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Asymptotic analysis of hydrodynamic forces in a Brinkman penalization method: case of an initial flow around an impulsively started rotating and translating circular cylinder

Y. Ueda[1],† and T. Kida[2]

[1]Department of Mechanical Engineering, Faculty of Science and Engineering, Setsunan University, 17–8 Ikeda-Nakamachi, Neyagawa, Osaka 572–8508, Japan

[2]Professor Emeritus, Osaka Metropolitan University, Osaka 599–8531, Japan

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The initial flow past an impulsively started rotating and translating circular cylinder is asymptotically analysed using a Brinkman penalization method on the Navier–Stokes equation. In our previous study (*J. Fluid Mech.*, vol. 929, 2021, A31), the asymptotic solution was obtained within the second approximation with respect to the small parameter, ϵ, that is of the order of $1/\lambda$. Here, λ is the penalization parameter. In addition, the Reynolds number based on the cylinder radius and the translating velocity is assumed to be of the order of ϵ. The previous study asymptotically analysed the initial flow past an impulsively started translating circular cylinder and investigated the influence of the penalization parameter λ on the drag coefficient. It was concluded that the drag coefficient calculated from the integration of the penalization term exhibits a half-value of the results of Bar-Lev & Yang (*J. Fluid Mech.*, vol. 72, 1975, pp. 625–647) as $\lambda \to \infty$. Furthermore, the derivative of vorticity in the normal direction was found to be discontinuous on the cylinder surface, which is caused by the tangential gradient of the pressure on the cylinder surface. The present study hence aims to investigate the variance on the drag coefficient against the result of Bar-Lev & Yang (1975). First, we investigate the problem of an impulsively started rotating circular cylinder. In this situation, the moment coefficient is independent of the pressure on the cylinder surface so that we can elucidate the role of the pressure to the hydrodynamic coefficients. Then, the problem of an impulsively started rotating and translating circular cylinder is investigated. In this situation, the pressure force induced by the unsteady flow far from the cylinder is found to play a key role on the drag force for the agreement with the result of Bar-Lev & Yang (1975), whereas the variance still exists on the lift force. To resolve the variance, an alternative formula to calculate the hydrodynamic force is derived, assuming that there is the pressure jump between the outside and inside of the cylinder surface. The pressure jump is obtained in

† Email address for correspondence: yoshiaki.ueda@mec.setsunan.ac.jp

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this analysis asymptotically. Of particular interest is the fact that this pressure jump can cause the variance on the lift force calculated by the integration of the penalization term.

Key words: vortex dynamics, computational methods

1. Introduction

Computational fluid dynamics for simulating of a flow past complicated time-varying geometries is still a challenging issue and requires advanced numerical techniques. In a computational procedure the no-slip boundary condition is enforced on the surface of a bluff body and in, e.g. the vortex particle method, the boundary element method is employed to satisfy the no-slip condition (see Koumoutsakos, Leonard & Pépin 1994; Koumoutsakos & Leonard 1995). To overcome the difficulty of enforcing the no-slip condition, the Brinkman penalization strategy has been introduced together with grid-based numerical methods such as the volume penalization method (see, e.g. Schneider & Farge 2002; Kadoch *et al.* 2013; Engels *et al.* 2015; Uchiyama *et al.* 2020) and the vortex penalization method (see, e.g. Gazzola *et al.* 2011; Rasmussen, Cottet & Walther 2011; Hejlesen *et al.* 2015). In these studies, the penalization methods are shown to be a fruitful technique for the simulation of a complicated geometry.

As cited in our previous study (see Ueda & Kida 2021), the convergence analysis of the penalization methods for the Navier–Stokes flow is given by Angot, Bruneau & Fabrie (1999), and the error is estimated to be of the order of $1/\lambda^{1/4}$, where λ is the penalization parameter. In their paper, the numerical and theoretical results are given for the two-dimensional flow past a square cylinder in a channel. Also, Angot (2011) proposed a well-posed model for the Stokes flow with jump boundary conditions. Bost, Cottet & Maitre (2010) showed the convergence analysis, extending the analysis of Angot *et al.* (1999). Feireisl, Neustupa & Stebel (2011) showed the convergence for compressible flows. Kadoch *et al.* (2013) analysed a one-dimensional diffusion equation, and the error of the L^2-norm is shown to be $O(1/\sqrt{\lambda})$ in a fluid domain and $O(1/\lambda^{1/4})$ in a solid domain. Furthermore, they verified the computed flows around moving circular cylinders. Carbou & Fabrie (2003) analysed the penalization model of the viscous incompressible Navier–Stokes equations for a small parameter of $\epsilon = 1/\sqrt{\lambda}$ using the Wentzel–Kramers–Brillouin method, and the error estimation in a fluid domain is shown to be of the order of ϵ unlike Angot *et al.* (1999). Carbou (2004) studied the porous thin layer model for the interface of a solid boundary on the penalized Navier–Stokes equations and the error is shown to be of the order of ϵ in a fluid domain. Furthermore, Carbou studied the double penalization model in which one adds the penalization term to the thin porous layer, and showed that the error is ϵ in a fluid domain. Nguyen van yen, Kolomenskiy & Schneider (2014) analysed the Laplace and Stokes operators with Dirichlet boundary conditions of the volume penalization using a spectral approach and carried out the dipole–wall collision numerically (see Nguyen van yen & Farge 2011). In their studies, they derived the Navier boundary condition for the tangential velocity.

In the numerical studies of the volume and vortex penalization methods, the initial flow past an impulsively started circular cylinder is often selected as a benchmark target (see, e.g. Schneider & Farge 2002; Rasmussen *et al.* 2011; Hejlesen *et al.* 2015; Rossinelli *et al.* 2010; Verma *et al.* 2017; Mimeau, Cottet & Mortazavi 2015). This flow is solved analytically by Bar-Lev & Yang (1975) using the method of matched asymptotic expansions based on the conventional Navier–Stokes equations. Also, Collins & Dennis

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(1973) investigated the flow that is valid for time beyond the first separation. In their analysis, the Fourier series, employed with respect to two space variables and time, was truncated to a finite number of terms. The initial stage of a flow caused by an impulsively started rotating and translating circular cylinder was solved by Badr & Dennis (1985) along the lines of the methodology of Collins & Dennis (1973). In unsteady flow past an impulsively started rotating and translating circular cylinder, we also obtained the asymptotic solutions at a low Reynolds number (see Ueda *et al.* 2001; Ueda & Kida 2002*a*) and high Reynolds number but at the early stage of motion (see Ueda & Kida 2002*b*). Chang & Chern (1991) computed the vortex shedding from the cylinder that is impulsively started with translating and rotating velocities, employing the hybrid vortex method. The flow around a rotating cylinder was studied by Mittal & Kumar (2003). Al-Mdallal (2012) and Mittal, Ray & Al-Mdallal (2017) treated the initial stage of a circular cylinder impulsively started with rotational oscillation. Rotational oscillating flow around a circular cylinder was studied by Lu & Sato (1996) numerically. Dennis, Nguyen & Kocabiyik (2000) also studied the flow caused by a rotationally oscillating and translating circular cylinder using the same approach as Collins & Dennis (1973).

The present study investigates the initial flow past an impulsively started rotating and translating circular cylinder from rest, employing the same approach as our previous study (see Ueda & Kida 2021), i.e. the method of matched asymptotic expansions for the Brinkman penalization model of the full Navier–Stokes equations. A particular finding of our previous study is the fact that the drag coefficient obtained from the integration of the penalization term exhibits the half of the results of Bar-Lev & Yang (1975) as the penalization parameter $\lambda \to \infty$. Also, this variance was deduced to arise from the discontinuity of the gradient of vorticity on the cylinder surface. This study aims to elucidate the reason of the variance. The gradient of vorticity on the cylinder surface is known to be related to the pressure force on the cylinder and, furthermore, the pressure force is independent of the moment. Therefore, this study first considers the problem that a circular cylinder impulsively rotates from rest. The second problem of an impulsively started rotating and translating circular cylinder from rest is then investigated. In this analysis, the drag and lift forces are obtained for $\lambda \to \infty$ and $t \ll 1$. On the basis of the analytical results, the above-mentioned variance will be discussed.

This paper organizes as follows. In § 2 we briefly address the governing equations of the Brinkman penalization method. In this study the relative coordinate system fixed with the cylinder is taken for the translating movement of the cylinder along the lines with our previous study and Bar-Lev & Yang (1975). The problem of an impulsively started translating and rotating cylinder can therefore be replaced by the problem that the cylinder is impulsively immersed in a uniform flow and impulsively rotates with a constant angular velocity. The asymptotic solutions are obtained using the Laplace transform to the governing penalization equation of motion in § 3. In § 3.2 the impulsive rotation problem is analysed, and the moment due to the force on the cylinder is obtained: (I) from the time derivative of the tangential component of the momentum of the entire fluid, (II) from the shear stress on the cylinder surface, (III) by integrating the penalization layer. The results of the moment obtained by these three approaches are shown to be the same, and it can therefore be found that the pressure field plays a key role to the variance on the drag force against the result obtained by integrating the penalization layer. The second problem of an impulsively rotating and translating circular cylinder is then investigated using the method of the matched asymptotic expansion in § 3.3. In § 3.4 the drag and lift forces are obtained for $\lambda \gg 1$ and $t \ll 1$ by the following: (i) the momentum of the whole fluid domain and the pressure sufficiently far from the cylinder, (ii) the integration of the penalization layer.

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Although the unsteady pressure force of the fluid domain is found to cause the variance for the drag force in § 3.4, the variance between the two approaches still exists and the difference is not so small. To resolve the variance, the alternative formula to calculate the fluid force for a solid cylindrical body with the continuous boundary is derived based on the fact that there is the pressure jump on the cylinder surface. The accuracy for the impulsively started translating and rotating case is verified asymptotically in § 3.5. Taking into account the pressure jump, the variance can be reduced to a sufficiently small quantity. In § 4 we summarize our conclusions.

2. Governing equations based on the Brinkman penalization

We consider an unsteady incompressible viscous flow that is governed by the Navier–Stokes equation. In the Brinkman penalization model the governing equation based on the Navier–Stokes equation is written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \lambda \chi (\boldsymbol{u}_S - \boldsymbol{u}), \tag{2.1}$$

where $\boldsymbol{u}$ is the velocity that fulfils the divergence-free condition of $\nabla \cdot \boldsymbol{u} = 0$, p is the pressure, ρ is the fluid density that is assumed to be constant and ν is the kinematic viscosity. In addition, $\boldsymbol{u}_S$ denotes the velocity of a solid body. The present study adopts the relative coordinate system fixed with the centre of a body that moves with a translating velocity. We can therefore replace it by the problem that the body rotates with respect to the centre of the body without the translating motion, i.e. $\boldsymbol{u}_S = \varOmega \boldsymbol{e}_z \times \boldsymbol{x}$. Here, the third term on the right-hand side of (2.1) accounts for the penalization term to enforce the no-slip boundary condition. The characteristic function, χ, defines the domain of active penalization, i.e.

$$\chi = \begin{cases} 1, & \boldsymbol{x} \in \bar{\mathscr{S}}, \\ 0, & \boldsymbol{x} \in \mathscr{F}, \end{cases} \tag{2.2}$$

in which $\mathscr{S}$ denotes the domain occupied by the solid body and $\mathscr{F}$ is the domain occupied by the fluid. In addition, λ is the penalization parameter. The conventional Navier–Stokes equations are found to be recovered in the fluid domain. The penalization approach regards a solid body as a porous media with the permeability being vanishingly small. This results in the fluid velocity being zero at a solid/fluid interface so that the penalization parameter can be required to be a sufficiently large value. The present two-dimensional flow is considered on the orthogonal (x, y) plane, and the normal unit vector $\boldsymbol{e}_z$ is the z direction perpendicular to the (x, y) plane.

In the penalization method the total hydrodynamic force $\boldsymbol{F}$ is calculated by the integration of the penalization term (e.g. Angot *et al.* 1999; Hejlesen *et al.* 2015)

$$\boldsymbol{F} = -\rho \int_{\mathscr{D}} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \mathrm{d}V = -\rho \int_{\mathscr{D}} \lambda \chi (\boldsymbol{u}_S - \boldsymbol{u})\, \mathrm{d}V = -\rho \int_{\mathscr{S}} \lambda (\boldsymbol{u}_S - \boldsymbol{u})\, \mathrm{d}V, \tag{2.3}$$

where the domain $\mathscr{D}$ indicates $\mathscr{D} = \mathscr{S} \cup \mathscr{F}$. The moving velocity $\boldsymbol{u}_S$ of the solid body is obtained as the average velocity inside the domain $\mathscr{S}$, i.e.

$$\boldsymbol{u}_S = \frac{1}{M} \int_{\mathscr{S}} \rho \chi \boldsymbol{u}\, \mathrm{d}V, \tag{2.4}$$

where M is the mass of the solid body. As shown in the previous work (see Ueda & Kida 2021) that investigates an impulsively started translating circular cylinder without

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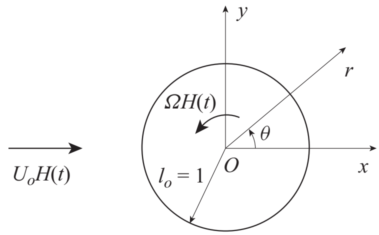

Figure 1. Notations for a circular cylinder impulsively immersed in a constant uniform stream U_o with a constant angular velocity $\Omega H(t)$.

rotation, the drag coefficient calculated by (2.3) exhibits a half-value of the result obtained by Bar-Lev & Yang (1975). It is found that a special notice, whether (2.3) is available or not, is required. This study therefore aims to elucidate this variance. To do so, we set the assumption that the velocity and the vorticity are continuous at the interface between $\mathscr{S}$ and $\mathscr{F}$. As will be shown in § 3.5 later, we will require the terms in addition to (2.3) due to the pressure jump between the outside and the inside of the cylinder.

3. Asymptotic analysis of an initial flow

3.1. *Problem settings and statement*

This section addresses the problem settings and the asymptotic analysis for the present target situations, i.e. initial flows around (1) an impulsively started rotating circular cylinder, and (2) an impulsively started rotating and translating circular cylinder. The governing equation for both problems is written by the nonlinear penalization equation of motion (2.1) with respect to time.

Similar to our previous study (see Ueda & Kida 2021), a small parameter, $\epsilon = U_t T_o / l_o$, is introduced in the present analysis, where l_o, T_o and U_t are the reference length, time and velocity, respectively. The angular velocity, Ωl_o, is selected as the reference velocity U_t for the first problem (pure rotating motion of a circular cylinder). The translating velocity U_o is also selected for the second problem (an impulsively started rotating and translating circular cylinder). In this study the radius of a cylinder is selected as the reference length l_o. Then, the actual time t^* is non-dimensionalized as $t = (U_t/l_o)t^*$ with respect to l_o and U_t.

This study considers an initial flow past an impulsively started rotating and translating circular cylinder from a quiescent state. On the relative coordinate system fixed with the centre of the cylinder, this flow is replaced by the situation that the circular cylinder is impulsively immersed in the uniform flow $U_o H(t)$ with a constant angular velocity $\Omega H(t)$ at $t = 0$. Here, the Heaviside step function $H(t)$ is defined as $H(t) = 0$ for $t \leq 0$ and $H(t) = 1$ for $t > 0$. The problem setting is then illustrated as figure 1, where (x, y) denotes the orthogonal coordinate system and (r, θ) denotes the polar coordinate system.

