Compositio Math. 141 (2005) 1103–1108
doi:10.1112/S0010437X05000291

On a result of Moonen on the moduli space of principally polarised abelian varieties

Andrei Yafaev

Abstract

This note extends a result of Moonen on the Zariski closure of some sets of special points in the moduli space of principally polarised abelian varieties of dimension g to the case of any Shimura variety, by a completely different method.

1. Introduction

In [Moo98], Moonen proves the following result.

Theorem 1.1 (Moonen). *Let $g \geqslant 1$ be an integer and let $\mathcal{A}_g$ be the moduli space of principally polarised abelian varieties of dimension g. Let Σ be a set of special points in $\mathcal{A}_g(\overline{\mathbb{Q}})$ such that there exists a prime p with the property that every s in Σ has good ordinary reduction at a place lying over p of which s is the canonical lift. Then all irreducible components of the Zariski closure of Σ are of Hodge type.*

This theorem is a special case of the André–Oort conjecture on subvarieties of Shimura varieties. For relevant terminology, statement and history, we refer to the introduction of [Edi01] and references contained therein. Among recent results on this conjecture, one can mention the articles [EY03] and [Yaf02].

Moonen's proof relies on the Serre–Tate theorem that states that the formal completion of $\mathcal{A}_g$ at an ordinary point is a formal torus. Such a result is presently unavailable in the case of an arbitrary Shimura variety, therefore Moonen's methods can not be used in the general situation of the André–Oort conjecture.

Our aim in this paper is to find an analog of Moonen's condition in the case of an arbitrary Shimura variety and to prove the corresponding statement using a method which involves only the geometry of Shimura varieties in characteristic zero. The statement of our main theorem is the following.

Theorem 1.2. *Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Let Σ be a set of special elements of X. Let V be a faithful representation of G. Via this representation, we view G as a closed subgroup of $\mathrm{GL}_{n\mathbb{Q}}$. For each s in Σ we let $\mathrm{MT}(s)$ be its Mumford–Tate group. We suppose that there is a prime p such that, for any s in Σ, the torus $\mathrm{MT}(s)_{\mathbb{Q}_p}$ has a subtorus M_s isomorphic to $\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}$ such that the weights in the representation $V_{\mathbb{Q}_p}$ of M_s are bounded uniformly in s and, for every nontrivial quotient T of $\mathrm{MT}(s)$, the image of M_s in $T_{\mathbb{Q}_p}$ is nontrivial. Furthermore, we assume that the Zariski closure in $\mathrm{GL}_{n\mathbb{Z}_p}$ of each M_s is isomorphic to $\mathbb{G}_{\mathrm{m}\mathbb{Z}_p}$.*

Then for any g in $G(\mathbb{A}_{\mathrm{f}})$ the irreducible components of the Zariski closure of the image of $\Sigma \times \{g\}$ in $\mathrm{Sh}_K(G, X)$ are of Hodge type.

Received 12 February 2003, accepted 12 February 2004, revised 1 March 2005, published online 1 September 2005.
2000 Mathematics Subject Classification 11G18, 14G35.
Keywords: Shimura variety, subvariety of Hodge type, André–Oort conjecture.

In the first section we prove that our theorem does indeed imply Moonen's result. More precisely, we will prove that the Mumford–Tate group of a CM abelian variety A, canonical at some place of characteristic p, satisfies the conditions of our statement.

The method used to prove Theorem 1.2 is that used in [EY03]. The possibility of doing this was suggested in the introduction to [EY03]. We let Z be an irreducible component of the Zariski closure of the image of $\Sigma \times \{g\}$. Suppose that G is semisimple of adjoint type and that Z is Hodge generic. Using the Galois action on special points, we prove that Z is contained in its image by a Hecke correspondence T_m with some element m in $G(\mathbb{Q}_p)$ satisfying a number of conditions.

In [EY03] we have characterised subvarieties of Hodge type by their property of being contained in their image by a suitable Hecke correspondence. Unfortunately, we can not apply this characterisation directly to the present case because in this characterisation the prime p was subject to a number of conditions. However, in the third section of this paper we refine this characterisation, making it applicable in our case.

2. Ordinary reduction of abelian varieties of CM type

We show that Theorem 1.2 implies Theorem 1.1.

