w \in E$ such that $g(w) = e$. Then $w \notin P(f)$ and $w \in \Omega(g) \subset CR(f)$. Hence $w \in CR(f) \setminus P(f)$ and $f^n(w) \in P(f)$. ∎

Proof of Theorem B: Assume that $h(f) = 0$ and there is $x \in CR(f) \setminus P(f)$ with $\omega(x,f) \cap P(f) \neq \emptyset$. Let $y \in \omega(x,f) \cap P(f)$ and let the period of y be n. As $\omega(x,f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$, there is i such that $y \in \omega\big(f^i(x), f^n\big)$. Hence $f^{n-i}(y) \in \omega(x, f^n)$. Note that $f^{n-i}(y)$ is a fixed point of f^n and $x \in CR(f) \setminus P(f) = CR(f^n) \setminus P(f^n)$, a contradiction. ∎

The sufficiency of the theorem follows from Lemma 2.7.

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