The present target problem mentioned in § 2 is formulated within a singular perturbation framework with respect to a perturbation parameter ϵ for the penalized Navier–Stokes equation (2.1). The time t is then stretched as $T = t/\epsilon$ (see Bar-Lev & Yang 1975). Because the penalization parameter λ is taken to be proportional to the inverse of the time increment δt (i.e. $\lambda = \alpha/\delta t$ with the relaxation coefficient α in Hejlesen *et al.* 2015), it is set as $\lambda = \lambda_o/\epsilon$ with $\lambda_o = O(1)$. To compare with the analytical results of Bar-Lev & Yang (1975), the Reynolds number *Re* is taken as a sufficiently large value so that the kinematic viscosity ν can be set as $\nu = \epsilon \nu_o$ with $\nu_o = O(1)$. The penalized governing equation (2.1) is then

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written as, on the polar coordinate system,

$$\frac{\partial u_r}{\partial T} + \epsilon\left(u_r\frac{\partial u_r}{\partial r} + u_\theta\frac{\partial u_r}{r\partial\theta} - \frac{u_\theta^2}{r}\right) = -\frac{\epsilon}{\rho}\frac{\partial p}{\partial r} + \lambda_o\chi(u_{sr} - u_r) + \epsilon^2\nu_o\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right), \tag{3.1a}$$

$$\frac{\partial u_\theta}{\partial T} + \epsilon\left(u_r\frac{\partial u_\theta}{\partial r} + u_\theta\frac{\partial u_\theta}{r\partial\theta} + \frac{u_r u_\theta}{r}\right) = -\frac{\epsilon}{\rho}\frac{\partial p}{r\partial\theta} + \lambda_o\chi(u_{s\theta} - u_\theta) + \epsilon^2\nu_o\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right), \tag{3.1b}$$

where ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{3.2}$$

In addition, the equation of continuity is written as

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta} = 0. \tag{3.3}$$

Note that (3.1*a*)–(3.3) are written as the dimensionless form with respect to U_t and l_o and, therefore, ν_o is found to be the dimensionless kinematic viscosity. Furthermore, the velocity of the solid body is written as $\boldsymbol{u}_S = r\Omega\boldsymbol{e}_\theta H(T)$ on the relative coordinate system fixed with the centre of the cylinder, where $\boldsymbol{e}_\theta$ denotes the unit vector in the θ direction.

3.2. *First problem: an impulsively rotating circular cylinder*

In this subsection we first consider the problem that a circular cylinder impulsively rotates with a constant angular velocity Ω without translating motion. In this problem, the moment exerted by the cylinder can be obtained from the tangential force acting on the cylinder surface, i.e. the force due to the pressure is independent of the moment. This problem makes it possible to investigate the contribution from the velocity field on the hydrodynamic force acting on the cylinder because it is free from the pressure contribution.

3.2.1. *Outer solutions*

We attempt to obtain the outer solutions $\boldsymbol{u}^o$, denoted by the superscript 'o', of the governing equations (3.1*a*)–(3.3). The outer domain is defined as the outside or the inside of the cylinder with respect to ϵ, i.e. $r - 1 = O(1)$. The outer solutions cannot satisfy the boundary condition on the solid–fluid boundary, as mentioned below. The inner solution near the boundary is therefore defined as $r - 1 = O(\epsilon)$ in the subsequent § 3.2.2.

Following our previous study (see Ueda & Kida 2021), the velocities, u_r^o and u_θ^o, in the radial (r) and tangential (θ) directions are asymptotically represented by

$$u_r^o = \sum_{n=0}^{\infty}\epsilon^n u_{r,n}^o, \quad u_\theta^o = \sum_{n=0}^{\infty}\epsilon^n u_{\theta,n}^o. \tag{3.4a,b}$$

Also, the pressure p^o is asymptotically written as

$$p^o = \sum_{n=0}^{\infty}\epsilon^n p_n^o. \tag{3.5}$$

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Then, (3.1*a*) and (3.1*b*) reduce to the following for the first two lowest-order solutions in the r and θ directions, respectively;

$$\frac{\partial u^o_{r0}}{\partial T} = \lambda_o \chi(-u^o_{r0}), \tag{3.6a}$$

$$\frac{\partial u^o_{r1}}{\partial T} + u^o_{r0}\frac{\partial u^o_{r0}}{\partial r} + u^o_{\theta 0}\frac{\partial u^0_{r0}}{r\partial \theta} - \frac{u^{o2}_{\theta 0}}{r} = -\frac{1}{\rho}\frac{\partial p^o_0}{\partial r} + \lambda_o \chi(-u^o_{r1}), \tag{3.6b}$$

$$\frac{\partial u^o_{\theta 0}}{\partial T} = \lambda_o \chi(\varOmega r H(T) - u^o_{\theta 0}), \tag{3.6c}$$

$$\frac{\partial u^o_{\theta 1}}{\partial T} + u^o_{r0}\frac{\partial u^o_{\theta 0}}{\partial r} + u^o_{\theta 0}\frac{\partial u^0_{\theta 0}}{r\partial \theta} + \frac{u^o_{r0}u^o_{\theta 0}}{r} = -\frac{1}{\rho}\frac{\partial p^o_0}{r\partial \theta} + \lambda_o \chi(-u^o_{\theta 1}). \tag{3.6d}$$

In the fluid domain, we have $\partial u^o_{r,0}/\partial T = \partial u^o_{\theta,0}/\partial T = 0$, because of $\chi = 0$. This gives $u^o_{r,0} = u^0_{\theta,0} = 0$ because $u^o_{r,0}$ and $u^o_{\theta,0}$ are zero at $T = 0$. We can obtain the outer solutions in the fluid domain by the recursive calculations for the higher-order approximation of (3.1*a*) and (3.1*b*) such as

$$u^o_{r,i} = u^o_{\theta,i} = 0 \quad \text{for } i = 0, 1, 2, \ldots. \tag{3.7}$$

Assuming that the pressure sufficiently far from the cylinder is zero, we have also

$$p^o_i = 0 \quad \text{for } i = 0, 1, 2, \ldots. \tag{3.8}$$

In contrast, since $\chi = 1$ in the body domain $\mathscr{S}$, we have $u^o_{r,i} = 0$ and $u^o_{\theta,i} = \varOmega r H(T)\delta_{i0}$ for $i = 0, 1, 2, \ldots$. This relation gives, in the body domain,

$$u^o_r = 0, \quad u^o_\theta = \varOmega r H(T), \quad p^o = \frac{\rho}{2}\varOmega^2 r^2 H(T) + C, \tag{3.9a–c}$$

where C is an integral constant.

3.2.2. *Inner solutions*

The outer flow described by (3.7) and (3.9*a–c*) causes the slip velocity on the surface of the cylinder (at $r = 1$) so that a non-uniform domain can exist near the cylinder surface. To remedy this, the radial coordinate r is stretched as $R = (r-1)/\epsilon$ where the solution in the stretched domain is called the inner solution that is denoted by the superscript 'i'. The governing equations and the continuity equation of (3.1*a*)–(3.3) are then described as, in the inner domain,

$$\frac{\partial u^i_r}{\partial T} + u^i_r\frac{\partial u^i_r}{\partial R} + \frac{\epsilon}{1+\epsilon R}u^i_\theta\frac{\partial u^i_r}{\partial \theta} - \epsilon\frac{u^{i2}_\theta}{1+\epsilon R} = -\frac{1}{\rho}\frac{\partial p^i}{\partial R} - \lambda_o\chi u^i_r$$
$$+ \nu_o\left[\left(\frac{\partial^2}{\partial R^2} + \frac{\epsilon}{1+\epsilon R}\frac{\partial}{\partial R} + \frac{\epsilon^2}{(1+\epsilon R)^2}\frac{\partial^2}{\partial \theta^2}\right)u^i_r - \frac{\epsilon^2}{(1+\epsilon R)^2}\left(u^i_r + 2\frac{\partial u^i_\theta}{\partial \theta}\right)\right], \tag{3.10a}$$

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$$\frac{\partial u_\theta^i}{\partial T} + u_r^i \frac{\partial u_\theta^i}{\partial R} + \frac{\epsilon}{1+\epsilon R} u_\theta^i \frac{\partial u_\theta^i}{\partial \theta} + \epsilon \frac{u_r^i u_\theta^i}{1+\epsilon R}$$
$$= -\frac{1}{\rho}\frac{\epsilon}{1+\epsilon R}\frac{\partial p^i}{\partial \theta} + \lambda_o \chi [(1+\epsilon R)\Omega H(T) - u_\theta^i]$$
$$+ \nu_o \left[\left(\frac{\partial^2}{\partial R^2} + \frac{\epsilon}{1+\epsilon R}\frac{\partial}{\partial R} + \frac{\epsilon^2}{(1+\epsilon R)^2}\frac{\partial^2}{\partial \theta^2} \right) u_\theta^i - \frac{\epsilon^2}{(1+\epsilon R)^2} \left(u_\theta^i - 2\frac{\partial u_r^i}{\partial \theta} \right) \right], \tag{3.10b}$$

$$\frac{\partial u_r^i}{\partial R} + \frac{\epsilon}{1+\epsilon R} u_r^i + \frac{\epsilon}{1+\epsilon R}\frac{\partial u_\theta^i}{\partial \theta} = 0, \tag{3.10c}$$

in which u_r^i, u_θ^i and p^i are assumed to be asymptotically represented as

$$u_r^i = \sum_{n=0}^{\infty} \epsilon^n u_{rn}^i, \quad u_\theta^i = \sum_{n=0}^{\infty} \epsilon^n u_{\theta n}^i, \quad p^i = \sum_{n=0}^{\infty} \epsilon^n p_n^i. \tag{3.11a–c}$$

From (3.10c), we have $\partial u_{r0}^i/\partial R = 0$ and, therefore, the first-order solution of u_r^i is found to be independent of R. The matching procedure to the outer solution inside the body domain (i.e. $u_r^{io} = u_r^{oi} = 0$) yields

$$u_{r0}^i = 0. \tag{3.12}$$

Equation (3.12) is found to fulfil the matching condition to the outer solution of the fluid domain. The first approximations of (3.10a) and (3.10b) are written as

$$0 = -\frac{1}{\rho}\frac{\partial p_0^i}{\partial R}, \tag{3.13a}$$

$$\frac{\partial u_{\theta 0}^i}{\partial T} = \lambda_o \chi (\Omega H(T) - u_{\theta 0}^i) + \nu_o \frac{\partial^2 u_{\theta 0}^i}{\partial R^2}. \tag{3.13b}$$

From (3.13a), p_0^i is found to be independent of R. Furthermore, from the matching procedure to p_i^o of (3.8) in the flow domain (i.e. $p^{io} = p_0^i + O(\epsilon) = p^{oi} = 0$), we have $p_0^i = 0$ for $R \geq 0$. For $R < 0$, (3.9a–c) becomes $p^{oi} = (\rho/2)\Omega^2 H(T) + C$ and we have

$$p_0^i = \begin{cases} 0 & \text{for } R \geq 0, \\ \dfrac{\rho}{2}\Omega^2 H(T) + C & \text{for } R < 0. \end{cases} \tag{3.14}$$

For $R \geq 0$, the governing equation of (3.13b) with respect to $u_{\theta 0}^i$ is written as

$$\frac{\partial u_{\theta 0}^i}{\partial T} = \nu_o \frac{\partial^2 u_{\theta 0}^i}{\partial R^2}. \tag{3.15}$$

The matching condition to the fluid domain gives the following boundary condition:

$$u_{\theta 0}^i \to 0 \quad \text{as } R \to \infty. \tag{3.16}$$

To solve (3.15) with (3.16), we employ the Laplace transform, $U_{\theta 0} = \mathcal{L}(u_{\theta 0}^i) = \int_0^\infty \mathrm{e}^{-sT} u_{\theta 0}^i(R, \theta; T)\,\mathrm{d}T$, similar to our previous study (see Ueda & Kida 2021). Then, the solution to (3.15) is easily derived as

$$U_{\theta 0} = A(s, \theta)\mathrm{e}^{-aR} \quad \text{with } a = \sqrt{\frac{s}{\nu_o}}, \tag{3.17}$$

where A is an integral constant that is a function of s and θ.

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For $R < 0$, the governing equation of (3.13*b*) is also written as

$$\frac{\partial u^i_{\theta 0}}{\partial T} = \lambda_o \Omega H(T) - \lambda_o u^i_{\theta 0} + \nu_o \frac{\partial^2 u^i_{\theta 0}}{\partial R^2}. \tag{3.18}$$

Taking into account the initial condition of $u^i_{\theta 0} = \Omega$, the Laplace transform to (3.18) is written as

$$sU_{\theta 0} - \Omega = \lambda_o \frac{\Omega}{s} - \lambda_o U_{\theta 0} + \nu_o \frac{\partial^2 U_{\theta 0}}{\partial R^2}. \tag{3.19}$$

The boundary condition as $R \to -\infty$ is obtained from the matching condition to the outer solution inside the body domain, i.e. $u^i_{\theta 0} \to \Omega H(T)$. Then, the solution to (3.19) is written as

$$U_{\theta 0} = B(s, \theta) \mathrm{e}^{\bar{a}R} + \frac{\Omega}{s} \quad \text{with } \bar{a} = \sqrt{\frac{s + \lambda_o}{\nu_o}}, \tag{3.20}$$

where B is an integral constant that is a function of s and θ. The integral constants, A and B, can be determined as $A = B + \Omega/s$ and $-aA = \bar{a}B$ by the enforcements of the continuity of the velocity and its gradient with respect to R at $R = 0$. Therefore, we have, for (3.17) and (3.20),

$$U_{\theta 0} = \begin{cases} \dfrac{\bar{a}}{a + \bar{a}} \dfrac{\Omega}{s} \mathrm{e}^{-aR} & \text{for } R \geq 0, \\ -\dfrac{a}{a + \bar{a}} \dfrac{\Omega}{s} \mathrm{e}^{\bar{a}R} + \dfrac{\Omega}{s} & \text{for } R < 0. \end{cases} \tag{3.21}$$

Using the relations of (A1) and (A2) in Appendix A, (3.21) becomes, for $R \geq 0$,

$$u^i_{\theta 0} = \Omega \left\{ \mathrm{erfc}\left(\frac{R}{2\sqrt{\nu_o T}} \right) - \frac{R}{4\sqrt{\pi \nu_o}} \int_0^T \frac{\exp\left(-\dfrac{\lambda_o}{2} \xi \right)}{(T - \xi)^{3/2}} \left[\mathrm{I}_1(\lambda_o \xi/2) + \mathrm{I}_0(\lambda_o \xi/2) \right] \right.$$
$$\left. \times \exp\left(-\frac{R^2}{4\nu_o (T - \xi)} \right) \mathrm{d}\xi \right\}, \tag{3.22}$$

for $R < 0$,

$$u^i_{\theta 0} = \Omega \left\{ 1 - \frac{|R|}{4\sqrt{\pi \nu_o}} \int_0^T \frac{\exp(-\lambda_o (T - \xi/2))}{(T - \xi)^{3/2}} \left[\mathrm{I}_1(\lambda_o \xi/2) + \mathrm{I}_0(\lambda_o \xi/2) \right] \right.$$
$$\left. \times \exp\left(-\frac{R^2}{4\nu_o (T - \xi)} \right) \mathrm{d}\xi \right\}. \tag{3.23}$$

Here, $\mathrm{I}_0(z)$ and $\mathrm{I}_1(z)$ are the zeroth- and the first-order modified Bessel functions of the first kind, respectively.