Let A be an abelian variety over $\overline{\mathbb{Q}}$, of CM-type. Let p be a prime number. We choose embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. We call v the place of $\overline{\mathbb{Q}}$ above p thus obtained. We put $V := \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$. The Hodge structure on V is given by a morphism $h\colon \mathbb{S} \to \mathbf{GL}(V_{\mathbb{R}})$. We let $T \subset \mathbf{GL}(V_{\mathbb{Q}})$ be the Mumford–Tate group of h. We view $h_{\mathbb{C}}$ as a morphism from $\mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}}$ to $T_{\mathbb{C}}$, and let μ be the restriction of $h_{\mathbb{C}}$ to the first of the two factors. Then μ has weights 1 and 0 on $V_{\mathbb{C}}$, each occurring with multiplicity $g := \dim(A)$. Let $E \subset \mathbb{C}$ be the reflex field, i.e. the field of definition of μ. Then $E \subset \overline{\mathbb{Q}}$ because T is a torus.

Proposition 2.1. *The abelian variety A is ordinary at v if and only if E is split at v, i.e. if and only if E is contained in $\mathbb{Q}_p$. Assume now that A is ordinary at v. Then A is canonical at v if and only if the morphism $\mu\colon \mathbb{G}_{\mathrm{m}\mathbb{Q}_p} \to \mathbf{GL}(V_{\mathbb{Q}_p})$ extends to a morphism $\mu\colon \mathbb{G}_{\mathrm{m}\mathbb{Z}_p} \to \mathbf{GL}(V_{\mathbb{Z}_p})$. For any nontrivial quotient T' of T the image of $\mu(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p})$ in $T'_{\mathbb{Q}_p}$ is nontrivial.*

Proof. The chosen embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ give an isomorphism of $\mathbb{Z}_p$-modules between $V_{\mathbb{Z}_p}$ and the Tate module $\mathrm{T}_p(A_{\overline{\mathbb{Q}}_p})$. We choose a finite extension F of E such that A has a model over F with good reduction over the ring of integers O_{F_v} of F_v. Then A is ordinary at v if and only if $V_{\mathbb{Q}_p}$ splits as a direct sum into two subspaces of dimension g on which the inertia subgroup I_{F_v} of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v)$ acts through the characters χ^0 and χ^1, with χ the p-adic cyclotomic character.

The theory of complex multiplication (see [ST68, Theorem 11 and Corollary 2] or [Ser68, II.2.8]) implies that the action of I_v is given, via local class field theory, by the map

$$O_{F_v}^* \xrightarrow{\mathrm{N}_{F_v/E_v}} E_v^* \xrightarrow{\mu} T(E_v) \xrightarrow{\mathrm{N}_{E_v/\mathbb{Q}_p}} T(\mathbb{Q}_p).$$

If E is split at v, i.e. if $E_v = \mathbb{Q}_p$, then indeed $V_{\mathbb{Q}_p}$ splits as desired for the action of I_{F_v}, because it does so for the action via μ, and hence A is ordinary.

Now suppose that E_v is of degree at least two over $\mathbb{Q}_p$. Then we claim that the Zariski closure of the image of $O_{F_v}^*$ in $T(\mathbb{Q}_p)$ has dimension at least two: this results from the fact that any two distinct embeddings τ and τ' of E into $\overline{\mathbb{Q}}$ induce distinct conjugates μ_τ and $\mu_{\tau'}$ in the group of cocharacters of T. It follows that A is not ordinary at v.

Now we assume that A is ordinary at v. By definition, A is canonical at v if and only if $V_{\mathbb{Z}_p}$ splits as a direct sum into two submodules of rank g, on which I_{F_v} acts through the characters χ^0 and χ^1. This splitting condition is equivalent to μ extending over $\mathbb{Z}_p$.

https://doi.org/10.1112/S0010437X05000291 Published online by Cambridge University Press

The last statement of Proposition 2.1 follows from the fact that T is the smallest algebraic subgroup of $\mathbf{GL}(V_{\mathbb{Q}})$ such that $T_{\mathbb{Q}_p}$ contains the image of $\mu(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p})$. $\square$

3. Irreducibility and density

In this section we adapt some results of [EY03, §§ 5 and 6] in order to apply them to our situation.