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Let us carry out further analysis to the second approximation for seeking the solutions of u^i_{r1} and $u^i_{\theta 1}$. Taking into account (3.12) and (3.22)–(3.23), (3.10*c*) is written as

$$\frac{\partial u^i_{r1}}{\partial R} = -\frac{\partial u^i_{\theta 0}}{\partial \theta} = 0. \tag{3.24}$$

Hence, we find that

$$u^i_{r1} = C(T, \theta), \tag{3.25}$$

where C is an integral constant that is a function of T and θ. The matching to the outer solution and the enforcement of the continuity of u^i_{r1} at $R = 0$ determine the integral constant of (3.25) as

$$u^i_{r1} = 0. \tag{3.26}$$

The second approximations of the governing equations (3.10*a*) and (3.10*b*) are written as, taking into account (3.12) and (3.14),

$$u^{i2}_{\theta 0} = \frac{1}{\rho}\frac{\partial p^i_1}{\partial R}, \tag{3.27a}$$

$$\frac{\partial u^i_{\theta 1}}{\partial T} = \lambda_o \chi (R\Omega H(T) - u^i_{\theta 1}) + \nu_o \left(\frac{\partial^2 u^i_{\theta 1}}{\partial R^2} + \frac{\partial u^i_{\theta 0}}{\partial R} \right). \tag{3.27b}$$

The matching conditions to the outer solutions in the fluid and the solid domains give $p^i_1 \to 0$ as $R \to \infty$, and $p^i_1 \to \rho R \Omega^2 H(T)$ and $u^i_{\theta 0} \to \Omega H(T)$ as $R \to -\infty$. From (3.27*a*), the pressure p^i_1 is then described as

$$p^i_1 = \begin{cases} -\rho \displaystyle\int_R^\infty u^{i2}_{\theta 0}\,\mathrm{d}R & \text{for } R \geq 0, \\ \rho \displaystyle\int_{-\infty}^R [u^{i2}_{\theta 0} - \Omega^2 H(T)]\,\mathrm{d}R + \rho \Omega^2 R H(T) & \text{for } R < 0. \end{cases} \tag{3.28}$$

Here, we introduce $\hat{u}^i_{\theta 1}$ as

$$u^i_{\theta 1} = \begin{cases} -\dfrac{1}{2} R u^i_{\theta 0} + \hat{u}^i_{\theta 1} & \text{for } R \geq 0, \\ -\dfrac{1}{2} R \left[u^i_{\theta 0} - \Omega H(T)\right] + R\Omega H(T) + \hat{u}^i_{\theta 1} & \text{for } R < 0. \end{cases} \tag{3.29}$$

Then, (3.27*b*) reduces to

$$\frac{\partial \hat{u}^i_{\theta 1}}{\partial T} = \begin{cases} \nu_o \dfrac{\partial^2 \hat{u}^i_{\theta 1}}{\partial R^2} & \text{for } R \geq 0, \\ \nu_o \dfrac{\partial^2 \hat{u}^i_{\theta 1}}{\partial R^2} - \lambda_o \hat{u}^i_{\theta 1} & \text{for } R < 0. \end{cases} \tag{3.30}$$

Employing the Laplace transform, the solution to (3.30) is obtained as

$$\hat{u}^i_{\theta 1} = \begin{cases} \mathcal{L}^{-1}\left(A_1(s, \theta)\mathrm{e}^{-aR}\right) & \text{for } R \geq 0, \\ \mathcal{L}^{-1}\left(B_1(s, \theta)\mathrm{e}^{\hat{a}R}\right) & \text{for } R < 0, \end{cases} \tag{3.31}$$

where A_1 and B_1 are respectively integral constants that are functions of s and θ. Here, for obtaining (3.31), the matching condition to the outer solution, i.e. $\hat{u}^i_{\theta 1} \to 0$ as $R \to \pm\infty$,

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is used. Equation (3.29) is therefore written as

$$u_{\theta 1}^i = \begin{cases} -\dfrac{1}{2}Ru_{\theta 0}^i + \mathcal{L}^{-1}(A_1(s,\theta)\mathrm{e}^{-aR}) & \text{for } R \geq 0, \\ -\dfrac{1}{2}R[u_{\theta 0}^i - \varOmega H(T)] + R\varOmega H(T) + \mathcal{L}^{-1}(B_1(s,\theta)\mathrm{e}^{\bar{a}R}) & \text{for } R < 0. \end{cases} \tag{3.32}$$

Here, the enforcements of the continuity of the velocity and its radial derivative at $R = 0$ determine the values of the integral constants A_1 and B_1 as follows:

$$A_1 = B_1 = -\frac{3}{2}\frac{\varOmega}{s(a+\bar{a})}. \tag{3.33}$$

Using the relations of

$$\frac{1}{s(a+\bar{a})} = \frac{\sqrt{\nu_o}}{\lambda_o}\left(\frac{1}{\sqrt{s+\lambda_o}} - \frac{1}{\sqrt{s}}\right) + \frac{\sqrt{\nu_o}}{s\sqrt{s+\lambda_o}} \tag{3.34}$$

and (A3) and (A4) in Appendix A, we finally obtain

$$u_{\theta 1}^i = -\frac{1}{2}Ru_{\theta 0}^i - \frac{3R\varOmega}{4\lambda_o \mathrm{\pi}}\int_0^T \frac{\exp(-\lambda_o\xi)-1}{\sqrt{\xi(T-\xi)^3}}\exp\left(-\frac{R^2}{4\nu_o(T-\xi)}\right)\mathrm{d}\xi$$
$$- \frac{3R\varOmega}{4\sqrt{\mathrm{\pi}\lambda_o}}\int_0^T \frac{\exp\left(-\dfrac{R^2}{4\nu_o(T-\xi)}\right)}{(T-\xi)^{3/2}}\mathrm{erf}(\sqrt{\lambda_o\xi})\,\mathrm{d}\xi \quad \text{for } R \geq 0, \tag{3.35a}$$

$$u_{\theta 1}^i = -\frac{1}{2}R\left(u_{\theta 0}^i - \varOmega H(T)\right) + R\varOmega H(T)$$
$$- \frac{3|R|\varOmega}{4\lambda_o\mathrm{\pi}}\int_0^T \frac{\exp(-\lambda_o\xi)-1}{\sqrt{\xi(T-\xi)^3}}\exp\left(-\lambda_o(T-\xi) - \frac{R^2}{4\nu_o(T-\xi)}\right)\mathrm{d}\xi$$
$$- \frac{3|R|\varOmega}{4\sqrt{\mathrm{\pi}\lambda_o}}\int_0^T \frac{\exp\left(-\lambda_o(T-\xi) - \dfrac{R^2}{4\nu_o(T-\xi)}\right)}{(T-\xi)^{3/2}}\mathrm{erf}(\sqrt{\lambda_o\xi})\,\mathrm{d}\xi \quad \text{for } R < 0. \tag{3.35b}$$

Figure 2 shows the comparison of $u_{\theta 0}^i/\varOmega$ (first-order solution) and $u_{\theta 1}^i/(R\varOmega)$ (second-order solution) among three values of λ with respect to $\eta = R/(2\sqrt{\nu_o T})$. To plot the data of figure 2, the numerical calculation of the following integral that is shown in (3.36) is needed. To do so, the integral variable ξ is changed by $\xi = Tx$ and, then, the variable x is changed to y by $x = 1 - \eta^2/[(y+\eta)^2]$;

$$\int_0^T \frac{f(\xi)}{(T-\xi)^{3/2}}\exp\left(-\frac{R^2}{4\nu_o(T-\xi)}\right)\mathrm{d}\xi$$
$$= 2\frac{\exp(-\eta^2)}{\eta}\int_0^\infty \exp(-y^2 - 2y\eta) f\left(1 - \frac{\eta^2}{(y+\eta)^2}\right)\mathrm{d}y. \tag{3.36}$$

In the numerical calculation of figure 2, the Hermite quadrature formula of $n = 9$ (see table 25.10 in Abramowitz & Stegun 1954) is used for the quadrature of the integral on the right-hand side of (3.36).

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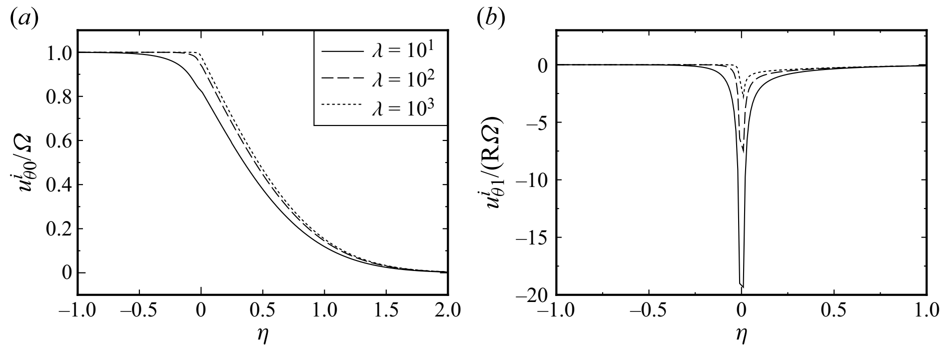

Figure 2. Comparison of the tangential velocities $u^i_{\theta 0}$ (first-order solution) and $u^i_{\theta 1}$ (second-order solution) among three values of λ with respect to $\eta = R/(2\sqrt{\nu_o T})$.

3.2.3. *Moment exerted by a cylinder*

In this subsection we attempt to obtain the moment exerted by the circular cylinder, using three kinds of approaches. Each approach is calculated (I) by the time derivative of the tangential component of the momentum of the entire fluid, (II) by the integration of the shear stress on the cylinder surface, and (III) by the integration of the penalization layer based on (2.3).

In approach (I) we consider the control surface that surrounds a circular domain having a radius of $r_\infty \gg 1$. Then, we can define the fluid domain as $\mathscr{F} = [(r, \theta)|1 \le r \le r_\infty, 0 \le \theta < 2\pi]$. The small fluid element $\mathrm{d}x\,\mathrm{d}y$ has the momentum $\rho u_\theta\,\mathrm{d}x\,\mathrm{d}y$ in the tangential direction. The moment induced by the fluid element is therefore described as $|\boldsymbol{x}|\rho[(\mathrm{d}/\mathrm{d}t)u_\theta]\,\mathrm{d}x\,\mathrm{d}y$. The moment M_1 of the flow exerted by the cylinder rotation is found to be calculated by, noting that the pressure on the control surface does not affect the moment,

$$M_1 = -\rho \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_1^{r_\infty} r^2 u_\theta\,\mathrm{d}r\,\mathrm{d}\theta. \tag{3.37}$$

To obtain M_1, we employ the Laplace transform to (3.37) and take the limit of $r_\infty \to \infty$. Then, we have, taking into account that the outer solution is described as $u^o_\theta = 0$ and u^i_θ is independent of θ,

$$\mathcal{L}(M_1) = -2\pi\rho s\int_0^\infty [U_{\theta 0} + \epsilon(2RU_{\theta 0} + U_{\theta 1}) + O(\epsilon^2)]\mathrm{d}R, \tag{3.38}$$

where $U_{\theta 1} = \mathcal{L}(u^i_{\theta 1})$. Using (3.21) and the first equation of (3.32), M_1 can be obtained as

$$\mathcal{L}(M_1) = -2\pi\rho\Omega\left[\frac{\bar{a}}{a(a+\bar{a})} + \frac{3}{2}\epsilon\left(\frac{\bar{a}}{a^2(a+\bar{a})} - \frac{1}{a(a+\bar{a})}\right) + O(\epsilon^2)\right]. \tag{3.39}$$

In approach (II) the moment M_2 on the fluid is calculated by the integration of the shear stress on the cylinder surface

$$M_2 = \int_0^{2\pi} 1 \times \tau_{r\theta}\,\mathrm{d}\theta = \rho\nu\int_0^{2\pi}\left(\frac{\partial u_\theta}{\partial r} - u_\theta + \frac{\partial u_r}{\partial \theta}\right)_{r=1+0}\mathrm{d}\theta. \tag{3.40}$$

The Laplace transform to (3.40) becomes, using the relations of $U_{r0} = U_{r1} = 0$,

$$\mathcal{L}(M_2) = 2\pi\rho\nu_o\left[\frac{\partial}{\partial R}(U_{\theta 0} + \epsilon U_{\theta 1}) - \epsilon U_{\theta 0} + O(\epsilon^2)\right]_{R=+0}. \tag{3.41}$$

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Substituting (3.21) and the first equation of (3.32) into (3.41), we have

$$\mathcal{L}(M_2) = 2\pi\Omega\left[-\frac{\bar{a}}{a(a+\bar{a})} + \epsilon\left(-\frac{3}{2}\frac{\bar{a}}{a^2(a+\bar{a})} + \frac{3}{2}\frac{1}{a(a+\bar{a})}\right) + O(\epsilon^2)\right]. \tag{3.42}$$

Comparing the results between (3.39) and (3.42), it is found that M_1 is identical with M_2 within the order of ϵ, i.e. the moment calculated from the entire fluid domain is the same as that calculated from the tangential force on the cylinder surface. Making use of the inverse Laplace transform, the moment M_1 (or M_2) is obtained as

$$\begin{aligned} M_1 = M_2 = -2\pi\rho\Omega\Bigg\{&\sqrt{\frac{\nu_o}{\pi T}} + \frac{1}{2}\sqrt{\frac{\nu_o}{\pi T^3}}\frac{1-\mathrm{e}^{-\lambda_o T}}{\lambda_o} \\ &+ \frac{3}{2}\epsilon\nu_o[H(T) - \mathrm{e}^{-\lambda_o T/2}(\mathrm{I}_0(\lambda_o T/2) + \mathrm{I}_1(\lambda_o T/2))] + O(\epsilon^2)\Bigg\}. \end{aligned} \tag{3.43}$$

In approach (III) the moment M_3 on the fluid is calculated by the integration of the penalization layer, based on (2.3),

$$\begin{aligned} M_3 &= -\rho\lambda\int_0^{2\pi}\int_0^1 r(r\Omega - u_\theta)r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\rho\lambda\epsilon\int_0^{2\pi}\int_{-\infty}^0 [(1+\epsilon R)\Omega - u_\theta^i](1+\epsilon R)^2\,\mathrm{d}R\,\mathrm{d}\theta. \end{aligned} \tag{3.44}$$

Similar to the calculation of M_1 or M_2, the Laplace transform to M_3 is written as, using (3.21) and the second equation of (3.32),

$$\mathcal{L}(M_3) = -2\pi\rho\lambda_o\Omega\left[\frac{a}{s\bar{a}(a+\bar{a})} + \frac{3}{2}\epsilon\left(-\frac{a}{s\bar{a}^2(a+\bar{a})} + \frac{1}{s\bar{a}(a+\bar{a})}\right)\right]. \tag{3.45}$$

The inverse Laplace transform to (3.45) yields

$$M_3 = -2\pi\rho\Omega\sqrt{\nu_o}\left\{\frac{1-\mathrm{e}^{-\lambda_o T}}{\sqrt{\pi T}} + \epsilon\frac{3\nu_o}{2\lambda_o}[H(T) + \mathrm{e}^{-\lambda_o T} - 2\mathrm{e}^{-\lambda_o T/2}\mathrm{I}_0(\lambda_o T/2)]\right\}. \tag{3.46}$$

From the results of (3.43) and (3.46), each moment is found to behave like, as $\lambda_o \to \infty$,

$$M_1,\quad M_2 \to -2\pi\rho\Omega\sqrt{\frac{\nu_o}{\pi T}},\quad M_3 \to -2\pi\rho\Omega\sqrt{\frac{\nu_o}{\pi T}}. \tag{3.47a–c}$$

It can therefore be found that the moment M_3 obtained by the integration of the penalization layer is identical to M_1 and M_2 as $\lambda_o \to \infty$. Because the pressure is independent of the moment in this pure rotation problem, the pressure is found to play an important role to the variance of the drag force, which is demonstrated in our previous study (see Ueda & Kida 2021). Note that (3.47*a–c*) is the same as the result of Badr & Dennis (1985). Figure 3 shows the comparisons of the values between M_1 and M_3 with respect to t among the three values of λ at $Re = 100$ (i.e. $\nu = 1/100$). It seems that the variance between M_1 (or M_2) and M_3 decreases exponentially with an increase in the value of λ.

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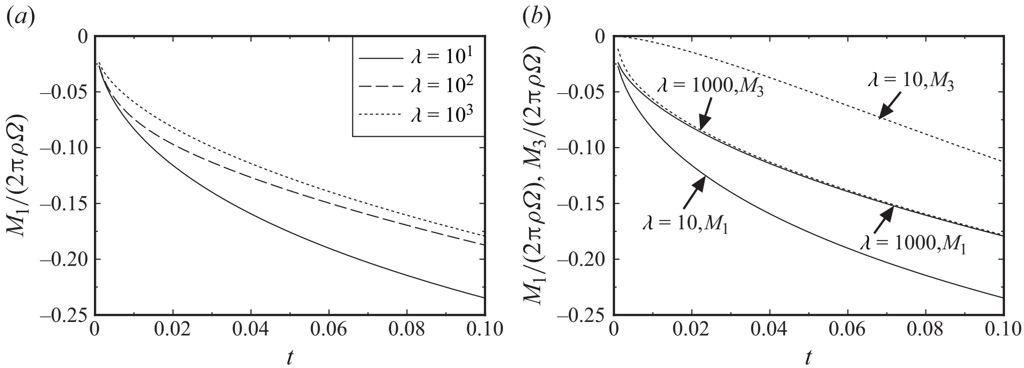

Figure 3. Initial behaviours of the moments M_1 and M_3 with respect to t among the different values of λ at $\nu = 1/100$.

3.3. *Second problem: an impulsively rotating and translating circular cylinder*

In this subsection we consider the problem that a circular cylinder impulsively starts with the angular velocity Ω and the translating velocity U_o. The velocity is normalized with U_o and, then, the translating velocity and the rotational angular velocity of the cylinder are non-dimensionalized as 1 and Ω, respectively. Similar to the analysis in § 3.2, the governing equations (3.1*a*)–(3.3) are adopted in this problem.

3.3.1. *Outer solutions*

The outer solutions in the fluid domain are described by (3.6*a*)–(3.6*d*) in the same manner as § 3.2.1, and the first approximation reduces to

$$\frac{\partial u^o_{r0}}{\partial T} = \frac{\partial u^o_{\theta 0}}{\partial T} = 0. \tag{3.48}$$

The velocities u^0_{r0} and $u^o_{\theta 0}$ are then found to be independent of T. The first-order outer solution in the fluid domain is known to exhibit a potential flow (i.e. inviscid solution). This fact can be confirmed by performing the recursive calculation. Based on this fact and the initial condition of the fluid velocity that $u^o_r = \cos\theta H(T)$ and $u^0_\theta = -\sin\theta H(T)$ for $r \geq 1$, the outer streamfunction ψ^o can be described as

$$\psi^o = r\sin\theta H(T) + \sum_{n=1}^{\infty} \frac{1}{r^n}(a^c_n \cos n\theta + a^s_n \sin n\theta), \tag{3.49}$$

where a^c_n and a^s_n are functions of T and ϵ, respectively.

The outer solutions inside the body domain are also described by (3.6*a*)–(3.6*d*). Taking into account that the initial condition is given by $u^o_r = 0$ and $u^o_\theta = r\Omega H(T)$, we readily have

$$u^o_r = 0, \quad u^o_\theta = r\Omega H(T), \quad \frac{p^o}{\rho} = \frac{1}{2}\Omega^2 r^2 H(T) + C, \tag{3.50a–c}$$

where C is an integral constant that is a function of T.

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3.3.2. *Inner solutions*

The inner solutions are described by the same governing equations as (3.10*a*)–(3.10*c*). The continuity equation (3.10*c*) gives

$$\frac{\partial u^i_{r0}}{\partial R} = 0, \tag{3.51}$$

in which u^i_{r0} is found to be independent of R (i.e. a function of T and θ). Because the outer solutions inside the body domain were $u^o_r = 0$ and $u^o_\theta = r\varOmega H(T)$ as obtained in (3.50*a–c*), we have the following first-order inner solution, for $R \geq 0$ and $R < 0$, taking into account the matching condition to the outer solutions:

$$u^i_{r0} = 0. \tag{3.52}$$

The first-order outer solution in the fluid domain described by (3.49) can be determined from the matching procedure to the inner solution of (3.52) and, therefore, we have $u^{oi}_{r0} = u^{io}_{r0} = 0$. Here, the inner (or outer) expansion of the outer (or inner) solution is denoted by the superscript '*oi*' (or '*io*'). Taking into account that the outer solution of ψ^o is written by (3.49), the above-mentioned matching condition (i.e. $u^{oi}_{r0} = u^{io}_{r0} = 0$) yields the first-order streamfunction in the outer fluid domain as

$$\psi^o_0 = \left(r - \frac{1}{r}\right) H(T) \sin\theta, \tag{3.53}$$

which is known to be the solution of an inviscid uniform flow past a circular cylinder. The governing equations that describe the first-order solutions in the inner fluid domain are the same as (3.13*a*) and (3.13*b*), i.e.

$$0 = -\frac{1}{\rho}\frac{\partial p^i_0}{\partial R}, \tag{3.54a}$$

$$\frac{\partial u^i_{\theta 0}}{\partial T} = \lambda_o \chi (\varOmega H(T) - u^i_{\theta 0}) + \nu_o \frac{\partial^2 u^i_{\theta 0}}{\partial R^2}. \tag{3.54b}$$

Here, let us consider $u^i_{\theta 0}$ of (3.54*b*). From (3.53), the inner expansion of $u^o_{\theta 0}$ in the fluid domain is written as $u^{oi}_{\theta 0} = -2\sin\theta H(T)$ and, therefore, we can write

$$u^i_{\theta 0} \to -2\sin\theta H(T) \quad \text{as } R \to \infty. \tag{3.55}$$

In contrast, since $u^o_{\theta 0} = r\varOmega H(T)$ inside the body domain, we can write

$$u^i_{\theta 0} \to \varOmega H(T) \quad \text{as } R \to -\infty. \tag{3.56}$$

The Laplace transform to (3.54*b*) for $R \geq 0$ is written as, taking into account the initial condition of $u^i_{\theta 0} = -2\sin\theta$,

$$sU_{\theta 0} + 2\sin\theta = \nu_o \frac{\partial^2 U_{\theta 0}}{\partial R^2}, \tag{3.57}$$

where $U_{\theta 0} = \mathcal{L}(u^i_{\theta 0})$. The solution to (3.55) can be then obtained as (see Appendix B for the detailed derivation)

$$U_{\theta 0} = -\frac{2}{s}\sin\theta + A_0(s,\theta)\mathrm{e}^{-aR} \quad \text{with } a = \sqrt{\frac{s}{\nu_o}}, \tag{3.58}$$

where A_0 is an integral constant that is a function of s and θ. For $R < 0$, the governing equation is given in (3.54*b*) as $\chi = 1$, and the initial condition is written as $u^i_{\theta 0} = \varOmega$.

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Therefore, the Laplace transform to the governing equation yields, for $R < 0$,

$$sU_{\theta 0} - \varOmega = \lambda_o \left(\frac{\varOmega}{s} - U_{\theta 0} \right) + \nu_o \frac{\partial^2 U_{\theta 0}}{\partial R^2}. \tag{3.59}$$

The solution to (3.59) is obtained as, taking into account (3.56) (see Appendix B),

$$U_{\theta 0} = B_0(s,\theta)\mathrm{e}^{\bar{a}R} + \frac{\varOmega}{s} \quad \text{with } \bar{a} = \sqrt{\frac{s+\lambda_o}{s}}, \tag{3.60}$$

where B_0 is an integral constant that is a function of s and θ. The indeterminate functions, A_0 and B_0, are determined by the enforcements of the continuity of the velocity and its derivative with respect to R at $R = 0$. Therefore, we can finally obtain

$$U_{\theta 0} = \begin{cases} \dfrac{\bar{a}}{a+\bar{a}} \dfrac{\varOmega + 2\sin\theta}{s} \mathrm{e}^{-aR} - \dfrac{2}{s}\sin\theta & \text{for } R \geq 0, \\ -\dfrac{a}{a+\bar{a}} \dfrac{\varOmega + 2\sin\theta}{s} \mathrm{e}^{\bar{a}R} + \dfrac{\varOmega}{s} & \text{for } R < 0. \end{cases} \tag{3.61}$$

By virtue of the form of (3.61) in the Laplace space, the solution $u^i_{\theta 0}$ in the real space is found to be written as

$$u^i_{\theta 0} = \hat{u}^0_{\theta 0} + \hat{u}^s_{\theta 0} \sin\theta. \tag{3.62}$$

Here, $\hat{u}^0_{\theta 0}$ is the same as (3.22) and (3.23) that are the solutions for the pure rotation problem analysed in § 3.2. In addition, $\hat{u}^s_{\theta 0}$ is the same as the solution for the pure translating motion that was derived in our previous study (see Ueda & Kida 2021).

For the second approximation, u^i_{r1} is described by (3.10*c*), and it reduces to

$$\frac{\partial u^i_{r1}}{\partial R} = -\frac{\partial u^i_{\theta 0}}{\partial \theta}. \tag{3.63}$$

Using the matching condition to the outer solution inside the body domain ($u^i_{r1} \to 0$ as $R \to -\infty$), (3.63) becomes

$$u^i_{r1} = -\frac{\partial}{\partial \theta} \int_{-\infty}^{R} u^i_{\theta 0}\,\mathrm{d}R. \tag{3.64}$$

Making use of the Laplace transform and the first-order solution of (3.61) in the Laplace space, (3.64) can be written as, for $R < 0$,

$$U_{r1} = -\frac{\partial}{\partial \theta} \int_{-\infty}^{R} U_{\theta 0}\,\mathrm{d}R = 2\frac{a}{\bar{a}(a+\bar{a})} \frac{1}{s} \mathrm{e}^{\bar{a}R} \cos\theta, \tag{3.65}$$

where $U_{r1} = \mathcal{L}(u^i_{r1})$. For $R \geq 0$, the function U_{r1} is continuous at $R = 0$ and, therefore, (3.64) can be written as

$$\begin{aligned} U_{r1} &= -\frac{\partial}{\partial \theta} \left(\int_{-\infty}^{0} U_{\theta 0}\,\mathrm{d}R + \int_{0}^{R} U_{\theta 0}\,\mathrm{d}R \right) \\ &= 2\frac{a}{\bar{a}(a+\bar{a})} \frac{1}{s} \cos\theta - 2\frac{\bar{a}}{a(a+\bar{a})} \frac{1 - \mathrm{e}^{-aR}}{s} \cos\theta + \frac{2}{s} R \cos\theta. \end{aligned} \tag{3.66}$$

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By virtue of (3.65) and (3.66), the function u^i_{r1} is found to be expressed as the form of

$$u^i_{r1} = \hat{u}_{r1}\cos\theta. \tag{3.67}$$

Here, we consider the matching to the outer solution. Equation (3.66) tends to, as $R \to \infty$,

$$U_{r1} \to \frac{2}{s}R\cos\theta + 2\frac{a-\bar{a}}{a\bar{a}s}\cos\theta. \tag{3.68}$$

Taking into account that the outer streamfunction is asymptotically represented as $\psi^o = \psi^o_0 + \epsilon\psi^o_1 + O(\epsilon^2)$, i.e.

$$\psi^o = \left(r - \frac{1}{r}\right)H(T)\sin\theta + \epsilon\sum_{n=1}^{\infty}\frac{1}{r^n}(a^c_n\cos n\theta + a^s_n\sin n\theta) + O(\epsilon^2), \tag{3.69}$$

the matching procedure between (3.68) and (3.69) determines the values of the coefficients as

$$a^c_n = 0, \quad \mathcal{L}(a^s_n) = 2\frac{a-\bar{a}}{a\bar{a}s}\delta_{n1}. \tag{3.70a,b}$$

The streamfunction in the outer fluid domain is therefore written as

$$\psi^o = \left(r - \frac{1}{r}\right)H(T)\sin\theta + 2\epsilon\frac{A}{r}\sin\theta + O(\epsilon^2), \tag{3.71}$$

where A is defined by

$$A = \mathcal{L}^{-1}\left(\frac{a-\bar{a}}{a\bar{a}s}\right). \tag{3.72}$$

The inner expansion of the outer solution (3.71) is calculated as, in the Laplace space,

$$\mathcal{L}(u^{oi}_\theta) = -2\frac{1}{s}\sin\theta + 2\epsilon\frac{R}{s}\sin\theta + 2\epsilon\frac{a-\bar{a}}{a\bar{a}s}\sin\theta. \tag{3.73}$$

The matching to (3.73) (i.e. $\mathcal{L}(u^{oi}_\theta) = \mathcal{L}(u^{io}_\theta)$) gives the following boundary condition of $u^i_{\theta 1}$ as $R \to \infty$:

$$\mathcal{L}(u^i_{\theta 1}) \to 2\frac{R}{s}\sin\theta + 2\frac{a-\bar{a}}{a\bar{a}s}\sin\theta \quad \text{as } R \to \infty. \tag{3.74}$$

Similarly, since $u^o_\theta = r\Omega H(T)$ inside the body domain, we have

$$\mathcal{L}(u^i_{\theta 1}) \to \frac{R}{s}\Omega \quad \text{as } R \to -\infty. \tag{3.75}$$

The second approximation of (3.10a) and (3.10b) are written as

$$\frac{\partial u^i_{r1}}{\partial T} - u^{i2}_{\theta 0} = -\frac{1}{\rho}\frac{\partial p^i_1}{\partial R} - \lambda_o\chi u^i_{r1} + \nu_o\frac{\partial^2 u^i_{r1}}{\partial R^2}, \tag{3.76a}$$

$$\frac{\partial u^i_{\theta 1}}{\partial T} + u^i_{r1}\frac{\partial u^i_{\theta 0}}{\partial R} + u^i_{\theta 0}\frac{\partial u^i_{\theta 0}}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p^i_0}{\partial \theta} + \lambda_o\chi(R\Omega H(T) - u^i_{\theta 1}) + \nu_o\left(\frac{\partial^2 u^i_{\theta 1}}{\partial R^2} + \frac{\partial u^i_{\theta 0}}{\partial R}\right). \tag{3.76b}$$

The pressure p^i_0 is found to be independent of R from (3.54a), and p^o is given by (3.50a–c) in the outer domain inside the body. In the fluid domain, since the outer solution

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is described by a potential flow (inviscid solution), the following unsteady Bernoulli's equation is valid:

$$\frac{p^o}{\rho} = -\frac{\partial \phi^o}{\epsilon \partial T} - \frac{1}{2}(|\boldsymbol{u}|^2 - 1). \tag{3.77}$$

Here, ϕ^o is the velocity potential in the outer fluid domain, and it is given from (3.71), as

$$\phi^o = \left(r + \frac{1}{r}\right) H(T)\cos\theta - 2\epsilon A\frac{1}{r}\cos\theta + O(\epsilon^2). \tag{3.78}$$

Substituting (3.78) into (3.77) yields

$$\frac{p^o}{\rho} = \frac{2}{r}\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta + \left(\frac{1}{r^2} - \frac{1}{2r^4} - \frac{2}{r^2}\sin^2\theta\right) H(T) + O(\epsilon). \tag{3.79}$$

Setting $r = 1 + \epsilon R$, we have

$$\frac{p^{oi}}{\rho} = 2\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta - 2H(T)\sin^2\theta + \frac{1}{2}H(T) + O(\epsilon). \tag{3.80}$$

Since $p_0^{oi} = C + (\rho/2)\varOmega^2 H(T) + O(\epsilon)$ for $R < 0$ from (3.50*a–c*), the function p_0^i is written as, taking into account (3.54*a*),

$$\frac{p_0^i}{\rho} = \begin{cases} 2\dfrac{\mathrm{d}A}{\mathrm{d}T}\cos\theta - 2H(T)\sin^2\theta + \dfrac{1}{2}H(T) & \text{for } R \geq 0, \\ \dfrac{C}{\rho} + \dfrac{1}{2}\varOmega^2 H(T) & \text{for } R < 0. \end{cases} \tag{3.81}$$

Equation (3.76*b*) can then be rewritten as

$$\frac{\partial u_{\theta 1}^i}{\partial T} - \nu_o\frac{\partial^2 u_{\theta 1}^i}{\partial R^2} + \lambda_o\chi u_{\theta 1}^i = \left\{\begin{array}{ll} 2\dfrac{\mathrm{d}A}{\mathrm{d}T}\sin\theta + 4H(T)\sin\theta\cos\theta & \text{for } R \geq 0 \\ 0 & \text{for } R < 0 \end{array}\right\} - u_{r1}^i\frac{\partial u_{\theta 0}^i}{\partial R}$$
$$- u_{\theta 0}^i\frac{\partial u_{\theta 0}^i}{\partial \theta} + \nu_o\frac{\partial u_{\theta 0}^i}{\partial R} + \lambda_o\chi(\varOmega R H(T)). \tag{3.82}$$

The second-order solution $u_{\theta 1}^i$ is therefore found to be expressed as, with respect to θ,

$$u_{\theta 1}^i = \hat{u}_{\theta 1}^0 + \hat{u}_{\theta 1}^c\cos\theta + \hat{u}_{\theta 1}^s\sin\theta + \hat{u}_{\theta 1}^{cs}\sin\theta\cos\theta. \tag{3.83}$$

Substituting (3.83) into (3.82), we can write (3.82) separately:

$$\frac{\partial \hat{u}_{\theta 1}^0}{\partial T} - \nu_o\frac{\partial^2 \hat{u}_{\theta 1}^0}{\partial R^2} + \lambda_o\chi\hat{u}_{\theta 1}^0 = \nu_o\frac{\partial \hat{u}_{\theta 0}^0}{\partial R} + \lambda_o\chi R\varOmega H(T), \tag{3.84a}$$

$$\frac{\partial \hat{u}_{\theta 1}^c}{\partial T} - \nu_o\frac{\partial^2 \hat{u}_{\theta 1}^c}{\partial R^2} + \lambda_o\chi\hat{u}_{\theta 1}^c = -\hat{u}_{r1}\frac{\partial \hat{u}_{\theta 0}^0}{\partial R} - \hat{u}_{\theta 0}^0\hat{u}_{\theta 0}^s, \tag{3.84b}$$

$$\frac{\partial \hat{u}_{\theta 1}^s}{\partial T} - \nu_o\frac{\partial^2 \hat{u}_{\theta 1}^s}{\partial R^2} + \lambda_o\chi\hat{u}_{\theta 1}^s = \nu_o\frac{\partial \hat{u}_{\theta 0}^s}{\partial R} + \left\{\begin{array}{ll} 2\dfrac{\mathrm{d}A}{\mathrm{d}T} & \text{for } R \geq 0 \\ 0 & \text{for } R < 0 \end{array}\right\}, \tag{3.84c}$$

$$\frac{\partial \hat{u}_{\theta 1}^{cs}}{\partial T} - \nu_o\frac{\partial^2 \hat{u}_{\theta 1}^{cs}}{\partial R^2} + \lambda_o\chi\hat{u}_{\theta 1}^{cs} = -\hat{u}_{r1}\frac{\partial \hat{u}_{\theta 0}^s}{\partial R} - \hat{u}_{\theta 0}^{s2} + \left\{\begin{array}{ll} 4H(T) & \text{for } R \geq 0 \\ 0 & \text{for } R < 0 \end{array}\right\}. \tag{3.84d}$$

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The boundary condition are written as, $R \to \infty$,

$$\hat{u}^{0}_{\theta 1} \to 0, \quad \hat{u}^{c}_{\theta 1} \to 0, \quad \hat{u}^{s}_{\theta 1} \to 2RH(T) + 2A, \quad \hat{u}^{cs}_{\theta 1} \to 0. \tag{3.85a–d}$$

As $R \to -\infty$, the boundary conditions are also written as

$$\hat{u}^{0}_{\theta 1} \to R\Omega H(T), \quad \hat{u}^{c}_{\theta 1} \to 0, \quad \hat{u}^{s}_{\theta 1} \to 0, \quad \hat{u}^{cs}_{\theta 1} \to 0. \tag{3.86a–d}$$

Using the Laplace transformation and the similar procedure for the derivation of the first approximations, we finally obtain the following solutions (the detailed derivation is described in Appendix C):

$$U^{0}_{\theta 1} = \begin{cases} -\dfrac{1}{2}\dfrac{\Omega}{s(a+\bar{a})}(\bar{a}R - 3)\,\mathrm{e}^{-aR} & \text{for } R \geq 0, \\ \dfrac{1}{2}\dfrac{\Omega}{s(a+\bar{a})}(aR - 3)\,\mathrm{e}^{\bar{a}R} + \dfrac{\Omega}{s}R & \text{for } R < 0. \end{cases} \tag{3.87}$$

The velocity, $\hat{u}^{0}_{\theta 1}$, is then found to be the same as the solution to the pure rotation problem obtained in (3.32).