Theorem 3.1. *Let (G, X) be a Shimura datum with G semi-simple of adjoint type. Let K be a neat open compact subgroup of $G(\mathbb{A}_{\mathrm{f}})$ which is the product of compact open subgroups K_l of $G(\mathbb{Q}_l)$, and let X^+ be a connected component of X. Let S be the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$; then $S = \Gamma\backslash X^+$ where $\Gamma := G(\mathbb{Q})^+ \cap K$ (with $G(\mathbb{Q})^+$ being the stabiliser of X^+ in $G(\mathbb{Q})$). Let Z be an irreducible Hodge generic subvariety of S containing a smooth special point.*

Let p be a prime. There is a compact open subgroup K' of K (K' is again a product of compact open subgroups K'_l of $G(\mathbb{Q}_l)$) such that there is a component Z' of the preimage of Z in $S' := \Gamma'\backslash X^+$ (with $\Gamma' := K' \cap G(\mathbb{Q})^+$) with the property that for any q in $(G(\mathbb{Q}_p) \times \prod_{l \neq p} K'_l) \cap G(\mathbb{Q})^+$ (the intersection is being taken in $G(\mathbb{A}_{\mathrm{f}})$) the image $T_q Z'$ of Z' by the Hecke correspondence T_q is irreducible.

Proof. We refer to the beginning of the proof of [EY03, Theorem 5.1] for the definition of the Hecke correspondence T_q. We proceed as in the proof of that theorem.

Let ξ be a faithful representation of G on a $\mathbb{Q}$-vector space V. Then ξ gives a polarisable variation of Hodge structures on the constant sheaf V_X. Since Γ is neat, the sheaf V_{X^+} descends to S. Choose a Γ-invariant lattice $V_{\mathbb{Z}}$ in V. This gives a polarisable variation of $\mathbb{Z}$-Hodge structure $\mathcal{V}_{\mathbb{Z}}(\xi)$ over S and, by restriction, over the smooth locus Z^{sm} of Z. Let s be a Hodge generic point of the smooth locus Z^{sm} and x a point of X^+ lying over s. The choice of x gives an isomorphism of the fibre $\mathcal{V}_{\mathbb{Z}}(\xi)_s$ with $V_{\mathbb{Z}}$, the Mumford–Tate group of the Hodge structure $\mathcal{V}_{\mathbb{Z}}(\xi)_s$ is $\xi(G)$. The image of $\pi_1(Z^{\mathrm{sm}}, s)$ in $\mathrm{GL}(V_{\mathbb{Z}})$ is Zariski dense in $\xi(G)$ by a theorem of André (see [And92, Theorem 1.4]). The same is true for $\xi(\Gamma)$. By Nori's theorem (see [EY03, Theorem 5.2] for the statement we use), the closures Γ and of the image Π of $\pi_1(Z^{\mathrm{sm}}, s)$ in $\mathrm{GL}_n(\hat{\mathbb{Z}})$ are open in the closure of $G(\mathbb{Z})$. It follows that there is an integer M such that, for any integer m prime to M, Π and Γ have the same image in $\mathrm{GL}(V_{\mathbb{Z}}/mV_{\mathbb{Z}})$. If p does not divide M, then we can just take $K' = K$ and apply [EY03, Theorem 5.1]. We now suppose that p divides this integer M. $\square$

Lemma 3.2. *Let K'_p be the closure of Π in $\mathrm{GL}_n(\mathbb{Z}_p)$. The group K'_p is an open compact subgroup of $G(\mathbb{Q}_p)$.*

Proof. Since, by Nori, the closure of Π is open in the closure of $G(\mathbb{Z})$ in $\mathrm{GL}_n(\hat{\mathbb{Z}})$, it suffices to prove that the closure of $G(\mathbb{Z})$ is open in $G(\mathbb{Q}_p)$. Let $\rho\colon \tilde{G} \to G$ be the simply connected covering of G. Let $\tilde{\Gamma}$ be an arithmetic subgroup of $\tilde{G}$ such that $\rho\tilde{\Gamma}$ is contained in $G(\mathbb{Z})$. By strong approximation, the closure of the image of $\tilde{\Gamma}$ in $\tilde{G}(\mathbb{Q}_p)$ is open. The lemma then follows from the fact that the map ρ from $\tilde{G}(\mathbb{Q}_p)$ to $G(\mathbb{Q}_p)$ is open. $\square$