For $u^{s}_{\theta 1}$, we have

$$U^{s}_{\theta 1} = \begin{cases} 2\dfrac{R}{s} + 2\dfrac{a-\bar{a}}{sa\bar{a}} - \dfrac{\bar{a}}{s(a+\bar{a})}R\mathrm{e}^{-aR} + D_s\mathrm{e}^{-aR} & \text{for } R \geq 0, \\ \dfrac{a}{s(a+\bar{a})}R\mathrm{e}^{\bar{a}R} + E_s\mathrm{e}^{\bar{a}R} & \text{for } R < 0, \end{cases} \tag{3.88}$$

where

$$D_s = \frac{2}{sa} - \frac{3}{s(a+\bar{a})}, \quad E_s = \frac{2}{s\bar{a}} - \frac{3}{s(a+\bar{a})}. \tag{3.89a,b}$$

It is then found that the velocity $\hat{u}^{s}_{\theta 1}$ is the same as the solution to the pure translation problem analysed in our previous paper (see Ueda & Kida 2021).

We define the following relations that are written as

$$U^{c}_{\theta 1} = \mathcal{L}(\hat{u}^{c}_{\theta 1}) \quad \text{and} \quad F^{c} = \mathcal{L}\left(\hat{u}_{r1}\frac{\partial \hat{u}^{0}_{\theta 0}}{\partial R} + \hat{u}^{0}_{\theta 0}\hat{u}^{s}_{\theta 0}\right). \tag{3.90a,b}$$

Then, we can have the following relations:

$$U^{c}_{\theta 1} = \begin{cases} -\dfrac{1}{2\nu_o a}\left(\mathrm{e}^{aR}\displaystyle\int_R^{\infty} F^{c}\mathrm{e}^{-aR}\,\mathrm{d}R + \mathrm{e}^{-aR}\int_0^{R} F^{c}\mathrm{e}^{aR}\,\mathrm{d}R\right) + D_c\mathrm{e}^{-aR}, & R \geq 0, \\ -\dfrac{1}{2\nu_o \bar{a}}\left(\mathrm{e}^{-\bar{a}R}\displaystyle\int_{-\infty}^{R} F^{c}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \mathrm{e}^{\bar{a}R}\int_0^{R} F^{c}\mathrm{e}^{-\bar{a}R}\,\mathrm{d}R\right) + E_c\mathrm{e}^{\bar{a}R}, & R < 0, \end{cases} \tag{3.91}$$

where D_c and E_c are respectively given by

$$D_c = \frac{\bar{a} - a}{2\nu_o a(a+\bar{a})}\int_0^{\infty} F^{c}\mathrm{e}^{-aR}\,\mathrm{d}R - \frac{1}{\nu_o(a+\bar{a})}\int_{-\infty}^{0} F^{c}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R, \tag{3.92a}$$

$$E_c = -\frac{1}{\nu_o(a+\bar{a})}\int_0^{\infty} F^{c}\mathrm{e}^{-aR}\mathrm{d}R + \frac{a-\bar{a}}{2\nu_o \bar{a}(a+\bar{a})}\int_{-\infty}^{0} F^{c}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R. \tag{3.92b}$$

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We also define $\hat{U}^{cs}_{\theta 1}$ and F^{cs} as

$$U^{cs}_{\theta 1} = \mathcal{L}(\hat{u}^{cs}_{\theta 1}) \quad \text{and} \quad F^{cs} = \mathcal{L}\left(\hat{u}_{r1}\frac{\partial \hat{u}^{s}_{\theta 0}}{\partial R} + \begin{Bmatrix} \hat{u}^{s2}_{\theta 0} - 4H(T) & \text{for } R \geq 0 \\ \hat{u}^{s2}_{\theta 0} & \text{for } R < 0 \end{Bmatrix}\right). \tag{3.93a,b}$$

Then, we can obtain

$$U^{cs}_{\theta 1} = \begin{cases} -\dfrac{1}{2\nu_o a}\left(\mathrm{e}^{aR}\displaystyle\int_R^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \mathrm{e}^{-aR}\int_0^R F^{cs}\mathrm{e}^{aR}\,\mathrm{d}R\right) + D_{cs}\mathrm{e}^{-aR} & \text{for } R \geq 0, \\ -\dfrac{1}{2\nu_o \bar{a}}\left(\mathrm{e}^{-\bar{a}R}\displaystyle\int_{-\infty}^R F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \mathrm{e}^{\bar{a}R}\int_0^R F^{cs}\mathrm{e}^{-\bar{a}R}\,\mathrm{d}R\right) + E_{cs}\mathrm{e}^{\bar{a}R} & \text{for } R < 0, \end{cases} \tag{3.94}$$

in which the constants D_{cs} and E_{cs} are respectively given by

$$D_{cs} = \frac{\bar{a}-a}{2\nu_o a(a+\bar{a})}\int_0^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R - \frac{1}{\nu_o(a+\bar{a})}\int_{-\infty}^0 F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R, \tag{3.95a}$$

$$E_{cs} = -\frac{1}{\nu_o(a+\bar{a})}\int_0^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{a-\bar{a}}{2\nu_o\bar{a}(a+\bar{a})}\int_{-\infty}^0 F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R. \tag{3.95b}$$

Based on the solutions obtained, each velocity component is found to asymptotically behave like, for $\lambda_o \gg 1$,

$$\hat{u}^{0}_{\theta 0} \sim \begin{cases} \varOmega\,\mathrm{erfc}\left(\dfrac{R}{2\sqrt{\nu_o T}}\right): & R \geq 0, \\ \varOmega H(T): & R < 0. \end{cases} \tag{3.96a}$$

$$\hat{u}^{s}_{\theta 0} \sim \begin{cases} 2\,\mathrm{erfc}\left(\dfrac{R}{2\sqrt{\nu_o T}}\right) - 2H(T): & R \geq 0, \\ 0: & R < 0. \end{cases} \tag{3.96b}$$

$$\hat{u}_{r1} \sim \begin{cases} 2RH(T) - 4\sqrt{\dfrac{\nu_o T}{\pi}} + 4\sqrt{\dfrac{\nu_o T}{\pi}}\exp\left(-\dfrac{R^2}{4\nu_o T}\right) - 2R\,\mathrm{erfc}\left(\dfrac{R}{2\sqrt{\nu_o T}}\right): & R \geq 0, \\ 0: & R < 0. \end{cases} \tag{3.96c}$$

$$\hat{u}^{0}_{\theta 1} \sim \begin{cases} -\dfrac{1}{2}\varOmega R\,\mathrm{erfc}\left(\dfrac{R}{2\sqrt{\nu_o T}}\right): & R \geq 0, \\ \varOmega RH(T): & R < 0. \end{cases} \tag{3.96d}$$

$$\hat{u}^{s}_{\theta 1} \sim \begin{cases} 2RH(T) - 4\sqrt{\dfrac{\nu_o T}{\pi}} + 4\sqrt{\dfrac{\nu_o T}{\pi}}\exp\left(-\dfrac{R^2}{4\nu_o T}\right) - 3R\,\mathrm{erfc}\left(\dfrac{R}{2\sqrt{\nu_o T}}\right): & R \geq 0, \\ 0: & R < 0. \end{cases} \tag{3.96e}$$

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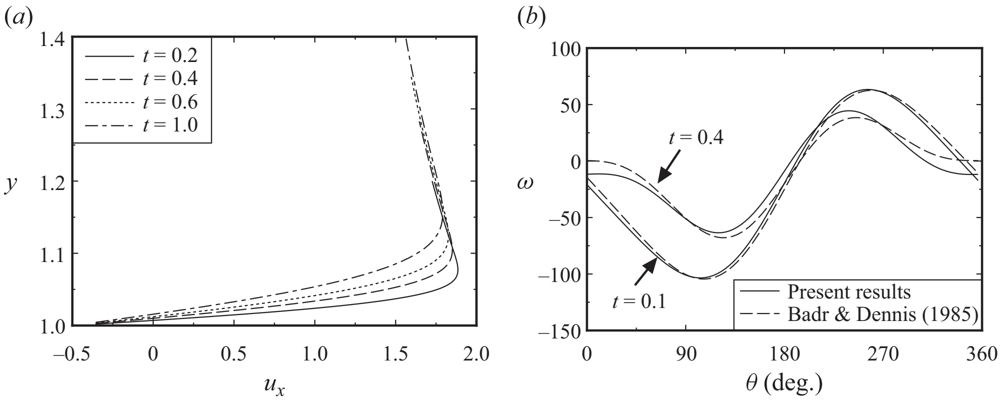

Figure 4. Velocity distribution of u_x on the y axis (a) and the vorticity distribution on the cylinder surface (b) in the case of $\nu = 1/500$ and $\Omega = 0.5$.

$$\hat{u}^{c}_{\theta 1} \sim \begin{cases} -\dfrac{1}{2\sqrt{\pi \nu_o}} \displaystyle\int_0^\infty \mathrm{d}R' \int_0^T \dfrac{f^c(\xi, R')}{\sqrt{T-\xi}} \left[\exp\left(-\dfrac{(R-R')^2}{4\nu_o(T-\xi)}\right) \right. \\ \qquad \left. - \exp\left(-\dfrac{(R+R')^2}{4\nu_o(T-\xi)}\right)\right] \mathrm{d}\xi : & R \geq 0, \\ 0 : & R < 0. \end{cases} \tag{3.96f}$$

$$\hat{u}^{cs}_{\theta 1} \sim \begin{cases} -\dfrac{1}{2\sqrt{\pi \nu_o}} \displaystyle\int_0^\infty \mathrm{d}R' \int_0^T \dfrac{f^{cs}(\xi, R')}{\sqrt{T-\xi}} \left[\exp\left(-\dfrac{(R-R')^2}{4\nu_o(T-\xi)}\right) \right. \\ \qquad \left. - \exp\left(-\dfrac{(R+R')^2}{4\nu_o(T-\xi)}\right)\right] \mathrm{d}\xi : & R \geq 0, \\ 0 : & R < 0. \end{cases} \tag{3.96g}$$

Figure 4(a) shows the velocity distribution of u_x on the y axis as $\lambda_o \to \infty$ in the case of $\nu = 1/500$ and $\Omega = 0.5$. Note here that the definition of the Reynolds number accounts for the fact that the quantity $1/\nu$ is replaced by $2/\nu$ for the Reynolds number used in Badr & Dennis (1985). Since the velocity u_x on the y axis is equal to u_θ at $\theta = \pi/2$, it can be obtained as the composite velocity, $u^i_\theta + u^o_\theta - u^{oi}_\theta$ at $\theta = \pi/2$, i.e.

$$u_x \approx (\Omega + 2)\,\mathrm{erfc}(\eta) - 1 - \frac{1}{r^2} - \frac{\Omega}{2}(r-1)\,\mathrm{erfc}(\eta) \tag{3.97}$$

$$+ 4\sqrt{\frac{\nu t}{\pi}}\left(\mathrm{e}^{-\eta^2} - \frac{1}{r^2}\right) - 3(r-1)\,\mathrm{erfc}(\eta) \quad \text{with } \eta = \frac{r-1}{2\sqrt{\nu t}}. \tag{3.98}$$

The vorticity distribution on the cylinder surface is shown on figure 4(b). The solid line denotes the present results and the dotted line is the result of Badr & Dennis (1985) that is written as

$$\omega|_{r=1} = \frac{\Omega}{\sqrt{\nu t}}\left(\frac{1}{\sqrt{\pi}} - \frac{\sqrt{\nu t}}{2}\right) + \left(\frac{2}{\sqrt{\pi \nu t}} + 1\right)\sin\theta \tag{3.99}$$

$$+ \Omega t\left(2.7844 - \frac{8}{3\sqrt{\pi \nu t}}\right)\cos\theta + t\left[6.5577 - \frac{2}{\sqrt{\pi \nu t}}\left(1 + \frac{4}{3\pi}\right)\right]\sin 2\theta. \tag{3.100}$$

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Since the vorticity distribution is then obtained from the inner solutions, i.e.

$$\omega = \frac{1}{\epsilon}\frac{\partial u^i_{\theta 0}}{\partial R} + \frac{\partial u^i_{\theta 1}}{\partial R} + u^i_{\theta 0} + O(\epsilon), \tag{3.101}$$

we have

$$\omega|_{r=1} \approx -\frac{\Omega + 2\sin\theta}{\sqrt{\pi \nu t}} - \frac{1}{2}\Omega - \sin\theta - 8\Omega\sqrt{\frac{t}{\pi\nu}}c_1\cos\theta - 2\sqrt{\frac{t}{\pi\nu}}c_2\sin 2\theta, \tag{3.102}$$

in which the coefficients c_1 and c_2 are given as $c_1 = 0.0452900$ and $c_2 = -1.34657$. We find that the leading term of the vorticity distribution is equal to the results of Badr & Dennis (1985).