We now return to the proof of the theorem. Let K' be the open compact subgroup of $G(\mathbb{A}_{\mathrm{f}})$, the product of compact open subgroups of $G(\mathbb{Q}_p)$ which is K_l at the place $l \neq p$ and K'_p at the place p. Let Γ' be the intersection of $G(\mathbb{Q})^+$ with K'. Let $S' := \Gamma'\backslash X^+$. Let $\tilde{Z}$ be the preimage of Z^{sm} in S'. The covering $\tilde{Z} \longrightarrow Z^{\mathrm{sm}}$ corresponds to the Π-set Γ/Γ'. Let γ_i be elements of Γ such that

$$\Gamma = \coprod_i \Pi\gamma_i\Gamma'.$$

The set of irreducible components of $\tilde{Z}$ corresponds to the set of γ_i and the fundamental group of the irreducible component Z_i corresponding to γ_i is $\Pi_i := \Gamma' \cap \gamma_i\Pi\gamma_i^{-1}$. We let Z' be the component Z_1

corresponding to $\gamma_i = 1$. The group $\Pi' := \Pi_1$ is $\Pi \cap \Gamma' = \Pi$ and it has the property that for any $k > 0$ the images of Π' and Γ' in $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ coincide.

Now let q be in $(G(\mathbb{Q}_p) \times \prod_{l \neq p} K'_l) \cap G(\mathbb{Q})^+$ and let m_p be an integer sufficiently large such that

$$\ker(\Gamma' \longrightarrow \mathrm{GL}(V_{\mathbb{Z}/p^{m_p}\mathbb{Z}})) \subset q^{-1}\Gamma' q.$$

The Π'-set that corresponds to the covering that gives $T_q Z'$ is $\Gamma'/(\Gamma' \cap q^{-1}\Gamma' q)$. Since the images in $\mathrm{GL}(V_{\mathbb{Z}/p^{m_p}\mathbb{Z}})$ of Γ' and Π' are the same, the group Π' acts transitively on this Π'-set. The irreducibility follows. This concludes the proof of Theorem 3.1. □

Proposition 3.3. *Let K be an open compact subgroup of $G(\mathbb{A}_\mathrm{f})$ which is a product of compact open subgroups K_l of $G(\mathbb{Q}_l)$ and let Γ be $G(\mathbb{Q})^+ \cap K$. Let p be a prime. Let q be an element of $G(\mathbb{Q})^+$ such that the image of q in $G(\mathbb{Q}_p)$ satisfies the condition that, for every simple factor G_j of G, the projection of q to $G_j(\mathbb{Q}_p)$ is not contained in a compact subgroup and all of its images in $G(\mathbb{Q}_l)$ for $l \neq p$ are in K_l. Then all $T_q + T_{q^{-1}}$ orbits are dense in $S := \Gamma\backslash X^+$.*

Proof. We proceed as in the proof of [EY03, Theorem 6.1]. We can (and do) assume that G is simple. It suffices to prove that the group Γ_q generated by Γ and q contains Γ with an infinite index. We can not apply [EY03, Proposition 6.2] directly because there is the condition that p has to be large enough. Suppose that the index is finite. Then some q^n with $n > 0$ belongs to Γ, and hence to K. Therefore the closure in $G(\mathbb{Q}_p)$ of the subgroup generated by q is compact. This contradicts the assumption made on q. □

4. Proof of the main result, Theorem 1.2

Let G, X, K, Σ, V, p and the M_s be as in the statement of Theorem 1.2. For each s in Σ we choose one of the two isomorphisms from $\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}$ to M_s and we call it α_s. Let Z be a component of the Zariski closure of $\Sigma \times \{1\}$. To prove Theorem 1.2, it is enough to prove that Z is of Hodge type. Indeed, Z is of Hodge type if and only if an irreducible component of its image by some Hecke correspondence is of Hodge type. We suppose that the compact open subgroup K is neat and is a product of open compact subgroups K_l of $G(\mathbb{Q}_l)$.