We obtain the second-order solution of the pressure p^i_1 that is governed by

$$\frac{\partial u^i_{r1}}{\partial T} - u^{i2}_{\theta 0} = -\frac{1}{\rho}\frac{\partial p^i_1}{\partial R} - \lambda_o \chi u^i_{r1} + \nu_o \frac{\partial^2 u^i_{r1}}{\partial R^2}. \tag{3.103}$$

Inside the body domain (i.e. $R < 0$), the inner expansion of the outer solution of the pressure is readily found to be written as

$$\frac{p^{io}_1}{\rho} = \Omega^2 R H(T). \tag{3.104}$$

Furthermore, because of $u^{io}_{\theta 0} = \Omega H(T)$ and $u^{io}_{r1} = 0$, (3.103) is rewritten as

$$\frac{p^i_1}{\rho} = -\frac{\partial}{\partial T}\int_{-\infty}^{R} u^i_{r1}\,\mathrm{d}R - \lambda_o \int_{-\infty}^{R} u^i_{r1}\,\mathrm{d}R + \nu_o \frac{\partial u^i_{r1}}{\partial R}$$
$$+ \int_{-\infty}^{R} (u^{i2}_{\theta 0} - \Omega^2 H(T))\,\mathrm{d}R + \Omega^2 R H(T). \tag{3.105}$$

For $R < 0$, the functions u_{r1} and $u_{\theta 0}$ are given by (3.96*a*)–(3.96*c*) for $\lambda_o \gg 1$. Therefore, (3.105) can be calculated as

$$\frac{p^i_1}{\rho} \sim \Omega^2 R H(T) + O(1/\sqrt{\lambda_o}) \quad \text{for } \lambda_o \gg 1. \tag{3.106}$$

For $R > 0$, we find that $u^i_{r1} \sim 2(A + RH(T))\cos\theta$ and $u^i_{\theta 0} \sim -2\sin\theta H(T)$ as $R \to \infty$ from (3.61)–(3.68). Therefore, (3.103) is rewritten as

$$\frac{1}{\rho}\frac{\partial p^i_1}{\partial R} = -\frac{\partial}{\partial T}[u^i_{r1} - (2A + 2RH(T))\cos\theta] + (u^{i2}_{\theta 0} - 4H(T)\sin^2\theta)$$
$$+ \nu_o \frac{\partial^2 u^i_{r1}}{\partial R^2} - 2\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta + 4H(T)\sin^2\theta. \tag{3.107}$$

Integrating (3.107) with respect to R, we have

$$\frac{p^i_1}{\rho} = \frac{\partial}{\partial T}\int_R^\infty [u^i_{r1} - (2A + 2RH(T))\cos\theta]\mathrm{d}R - \int_R^\infty [u^{i2}_{\theta 0} - 4H(T)\sin^2\theta]\mathrm{d}R$$
$$+ \nu_o \frac{\partial u^i_{r1}}{\partial R} - 2\frac{\mathrm{d}A}{\mathrm{d}T}R\cos\theta + 4RH(T)\sin^2\theta + C_1(\theta, T), \tag{3.108}$$

where C_1 is an integral constant that can be determined from the matching procedure to the inner expansion of the outer pressure p^{oi}. The derivations of the pressure p^o_1 and p^{oi}

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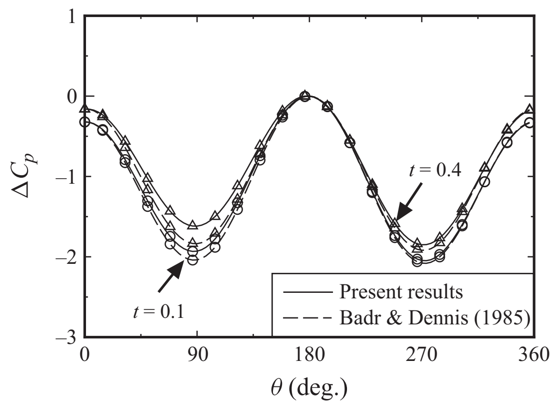

Figure 5. Comparison of the pressure distribution $\Delta C_p = (p(\theta) - p(\pi))/\rho$ on the cylinder surface between the present results and Badr & Dennis (1985) in the case of $\nu = 1/500$ and $\varOmega = 0.5$.

are described in Appendix D. Therefore, we have

$$C_1 = -2\nu_o H(T)\cos\theta + 4A\sin^2\theta - \frac{\mathrm{d}}{\mathrm{d}T}(a_1^c\cos\theta + a_1^s\sin\theta) - \frac{\mathrm{d}}{\mathrm{d}T}(a_2^c\cos 2\theta), \quad (3.109)$$

where a_1^c, a_1^s and a_2^c are given in Appendix D. Using (3.105) and (3.108), we have the pressure jump of $\Delta p_1 = p_1^i|_{r=1+0} - p_1^i|_{r=1-0}$ as

$$\frac{\Delta p_1}{\rho} = \frac{\partial}{\partial T}\mathrm{pf}\int_{-\infty}^{\infty} u_{r1}^i\,\mathrm{d}R + \lambda_o\int_{-\infty}^{0} u_{r1}^i\,\mathrm{d}R - \mathrm{pf}\int_{-\infty}^{\infty} u_{\theta 0}^{i2}\,\mathrm{d}R + C_1, \quad (3.110)$$

where the symbol pf denotes the integration in the finite part sense of Hadamard (see Hadamard 1932).

Figure 5 shows the pressure distribution on the cylinder surface. The present result is given by

$$\begin{aligned}\frac{p - p_\infty}{\rho} \approx{}& -2\sqrt{\frac{\nu}{\pi t}}\cos\theta + \frac{1}{2} - 2\sin^2\theta \\ & - 16\sqrt{\frac{\nu t}{\pi}}(\varOmega + \sin\theta)\left[\varOmega\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\sin\theta\right] \\ & - \nu\cos\theta - 8\sqrt{\frac{\nu t}{\pi}}\sin^2\theta + \left[4\varOmega\sqrt{\frac{\nu t}{\pi}}f_m^c - 4\varOmega\sqrt{\frac{\nu t}{\pi}}(3 - 2\sqrt{2})\right]\sin\theta \\ & + \left[2\sqrt{\frac{\nu t}{\pi}}f_m^{cs} - 8\sqrt{\frac{\nu t}{\pi}}(3 - 2\sqrt{2})\right]\cos 2\theta. \quad (3.111)\end{aligned}$$

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Thus, we have the alternative expression to compare with the result of Badr & Dennis (1985):

$$\Delta C_p = \frac{p(\theta) - p(\pi)}{\rho} \approx -2\sqrt{\frac{\nu}{\pi t}}(1 + \cos\theta) - 2\sin^2\theta - 16\Omega\sqrt{\frac{\nu t}{\pi}}(1 - \sqrt{2})\sin\theta$$
$$- 8\sqrt{\frac{\nu t}{\pi}}\sin^2\theta + \nu(\cos\theta - 1) + \sqrt{\frac{\nu t}{\pi}}[4\Omega f_m^c - 4\Omega(3 - 2\sqrt{2})]\sin\theta$$
$$+ \sqrt{\frac{\nu t}{\pi}}[2f_m^{cs} - 8(1 - 2\sqrt{2})](\cos 2\theta - 1). \quad (3.112)$$

Here $f_m^c = 0.21483$ and $f_m^{cs} = -2.69143$. In contrast, the result of Badr & Dennis (1985) is given by

$$\Delta C_p \approx -2\sqrt{\frac{\nu}{\pi t}}(1 + \cos\theta) - \left(1 - \frac{\sqrt{\nu t}}{4}q_2\right)(1 - \cos 2\theta) + \frac{1}{2}\sqrt{\nu t}\Omega q_1 \sin\theta, \quad (3.113)$$

in which the coefficients q_1 and q_2 are written as $q_1 = 5.79901$ and $q_2 = 14.3122$. Note here that the leading term is found to be the same between both results.

3.4. *Hydrodynamic forces*

This section attempts to investigate the variance of the drag force against the results of Bar-Lev & Yang (1975), which is demonstrated in our previous paper (see Ueda & Kida 2021). We calculate the hydrodynamic force $\boldsymbol{F} = (F_x, F_y)$ by the following two approaches: (i) by the time derivative of the momentum of the fluid flow and the pressure on the control surface, and (ii) by the integration of the penalization layer given by (2.3). In this section we intend to first investigate the drag force F_x and, then, the lift force F_y. In the subsequent § 3.5 the relation between approaches (i) and (ii) will be presented for the solid cylindrical body having a continuous contour.

In approach (i) we set the circular control surface around the circular cylinder, of which the centre is set at the origin, with a large radius of r_∞. The fluid domain is then defined as $\mathscr{F} = [(r, \theta)|1 \le r \le r_\infty, 0 \le \theta < 2\pi]$. The small fluid element $\mathrm{d}m = r\,\mathrm{d}r\,\mathrm{d}\theta$ in the fluid domain $\mathscr{F}$ has the momentum $\rho\boldsymbol{u}\,\mathrm{d}m$ and, therefore, the force of $-(\mathrm{d}/\mathrm{d}t)\int_1^{r_\infty}\rho\boldsymbol{u}r\,\mathrm{d}r\,\mathrm{d}\theta$ affects the fluid in $\mathscr{F}$. Furthermore, the fluid experiences the force exerted by the pressure on the control surface having the radius of r_∞, i.e. $-\int_0^{2\pi} p(\cos\theta, \sin\theta)r_\infty\,\mathrm{d}\theta$. Therefore, the hydrodynamic force $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = (F_x, F_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_1^{r_\infty}\rho(u, v)r\,\mathrm{d}r\,\mathrm{d}\theta - \int_0^{2\pi} p(\cos\theta, \sin\theta)r_\infty\,\mathrm{d}\theta, \quad (3.114)$$

where $\boldsymbol{u} = (u, v)$ is the velocity defined on the (x, y) plane.

In the outer fluid domain for $r > 1$, we have, from (D11),

$$\frac{\partial\phi^o}{\partial T} = -2\frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta + \epsilon\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}T}(a_1^c\cos\theta + a_1^s\sin\theta) + \epsilon\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}T}(a_2^c\cos 2\theta) + O(\epsilon^2). \quad (3.115)$$

Here, the r and θ components of the velocity (i.e. u_r^o and u_θ^o) in the outer fluid domain (for $r > 1$) are obtained by (D12*a*) and (D12*b*). Therefore, we find that

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$|\boldsymbol{u}|^2 = 1 + O(1/r^2) + O(\epsilon^2)$. Using (3.115) with (3.72), the Laplace transform to unsteady Bernoulli's equation yields the pressure for $r \gg 1$:

$$\mathcal{L}\left(\frac{p}{\rho}\right) \sim 2\frac{a-\bar{a}}{a\bar{a}}\frac{1}{r}\cos\theta - \epsilon[s\mathcal{L}(a_1^c)\cos\theta + s\mathcal{L}(a_1^s)\sin\theta]\frac{1}{r} + O(1/r^2). \tag{3.116}$$

The drag force F_x of (3.114) is calculated as, taking the limit of $r_\infty \gg 1$,

$$\begin{aligned} F_x = &-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_1^\infty \rho u r\,\mathrm{d}r\,\mathrm{d}\theta - 2\rho\sqrt{\frac{\nu_o\pi}{T}}\left(\mathrm{e}^{-\lambda_o T} - 1\right) \\ &+ \pi\rho\nu_o\epsilon[(1+\mathrm{e}^{-\lambda_o T})H(T) - 2\mathrm{e}^{-\lambda_o T/2}\mathrm{I}_0(\lambda_o T/2)] + O(\epsilon^2). \end{aligned} \tag{3.117}$$

Note that $s\mathcal{L}(a_1^c) = \nu_o[1/s + 1/(s+\lambda_o) - 2/\sqrt{s(s+\lambda_o)}]$ from (D9*a*). The first term on the right-hand side of (3.117), which is denoted by F_{x1}, is calculated as

$$\begin{aligned} F_{x1} &= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_1^\infty (u_r\cos\theta - u_\theta\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_1^\infty (u_r^o\cos\theta - u_\theta^o\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &\quad - \rho\frac{\mathrm{d}}{\mathrm{d}T}\int_0^{2\pi}\int_0^\infty [(u_r^i - u_r^{io})\cos\theta - (u_\theta^i - u_\theta^{io})\sin\theta](1+\epsilon R)\,\mathrm{d}R\,\mathrm{d}\theta. \end{aligned} \tag{3.118}$$

The first term on the right-hand side of (3.118) is calculated from the outer solution of the velocity u_r^o and u_θ^o:

$$\int_0^{2\pi}\int_1^\infty (u_r^o\cos\theta - u_\theta^o\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi + O(\epsilon^2). \tag{3.119}$$

It can be found that the outer solutions do not affect F_{x1} within the order of ϵ. Therefore, F_{x1} is calculated as, using the Laplace transform,

$$\begin{aligned} \mathcal{L}(F_{x1}) = &-\pi\rho s\int_0^\infty \{-U_{\theta 0}^s + U_{\theta 0}^{sio} + \epsilon[U_{r1} - U_{r1}^{io} - U_{\theta 1}^s + U_{\theta 1}^{sio} - R(U_{\theta 0}^s - U_{\theta 0}^{sio})]\}\mathrm{d}R \\ &+ O(\epsilon^2). \end{aligned} \tag{3.120}$$

Here, the following relations are obtained from the inner solutions:

$$\left.\begin{aligned} \int_0^\infty (U_{\theta 0}^s - U_{\theta 0}^{sio})\,\mathrm{d}R &= \int_0^\infty \frac{2\bar{a}}{s(a+\bar{a})}\mathrm{e}^{-aR}\,\mathrm{d}R = \frac{2\bar{a}}{sa(a+\bar{a})}, \\ \int_0^\infty R(U_{\theta 0}^s - U_{\theta 0}^{sio})\,\mathrm{d}R &= \int_0^\infty \frac{2\bar{a}}{s(a+\bar{a})}R\mathrm{e}^{-aR}\,\mathrm{d}R = \frac{2\bar{a}}{sa^2(a+\bar{a})}, \\ \int_0^\infty (U_{r1} - U_{r1}^{io})\,\mathrm{d}R &= \int_0^\infty \frac{2\bar{a}}{sa(a+\bar{a})}\mathrm{e}^{-aR}\,\mathrm{d}R = \frac{2\bar{a}}{sa^2(a+\bar{a})}, \\ \int_0^\infty (U_{\theta 1}^s - U_{\theta 1}^{sio})\,\mathrm{d}R &= \frac{\bar{a}}{sa^2(a+\bar{a})} + \frac{D_s}{a}. \end{aligned}\right\} \tag{3.121}$$

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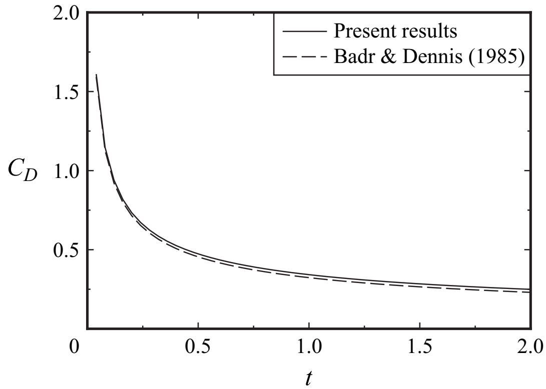

Figure 6. Comparison of the drag coefficient C_D with the result of Badr & Dennis (1985) in the case of $\nu = 1/500$ and $\Omega = 0.5$.

Taking the inverse Laplace transform to (3.120), we obtain

$$F_{x1} = \pi\rho\left\{2\sqrt{\frac{\nu_o}{\pi T}} - \sqrt{\frac{\nu_o}{\pi T^3}}\frac{1-\mathrm{e}^{-\lambda_o T}}{\lambda_o} + \epsilon\nu_o[3H(T) - 2\mathrm{e}^{-\lambda_o T/2}(\mathrm{I}_0(\lambda_o T/2) + \mathrm{I}_1(\lambda_o T/2))]\right\} + O(\epsilon^2). \quad (3.122)$$

Substituting (3.122) into (3.117), we find that F_x behaves like

$$F_x \sim 4\rho\sqrt{\frac{\pi\nu_o}{T}} \quad \text{for } \lambda_o \gg 1. \quad (3.123)$$

The result of (3.123) is found to be the same as that obtained by Bar-Lev & Yang (1975) and our previous study (see Ueda & Kida 2021) for an impulsively started translating circular cylinder without rotating motion. As found by the above analysis, the force exerted by the pressure far from the body plays an important role for the calculation of the hydrodynamic force. Figure 6 shows the comparison of the drag coefficient C_D, which is calculated by $C_D = F_x/\rho$ in the present paper, between the result of Badr & Dennis (1985) (i.e. $C_D = 4\sqrt{\pi\nu/t} + \pi\nu$) and the leading term of the present result as $\lambda_o \to \infty$. In the previous study (see Ueda & Kida 2021), the drag force F_x was obtained by the following two equations:

$$\frac{F_x}{\rho} = \nu\int_0^{2\pi}\left.\left(\frac{\partial\omega}{\partial r} - \omega\right)\right|_{r=1}\sin\theta\,\mathrm{d}\theta, \quad (3.124)$$

$$\frac{F_x}{\rho} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}} y\omega\,\mathrm{d}v. \quad (3.125)$$

These relations are derived by imposing the no-slip condition on the cylinder surface. In the penalization method the no-slip condition is not imposed. Therefore, these formulae are not directly available for the penalization method. In Appendix F we note an additional

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term to the above relations in the penalization method, i.e.

$$\frac{F_{bx}}{\rho} = \nu \int_0^{2\pi} \left(\frac{\partial \omega}{\partial r} - \omega \right)\bigg|_{r=1} \sin\theta \, d\theta - \frac{d}{dt} \int_0^{2\pi} u_\theta|_{r=1} \sin\theta \, d\theta - \int_0^{2\pi} u_r|_{r=1} \, u|_{r=1} \, d\theta. \tag{3.126}$$

Here, let us calculate the hydrodynamic force by approach (ii). The x component of the hydrodynamic force is denoted by F_{px} and, then, it is written as, from (2.3),

$$F_{px} = \pi\rho\lambda_o \int_{-\infty}^{0} [-\hat{u}^s_{\theta 0} + \epsilon(\hat{u}_{r1} - \hat{u}^s_{\theta 1} - R\hat{u}^s_{\theta 0}) + O(\epsilon^2)] dR. \tag{3.127}$$

The outer solution inside the body domain is governed by the rotational motion of the cylinder alone so that it cannot affect the x component of the force. Using the result of the Laplace transform to (3.127), which is written as

$$\mathcal{L}(F_{px}) = \pi\rho\lambda_o \left[2\frac{a}{s\bar{a}(a+\bar{a})} + \epsilon \frac{\bar{a}-a}{s\bar{a}(a+\bar{a})^2} + O(\epsilon^2) \right], \tag{3.128}$$

we have

$$F_{px} = \pi\rho \left[2\sqrt{\frac{\nu_o}{\pi T}} - 2\sqrt{\frac{\nu_o}{\pi T}} e^{-\lambda_o T} + \epsilon\nu_o \left(H(T) + e^{-\lambda_o T} - 2e^{-\lambda_o T/2} I_0(\lambda_o T/2) \right) + O(\epsilon^2) \right]. \tag{3.129}$$

Also, F_{px} behaves like

$$F_{px} \to 2\rho\sqrt{\frac{\pi\nu_o}{T}} \quad \text{as } \lambda_o \to \infty. \tag{3.130}$$

It can be found that the result of (3.130) exhibits the half-value of F_x, and this finding is the same as our previous study (see Ueda & Kida 2021).