Lemma 4.1. *We can assume that Z is Hodge generic, that G is semisimple of adjoint type and that the tori M_s lie in one K_p-conjugacy class as s ranges through Σ.*

Proof. Using [EY03, Proposition 2.1], we replace G with the generic Mumford–Tate group on Z; hence Z is now Hodge generic.

We now prove that we can replace Σ by a Zariski dense subset such that the tori M_s lie in one K_p-conjugacy class. By [DG70, exposé XI, Corollaire 4.2], the functors $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$ and $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, \mathrm{GL}_{n\mathbb{Q}_p})$ are representable by smooth $\mathbb{Q}_p$-schemes and the natural map from $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$ to $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, \mathrm{GL}_{n\mathbb{Q}_p})$ is a closed immersion. Furthermore, by [DG70, exposé XI, Corollaire 5.1], for any x in $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$, the morphism from $G_{\mathbb{Q}_p}$ to $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$ which sends g to gxg^{-1} is smooth. Hence the K_p-orbits in $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$ are open for the p-adic topology.

By the assumptions in Theorem 1.2, the α_s actually lie in $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Z}_p}, \mathrm{GL}_{n\mathbb{Z}_p})$. As the weights given by the α_s are uniformly bounded, all α_s lie in a compact subset of $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Z}_p}, \mathrm{GL}_{n\mathbb{Z}_p})$, hence of $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}, G_{\mathbb{Q}_p})$, and hence in only finitely many K_p-orbits. For at least one of those orbits, the image in Z of the corresponding set of s is Zariski dense. We replace Σ by that set of s.

It remains to see that we can replace G with the adjoint group G^{ad}. Let K^{ad} be a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}_\mathrm{f})$ which is a product of compact open subgoups K^{ad}_l of $G^{\mathrm{ad}}(\mathbb{Q}_l)$, such that K^{ad}

https://doi.org/10.1112/S0010437X05000291 Published online by Cambridge University Press

contains the image of K; this gives a finite morphism

$$\mathrm{Sh}_K(G,X) \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}).$$

Let Σ^{ad} be the image of Σ by this morphism and Z^{ad} the image of Z. Then Z is of Hodge type if and only if Z^{ad} is of Hodge type and the image of Σ^{ad} is Zariski dense in Z^{ad} (see [EY03, Proposition 2.2]). The tori M_s lie in one K_p-orbit, and hence their images in $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ lie in one K_p^{ad}-orbit. We replace G with G^{ad} and $\mathrm{Sh}_K(G,X)$ with $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$. After possibly having replaced K with a subgroup of finite index, we can assume K to be neat. □

We let X^+ be a connected component of X and we let S be the image of $X^+\times\{1\}$ in $\mathrm{Sh}_K(G,X)$. We can assume (by a suitable choice of X^+) that Z is contained in S. We replace K with the group K' and Z with Z' as in Theorem 3.1. We choose a number field F such that S has a canonical model over F and such that Z is defined over F (as an irreducible closed subscheme).

For s in Σ, we let L_s be the compositum (in $\mathbb{C}$) of the splitting field of the torus $\mathrm{MT}(s)$ and the field F. The orbit of the image of s in Z under $\mathrm{Gal}(\overline{\mathbb{Q}}/L_s)$ then consists of the images in Z of the (s,g) with g in the image of the reciprocity map $r_s\colon (\mathbb{A}_{\mathrm{f}}\otimes L_s)^* \to \mathrm{MT}(s)(\mathbb{A}_{\mathrm{f}})$.