Let us calculate the lift force F_y by approach (i) that uses the force F_{my} exerted by the momentum of the fluid flow and the force F_{y1} exerted by the pressure on the circular control surface at $r = r_\infty$. The lift force F_{my} exerted by the momentum of the fluid flow is written as

$$\begin{aligned} F_{my} &= -\rho \frac{d}{dt} \int_0^{2\pi} \int_1^{\infty} (u_r \sin\theta + u_\theta \cos\theta) r \, dr \, d\theta \\ &= -\rho\pi \frac{d}{dT} \int_0^{\infty} [\epsilon(\hat{u}^c_{\theta 1} - \hat{u}^{cio}_{\theta 1}) + O(\epsilon^2)] dR. \end{aligned} \tag{3.131}$$

Making use of the Laplace transform to (3.131) and the integration by parts, we have

$$\begin{aligned} \mathcal{L}(F_{my}) &= -\rho\pi\epsilon s \left[-\frac{1}{2a\nu_o} \int_0^{\infty} e^{aR} \left(\int_R^{\infty} F^c e^{-aR'} dR' \right) dR \right. \\ &\quad \left. - \frac{1}{2a\nu_o} \int_0^{\infty} e^{-aR} \left(\int_0^{R} F^c e^{aR'} dR' \right) dR + \frac{D_c}{a} \right] + O(\epsilon^2) \\ &= -\rho\pi\epsilon \left[\frac{\bar{a}}{a+\bar{a}} \int_0^{\infty} F^c e^{-aR} dR - \int_0^{\infty} F^c dR - \frac{a}{a+\bar{a}} \int_{-\infty}^{0} F^c e^{\bar{a}R} dR \right] + O(\epsilon^2). \end{aligned} \tag{3.132}$$

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The lift force F_{y1} exerted by the pressure on the circular control surface at $r = r_\infty$ is calculated as, by the use of (3.116),

$$\frac{1}{\rho}\mathcal{L}(F_{y1}) = -\frac{1}{\rho}\int_0^{2\pi} \mathcal{L}(p)|_{r=r_\infty} r_\infty \sin\theta \, \mathrm{d}\theta = \pi\epsilon s \mathcal{L}(a_1^s) + O(\epsilon^2)$$
$$= \pi\epsilon s\left[-\frac{1}{\nu_o(a+\bar{a})}\left(\frac{1}{\bar{a}} + \frac{\bar{a}}{a^2}\right)\int_0^\infty F^c \mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{s}\int_0^\infty F^c\,\mathrm{d}R\right.$$
$$\left. + \frac{1}{\nu_o(a+\bar{a})}\left(\frac{1}{a} - \frac{1}{\bar{a}}\right)\int_{-\infty}^0 F^c \mathrm{e}^{\bar{a}R}\,\mathrm{d}R + \frac{1}{s+\lambda_o}\int_{-\infty}^0 F^c\,\mathrm{d}R\right]. \tag{3.133}$$

Therefore, we have, for $\lambda_o \gg 1$,

$$\frac{F_{my}}{\rho} \sim \frac{F_{y1}}{\rho} \sim \pi\epsilon\mathcal{L}^{-1}\left(\int_0^\infty F^c\,\mathrm{d}R - \int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R\right)$$
$$\sim \pi\epsilon\left(\int_0^\infty f^c(T,R)\,\mathrm{d}R - \frac{1}{2\sqrt{\pi\nu_o}}\int_0^\infty R\,\mathrm{d}R\int_0^T f^c(T-\xi,R)\frac{\exp\left(-\dfrac{R^2}{4\nu_o\xi}\right)}{\xi^{3/2}}\,\mathrm{d}\xi\right), \tag{3.134}$$

where $f^c = \mathcal{L}^{-1}(F^c)$.

Let us calculate the lift force F_{py} by approach (ii) that uses the integration of the penalization layer, as shown in (2.3). Then, the Laplace transform of F_{py} is written as

$$\mathcal{L}(F_{py}) = \rho\pi\epsilon\lambda_o\int_{-\infty}^0 U_{\theta 1}^c\,\mathrm{d}R + O(\epsilon^2)$$
$$= \rho\pi\epsilon\lambda_o\left(\frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^0 F^c\mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^0 F^c\,\mathrm{d}R + \frac{E_c}{\bar{a}}\right) + O(\epsilon^2)$$
$$= \rho\pi\epsilon\lambda_o\left(\frac{a}{\nu_o\bar{a}^2(a+\bar{a})}\int_{-\infty}^0 F^c\mathrm{e}^{\bar{a}R}\,\mathrm{d}R\right.$$
$$\left. - \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^0 F^c\,\mathrm{d}R - \frac{1}{\nu_o\bar{a}(a+\bar{a})}\int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R\right) + O(\epsilon^2). \tag{3.135}$$

For $\lambda_o \gg 1$, we have

$$\mathcal{L}(F_{py}) \sim -\rho\pi\epsilon\left(\int_{-\infty}^0 F^c\,\mathrm{d}R + \int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R\right). \tag{3.136}$$

From (3.134) (note that $F_y = F_{my} + F_{y1}$) and the inverse Laplace transform to (3.136), we have, for $\lambda_o \gg 1$,

$$F_y \sim 2\rho\pi\epsilon\left(\int_0^\infty f^c(T,R)\,\mathrm{d}R - \frac{1}{2\sqrt{\nu_o\pi}}\int_0^\infty R\,\mathrm{d}R\int_0^T f^c(T-\xi,R)\frac{\exp\left(-\dfrac{R^2}{4\nu_o\xi}\right)}{\xi^{3/2}}\,\mathrm{d}\xi\right), \tag{3.137a}$$

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$$F_{py} \sim -\rho \pi \epsilon \left(\int_{-\infty}^{0} f^c(T, R)\, \mathrm{d}R + \frac{1}{2\sqrt{\nu_o \pi}} \int_0^\infty R\, \mathrm{d}R \int_0^T f^c(T-\xi, R) \frac{\exp\left(-\dfrac{R^2}{4\nu_o \xi}\right)}{\xi^{3/2}}\, \mathrm{d}\xi \right). \tag{3.137b}$$

Note that the asymptotic behaviours, for $\lambda_o \gg 1$, of (3.137*a*)–(3.137*b*) are obtained by using the result of Appendix E.

Here, let us consider the behaviour of F_y for $\lambda_o \gg 1$ and $T \ll 1$ to describe F_y explicitly. To calculate (3.137*a*)–(3.137*b*), we need to obtain the function f^c to which the Laplace transform is given by (C11*a*,*b*). Using (3.96*a*)–(3.96*g*), f^c $(= \hat{u}_{r1}(\partial \hat{u}^0_{\theta 0}/\partial R) + \hat{u}^0_{\theta 0}\hat{u}^s_{\theta 0})$ is written as

$$f^c(T, R) = \begin{cases} -\dfrac{4\Omega}{\sqrt{\pi}} \mathrm{e}^{-\eta^2} \left(\eta\, \mathrm{erf}(\eta) + \dfrac{\mathrm{e}^{-\eta^2} - 1}{\sqrt{\pi}} \right) \\ \quad -2\Omega\, \mathrm{erf}(\eta)(1 - \mathrm{erf}(\eta)) + O(1/\sqrt{\lambda_o}) & \text{for } R \geq 0, \\ O(1/\sqrt{\lambda_o}) & \text{for } R < 0, \end{cases} \tag{3.138}$$

where $\eta = R/(2\sqrt{\nu_o T})$. To calculate (3.137*a*)–(3.137*b*), we define the functions S_1 and S_2 as

$$S_1 = \int_0^\infty f^c(T, R)\, \mathrm{d}R, \tag{3.139a}$$

$$S_2 = \frac{1}{2\sqrt{\pi \nu_o}} \int_0^\infty \int_0^T f^c(T-\xi, R) \frac{\exp\left(-\dfrac{R^2}{4\nu_o \xi}\right)}{\xi^{3/2}} R\, \mathrm{d}R\, \mathrm{d}\xi. \tag{3.139b}$$

Substituting (3.138) into (3.139*a*), S_1 is written as

$$S_1 = -4\Omega\sqrt{\nu_o T} \int_0^\infty \left\{ \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\eta^2} \left(\eta\, \mathrm{erf}(\eta) + \frac{\mathrm{e}^{-\eta^2} - 1}{\sqrt{\pi}} \right) + \mathrm{erf}(\eta)[1 - \mathrm{erf}(\eta)] \right\} \mathrm{d}\eta \\ + O(1/\sqrt{\lambda_o}). \tag{3.140}$$

Using the relations, which are calculated by integration by parts,

$$\int_0^\infty \eta \mathrm{e}^{-\eta^2}\, \mathrm{erf}(\eta)\, \mathrm{d}\eta = \frac{1}{2^{3/2}}, \quad \int_0^\infty \mathrm{erf}(\eta)[1 - \mathrm{erf}(\eta)]\, \mathrm{d}\eta = \frac{\sqrt{2} - 1}{\sqrt{\pi}}, \tag{3.141a,b}$$

the function S_1 can be obtained as

$$S_1 = -8\Omega \sqrt{\frac{\nu_o T}{\pi}} (\sqrt{2} - 1) + O(1/\sqrt{\lambda_o}). \tag{3.142}$$

Similarly, the function S_2 is written as, substituting (3.138) into (3.139*b*) and taking into account $f^c(T - \xi, 0) = 0$,

$$S_2 = \sqrt{\frac{\nu_o}{\pi}} \int_0^T \frac{\mathrm{d}\xi}{\sqrt{\xi}} \int_0^\infty \frac{\partial f^c(T-\xi, R)}{\partial R} \exp\left(-\frac{R^2}{4\nu_o \xi}\right) \mathrm{d}R + O(1/\sqrt{\lambda_o}) \\ = \frac{1}{2\sqrt{\pi}} \int_0^T \frac{\mathrm{d}\xi}{\sqrt{\xi(T-\xi)}} \int_0^\infty \frac{\partial f^c(\eta')}{\partial \eta'} \exp\left(-\frac{R^2}{4\nu_o \xi}\right) \mathrm{d}R + O(1/\sqrt{\lambda_o}), \tag{3.143}$$

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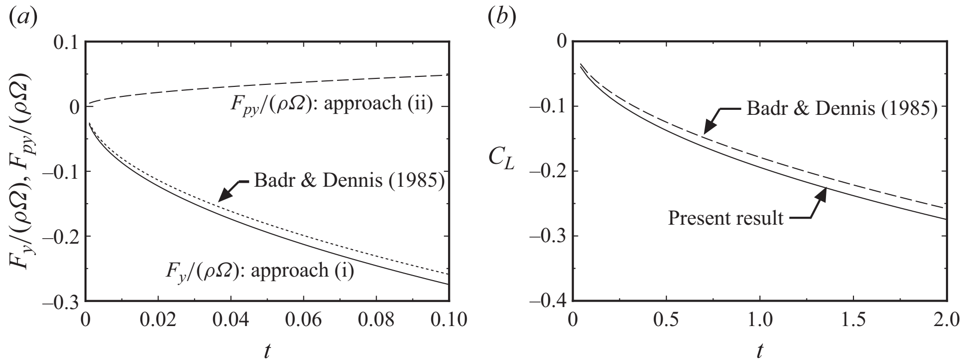

Figure 7. (*a*) Comparison of the lift force between F_y (by approach (i)) and F_{py} (by approach (ii)) against the asymptotic solution of Badr & Dennis (1985) at $Re = 100$ for $\lambda_o \gg 1$ and $t \ll 1$. (*b*) Comparison of the lift coefficient C_L with the result of Badr & Dennis (1985) in the case of $\nu = 1/500$ and $\Omega = 0.5$.

where $\eta' = \eta\sqrt{T/(T-\xi)}$. Note that the first integral in (3.143) is calculated by integration by parts. Changing the integral variable ξ to x by $x = \xi/T$, the function S_2 is rewritten as

$$S_2 = \sqrt{\frac{\nu_o T}{\pi}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} \int_0^\infty \frac{\partial f^c(\eta)}{\partial \eta} \exp\left(-\frac{1-x}{x}\eta^2\right) \mathrm{d}\eta + O(1/\sqrt{\lambda_o}). \tag{3.144}$$

Using the relation, which is calculated by changing the variable x to y by $x = \eta^2/y^2$,

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{e}^{-\eta^2/x}\, \mathrm{d}x = 2\eta \int_\eta^\infty \frac{\mathrm{e}^{-y^2}}{y^2}\, \mathrm{d}y = 2\mathrm{e}^{-\eta^2} - 2\sqrt{\pi}\eta[1 - \mathrm{erf}(\eta)], \tag{3.145}$$

the function S_2 reduces to

$$S_2 = 4\sqrt{\frac{\nu_o T}{\pi}} \int_0^\infty \frac{\mathrm{d}\hat{f}^c}{\mathrm{d}\eta} [\mathrm{e}^{-\eta^2} - \sqrt{\pi}\eta(1 - \mathrm{erf}(\eta))] \mathrm{e}^{\eta^2}\, \mathrm{d}\eta + O(1/\sqrt{\lambda_o}), \tag{3.146}$$

where

$$\frac{\mathrm{d}\hat{f}^c}{\mathrm{d}\eta} = \frac{4}{\sqrt{pi}}\eta^2 \mathrm{e}^{-\eta^2}\, \mathrm{erf}(\eta) + \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\eta^2}\, \mathrm{erf}(\eta) - \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\eta^2} + \frac{4}{\pi}\eta \mathrm{e}^{-2\eta^2} - \frac{4}{\pi}\eta \mathrm{e}^{-\eta^2}. \tag{3.147}$$

The value of the integral included in (3.146) can be estimated numerically by the Hermite quadrature formula of $n = 9$:

$$S_2 = 4\Omega\sqrt{\frac{\nu_o T}{\pi}} s_o + O(1/\sqrt{\lambda_o}), \quad s_o = -0.21578. \tag{3.148}$$

The lift forces F_y and F_{py} are therefore found to asymptotically behave like

$$F_y = 2\rho\pi\epsilon(S_1 - S_2) + O(\epsilon^2) \sim -8\rho\Omega\sqrt{\pi\nu t}[2(\sqrt{2} - 1) + s_o], \tag{3.149a}$$

$$F_{py} \sim -\rho\pi\epsilon S_2 + O(\epsilon^2) \sim -4\rho\Omega\sqrt{\pi\nu t} s_o. \tag{3.149b}$$

Figure 7 shows the comparison of the asymptotic behaviours of $F_y/(\rho\Omega)$ and $F_{py}/(\rho\Omega)$ given by (3.149*a*) and (3.149*b*) against the asymptotic solution obtained by Badr & Dennis (1985): $F_y/(\rho\Omega) \sim -2\sqrt{\nu t}[1.4499\pi - 4/(3\sqrt{\pi}) + 2.0 \times 0.6961\pi\sqrt{\nu t}]$. It seems that the

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result of approach (ii) (by the integration of the penalization layer) is completely different from the result of approach (i) (by the time derivative of the momentum of the fluid flow and the pressure on the control surface) that is similar to the asymptotic solution of Badr & Dennis (1985). In figure 7 the lift coefficient C_L, which is calculated by $C_L = F_y/\rho$ in the present paper, is also shown together with the result of Badr & Dennis (1985): $C_L = -8\Omega\sqrt{\pi \nu t}[2(\sqrt{2}-1) - 0.21578]$.