Lemma 4.2. *There is an element m in $G(\mathbb{Q}_p)$ such that the image of m in every $G_j(\mathbb{Q}_p)$ (as usual, G_j stands for the simple factors of G) is not contained in a compact subgroup and such that, for every s in Σ, there is an element k in K such that kmk^{-1} is contained in $M_s(\mathbb{Q}_p)\cap r_s((\mathbb{Q}_p\otimes L_s)^*)$.*

Proof. We proceed as in [EY03, § 7.3]. Since M_s is isomorphic to $\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}$, any element of $M_s(\mathbb{Q}_p)$ not contained in the maximal compact subgroup of $M_s(\mathbb{Q}_p)$ will have an image in each $G_i(\mathbb{Q}_p)$ which is not contained in a compact subgroup. By [Yaf02, Proposition 2.5], the index of $M_s(\mathbb{Q}_p)\cap r_s((\mathbb{Q}_p\otimes L_s)^*)$ in $M_s(\mathbb{Q}_p)$ divides a positive integer e independent of s. Let s_0 be in Σ, and let m be an element of $M_{s_0}(\mathbb{Q}_p)$ which is not in the maximal compact subgroup of $M_{s_0}(\mathbb{Q}_p)$ and which is in the image of the eth power map on $M_{s_0}(\mathbb{Q}_p)$. If s is in Σ and k is an element of K_p such that $M_s = kM_{s_0}k^{-1}$, then kmk^{-1} is in $M_s(\mathbb{Q}_p)\cap r_s((\mathbb{Q}_p\otimes L_s)^*)$. □

Let m be as in Lemma 4.2 and let s be in the image of Σ in Z. There is an element σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $\sigma(s)$ is in $T_m s$. As Σ is Zariski dense in Z and as both Z and $T_m Z$ are defined over F, Z is contained in $T_m Z$.

The connected components of the correspondence that T_m induces on S are the T_{m_i}, induced by m_i in $G(\mathbb{Q})^+$ acting on X^+, such that

$$G(\mathbb{Q})^+\cap KmK = \coprod_i \Gamma m_i^{-1}\Gamma.$$

Let q be one of the m_i such that $Z\subset T_q Z$. Since q^{-1} is in KmK, and m is in $G(\mathbb{Q}_p)$, the image of q in $G(\mathbb{Q}_l)$ is in K_l for all $l\neq p$. By Theorem 3.1, $T_q Z$ is irreducible, and hence $Z = T_q Z$. Now $T_{q^{-1}}Z = T_{q^{-1}}T_q Z \supset Z$. Since, again by Theorem 3.1, $T_{q^{-1}}Z$ is irreducible, we get $Z = T_{q^{-1}}Z = T_q Z$. It follows that Z contains $T_q + T_{q^{-1}}$ orbits which are all dense by Proposition 3.3, and hence $Z = S$.

Acknowledgements

The author would like to extend his deepest thanks to Bas Edixhoven for pointing out numerous mistakes and inaccuracies in the previous versions of this paper, suggesting ways of fixing them and eventually reworking the final version of the paper. Among many other things, Bas Edixhoven has replaced my awkward use of the Shimura–Taniyama theorem in § 2 by elegant conceptual arguments. I am grateful to Rutger Noot for pointing out a gap in a previous version of this paper. I thank Florian Breuer for carefully reading one of the previous versions and pointing out many inaccuracies and typos. The author was supported by an EPSRC grant and the European Research Training Network 'Arithmetic Algebraic Geometry'.

References

And92 Y. André, *Mumford–Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compositio Math. **82** (1992), 1–24.

DG70 M. Demazure and A. Grothendieck, *SGA 3: Schémas en groupes*, Lecture Notes in Mathematics, vols 151–153 (Springer, Berlin–New York, 1970).

EY03 S. J. Edixhoven and A. Yafaev, *Subvarieties of Shimura varieties*, Ann. of Math. (2) **157** (2003), 621–645.

Edi01 S. J. Edixhoven, *On the André–Oort conjecture for Hilbert modular surfaces*, in *Moduli of abelian varieties*, Texel Island, 1999, Progress in Mathematics, vol. 195 (Birkhäuser, Basel, 2001), 133–155.

Moo98 B. Moonen, *Linearity properties of Shimura varieties. II*, Compositio Math. **114** (1998), 3–35.

Ser68 J.-P. Serre, *Abelian l-adic representations and elliptic curves*, McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute (W. A. Benjamin, New York–Amsterdam, 1968).

ST68 J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

Yaf02 A. Yafaev, *A conjecture of Yves André*, Preprint (2002), math.NT/0302125.

Andrei Yafaev yafaev@math.ucl.ac.uk
University College London, Department of Mathematics, 25 Gordon Street, London WC1H 0AH, UK