3.5. *Alternative formula of hydrodynamic force*

As seen in the preceding subsection § 3.4, the lift force calculated by approach (ii) yields a completely different value from the result of approach (i). In particular, the result of approach (ii) exhibits the opposite sign against the result of approach (i) and the asymptotic solution of Badr & Dennis (1985) (see figure 7). To resolve the variance, we attempt to derive an alternative formula to calculate the hydrodynamic forces.

In general, let us consider a two-dimensional domain $\mathscr{D}$ that consists of two closed boundaries $\mathscr{C}_o$ and $\mathscr{C}_i$ such that $\mathscr{C}_i \subset \mathscr{C}_o$. The hydrodynamic force due to the momentum of the fluid flow in the domain $\mathscr{D}$ is defined as $(F_x, F_y)_{\mathscr{D}}$ and, then, it is written as

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{D}} (u, v)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathscr{D}} \left(\frac{\mathrm{D}u}{\mathrm{D}t}, \frac{\mathrm{D}v}{\mathrm{D}t}\right)\mathrm{d}x\,\mathrm{d}y. \tag{3.150}$$

Substituting (2.1) into (3.150), we have

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}} = \int_{\mathscr{D}} \frac{1}{\rho}\nabla p\,\mathrm{d}x\,\mathrm{d}y - \lambda\chi\int_{\mathscr{D}} (\boldsymbol{u}_s - \boldsymbol{u})\,\mathrm{d}x\,\mathrm{d}y - \nu\int_{\mathscr{D}} \nabla^2 \boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.151}$$

Using Green's theorem, we have the following relations with respect to the first and third terms on the right-hand side of (3.151):

$$\int_{\mathscr{D}} \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{C}_o - \mathscr{C}_i} (p\,\mathrm{d}y, -p\,\mathrm{d}x), \tag{3.152a}$$

$$\int_{\mathscr{D}} (\nabla^2 u, \nabla^2 v)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{C}_o - \mathscr{C}_i} (\omega\,\mathrm{d}x, \omega\,\mathrm{d}y). \tag{3.152b}$$

Here ω is the vorticity.

Here, in general, we look at a closed contour $\mathscr{C}$ that surrounds a two-dimensional fluid domain $\mathscr{D}'$, and we consider the hydrodynamic force (F_{bx}, F_{by}) on the fluid in $\mathscr{D}'$ exerted by the outer domain of $\mathscr{C}$. The small element $\mathrm{d}s$ on $\mathscr{C}$ is acted on by normal and tangential stresses and outflow momentum from $\mathscr{C}$. Note that the outflow momentum is written as $-\rho u_n(u, v)\,\mathrm{d}s$, where the subscript n indicates the outward normal direction of the contour $\mathscr{C}$. Then, the hydrodynamic force (F_{bx}, F_{by}) is described as

$$F_{bx} = \int_{\mathscr{C}} (-p\,\mathrm{d}y + \mu\omega\,\mathrm{d}x - \rho u_n u\,\mathrm{d}s), \tag{3.153a}$$

$$F_{by} = \int_{\mathscr{C}} (p\,\mathrm{d}x + \mu\omega\,\mathrm{d}y - \rho u_n v\,\mathrm{d}s). \tag{3.153b}$$

We apply (3.153a) and (3.153b) to the contours $\mathscr{C}_o$ and $\mathscr{C}_i$. Taking into account (3.152a)–(3.152b), (3.151) can be then described as

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}} = -\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_o - \mathscr{C}_i} - \int_{\mathscr{C}_o - \mathscr{C}_i} u_n(u, v)\,\mathrm{d}s - \lambda\chi\int_{\mathscr{D}} (\boldsymbol{u}_s - \boldsymbol{u})\,\mathrm{d}x\,\mathrm{d}y. \tag{3.154}$$

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Now, let us apply (3.154) to the fluid domain $\mathscr{D}_f$ that is surrounded by the control surface $\mathscr{C}_\infty$ and the body surface $\mathscr{C}_+$. Then, because of $\chi = 0$, (3.154) is rewritten as

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_f} = -\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_\infty - \mathscr{C}_+} - \int_{\mathscr{C}_\infty - \mathscr{C}_+} u_n(u, v)\, \mathrm{d}s. \tag{3.155}$$

Therefore, the hydrodynamic force on the cylinder, (F_{bx}, F_{by}), which is exerted from the fluid domain, is obtained as

$$\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+} = \left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_f} + \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_\infty} + \int_{\mathscr{C}_\infty - \mathscr{C}_+} u_n(u, v)\, \mathrm{d}s. \tag{3.156}$$

Because of $\omega \approx 0$ and $(u, v) = (1 + O(1/r^2), O(1/r^2))$ in the neighbourhood of the control surface $\mathscr{C}_\infty$, the pressure solely affects the second term on the right-hand side of (3.156), and the momentum of the fluid flow affects the first term (see § 3.4).

For the domain $\mathscr{D}_s$ inside the body whose surface is defined as $\mathscr{C}_-$, (3.154) is written as

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_s} = -\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_-} - \int_{\mathscr{C}_-} u_n(u, v)\, \mathrm{d}s - \lambda \int_{\mathscr{D}_s} (\boldsymbol{u}_s - \boldsymbol{u})\, \mathrm{d}x\, \mathrm{d}y. \tag{3.157}$$

From (3.155) and (3.157), we have

$$\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_f + \mathscr{D}_s} = -\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_\infty} + \left(\frac{F_{vx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+ - \mathscr{C}_-} - \int_{\mathscr{C}_\infty} u_n(u, v)\, \mathrm{d}s - \lambda \int_{\mathscr{D}_s} (\boldsymbol{u}_s - \boldsymbol{u})\, \mathrm{d}x\, \mathrm{d}y. \tag{3.158}$$

As shown in § 3.3.2, there is the pressure difference Δp (i.e. $\Delta p \neq 0$) between the inside and the outside of the cylinder surface $\mathscr{C}$ and, therefore, we can write

$$\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+} = \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_-} - \int_{\mathscr{C}} \left(\frac{\Delta p}{\rho} \mathrm{d}y, -\frac{\Delta p}{\rho}\, \mathrm{d}x\right), \tag{3.159}$$

where $\mathscr{C}_\pm$ denotes the contour of the cylinder surface on the side of $\mathscr{D}_f$ and $\mathscr{D}_s$, respectively. Taking into account (3.157), (3.159) is written as

$$\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+} = -\left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_s} - \int_{\mathscr{C}} u_n(u, v)\, \mathrm{d}s - \lambda \int_{\mathscr{D}_s} (\boldsymbol{u}_s - \boldsymbol{u})\, \mathrm{d}x\, \mathrm{d}y - \int_{\mathscr{C}} \left(\frac{\Delta p}{\rho}\, \mathrm{d}y, \frac{\Delta p}{\rho}\, \mathrm{d}x\right). \tag{3.160}$$

Approach (i) in the preceding § 3.4 is

$$\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathrm{(i)}} = \left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_f} + \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_\infty} = \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+} - \int_{\mathscr{C}_\infty - \mathscr{C}_+} u_n(u, v)\, \mathrm{d}s. \tag{3.161}$$

Therefore, we can see that $(F_{bx}/\rho, F_{by}/\rho)_{\mathrm{(i)}} = (F_{bx}/\rho, F_{by}/\rho)_{\mathscr{C}_+}$, since in the case of the present study of the circular cylinder, that is, because of $u_n = u_r^i = O(\epsilon)$ as shown in

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§ 3.3.2, the last term of (3.156) is found to be of the order of ϵ. It is therefore found that the asymptotic behaviour of F_{bx} for $\lambda_o \gg 1$ and $T \ll 1$ is the same as F_x obtained in § 3.4. For approach (ii) in the case of $u_n = 0$ on $\mathscr{C}_-$, we have from (3.159)

$$\left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{(ii)} = \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_-} + \left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_s}$$
$$= \left(\frac{F_{bx}}{\rho}, \frac{F_{by}}{\rho}\right)_{\mathscr{C}_+} + \left(\frac{F_x}{\rho}, \frac{F_y}{\rho}\right)_{\mathscr{D}_s} + \int_{\mathscr{C}} \left(\frac{\Delta p}{\rho}\,\mathrm{d}y, -\frac{\Delta p}{\rho}\,\mathrm{d}x\right). \quad (3.162)$$

We see that even when the velocity inside the cylinder is zero, there is the difference in the third term of the right-hand side of (3.162) between approach (i) and (ii) by the pressure jump. Since the present analysis takes the relative coordinate system fixed with the centre of the cylinder, the first term on the right-hand side of (3.160) becomes zero. It can then be found that the calculation of the hydrodynamic force by the integration of the penalization layer (i.e. (2.3)) is formulated, assuming that the pressure difference Δp and the force on the cylinder surface exerted by the momentum due to the fluid flow are zero. In this assumption, $u_n = 0$, $\Delta p = 0$ and $(F_x/\rho, F_y/\rho)_{\mathscr{D}_s} = 0$. Therefore, we have $(F_x/\rho, F_y/\rho)_{\mathrm{i}} = (F_x/\rho, F_y/\rho)_{\mathrm{ii}}$. Here, the pressure difference, Δp, between the inside and the outside of the cylinder surface is given from (3.81) and (3.110).

$$\frac{\Delta p}{\rho} = \frac{1}{\rho}\left(p|_{r\to 1+0} - p|_{r\to 1-0}\right)$$
$$= 2\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta - 2\sin^2\theta + \frac{1}{2}(1-\varOmega^2) - C$$
$$+ \epsilon\left\{\frac{\partial}{\partial T}\int_0^\infty [u^i_{r1} - 2(A+R)\cos\theta]\,\mathrm{d}R - \int_0^\infty (u^{i2}_{\theta 0} - 4\sin^2\theta)\,\mathrm{d}R + \nu_o \left.\frac{\partial u^i_{r1}}{\partial R}\right|_{R=+0}\right.$$
$$\left. - 2\nu_o\cos\theta - \frac{\mathrm{d}}{\mathrm{d}T}(a^c_1\cos\theta + a^s_1\sin\theta + a^c_2\cos 2\theta) + 4A\sin^2\theta\right\}. \quad (3.163)$$

For $\lambda_o \gg 1$, (3.96*a*)–(3.96*c*) and (D9*a*)–(D9*d*) are written as

$$u^i_{\theta 0} \sim -2\sin\theta + (\varOmega + 2\sin\theta)\,\mathrm{erfc}(\eta) + O(1/\sqrt{\lambda_o}), \quad (3.164a)$$

$$\frac{\partial}{\partial T}\int_0^\infty \left(u^i_{r1} - 2(A+R)\cos\theta\right)\mathrm{d}R \sim 2\nu_o\cos\theta + O(1/\sqrt{\lambda_o}), \quad (3.164b)$$

$$\left.\frac{\partial u^i_{r1}}{\partial R}\right|_{R=+0} \sim O(1/\sqrt{\lambda_o}), \quad (3.164c)$$

$$\frac{\mathrm{d}}{\mathrm{d}T}(a^c_1) \sim \nu_o + O(1/\sqrt{\lambda_o}), \quad (3.164d)$$

$$\frac{\mathrm{d}}{\mathrm{d}T}(a^s_1) \sim \mathcal{L}^{-1}\left(-\int_0^\infty F^c \mathrm{e}^{-aR}\,\mathrm{d}R + \int_0^\infty F^c \mathrm{d}R\right) + O(1/\sqrt{\lambda_o}), \quad (3.164e)$$

$$\frac{\mathrm{d}}{\mathrm{d}T}(a^c_2) \sim \mathcal{L}^{-1}\left(-\int_0^\infty F^{cs} \mathrm{e}^{-aR}\,\mathrm{d}R + \int_0^\infty F^{cs} \mathrm{d}R\right) + O(1/\sqrt{\lambda_o}), \quad (3.164f)$$

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where $\eta = r/(2\sqrt{\nu_o T})$. Then, the following integral involved in (3.163) is calculated as

$$\int_0^\infty (u_{\theta 0}^{i2} - 4\sin^2\theta)\,\mathrm{d}R \sim 2\sqrt{\nu_o T}\int_0^\infty (\varOmega + 2\sin\theta)\,\mathrm{erfc}(\eta) \times [-4\sin\theta + (\varOmega + 2\sin\theta)\,\mathrm{erfc}(\eta)]\,\mathrm{d}\eta = 2\sqrt{\frac{\nu_o T}{\pi}}[-4\sin\theta(\varOmega + 2\sin\theta) + (\varOmega + 2\sin\theta)^2(2-\sqrt{2})]. \tag{3.165}$$

Here, the following relations are used for the calculation of (3.165):

$$\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-aR}}{s}\right) = \mathrm{erfc}(\eta), \quad \int_0^\infty \mathrm{erfc}(\eta)\,\mathrm{d}\eta = \frac{1}{\sqrt{\pi}}, \quad \int_0^\infty \mathrm{erfc}^2(\eta)\,\mathrm{d}\eta = \frac{1}{\sqrt{\pi}}(2-\sqrt{2}). \tag{3.166a–c}$$

Taking into account $A \sim 2\sqrt{\nu_o T/\pi}$ and using (3.164a)–(3.164f) together with (3.165), (3.163) can be written as

$$\frac{\Delta p}{\rho} \sim 2\sqrt{\frac{\nu_o}{\pi T}}\cos\theta - 2\sin^2\theta + \frac{1}{2}(1-\varOmega^2) + \epsilon\left\{8\sqrt{\frac{\nu_o T}{\pi}}\sin^2\theta - \nu_o\cos\theta - 2\sqrt{\frac{\nu_o T}{\pi}}\left[-4\varOmega\sin\theta - 8\sin^2\theta + (2-\sqrt{2})(\varOmega^2 + 4\varOmega\sin\theta + 4\sin^2\theta)\right] + \mathcal{L}^{-1}\left(\int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R - \int_0^\infty F^c\,\mathrm{d}R\right)\sin\theta + \mathcal{L}^{-1}\left(\int_0^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R - \int_0^\infty F^{cs}\,\mathrm{d}R\right)\cos 2\theta\right\} + O(\epsilon^2, 1/\sqrt{\lambda_o}). \tag{3.167}$$

Integrating the x and y components of (3.155) acting on the cylinder surface, we have

$$\int_0^{2\pi}\frac{\Delta p}{\rho}\,\mathrm{d}x \sim 8\epsilon\sqrt{\nu_o\pi T}\varOmega(1-\sqrt{2}) - \pi\epsilon\mathcal{L}^{-1}\left(\int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R - \int_0^\infty F^c\,\mathrm{d}R\right), \tag{3.168a}$$

$$\int_0^{2\pi}\frac{\Delta p}{\rho}\,\mathrm{d}y \sim 2\pi\sqrt{\frac{\nu_o}{\pi T}} - \pi\epsilon\nu_o. \tag{3.168b}$$

Therefore, the hydrodynamic forces can be found to behave like, for $\lambda_o \gg 1$,

$$F_{bx} \sim 4\rho\sqrt{\frac{\nu}{\pi t}}, \tag{3.169a}$$

$$F_{by} \sim 2\rho\pi\epsilon\left(\frac{1}{2}\int_0^\infty F^c\mathrm{d}R - \int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R\right) + 8\rho\varOmega\sqrt{\pi\nu t}(1-\sqrt{2}). \tag{3.169b}$$

Note that (3.169a) is derived as follows:

$$F_{bx} \approx 2\rho\sqrt{\frac{\nu}{\pi t}} + 2\rho\sqrt{\frac{\nu}{\pi t}}\left(1-\mathrm{e}^{-\lambda_o T}\right) \sim 4\rho\sqrt{\frac{\nu}{\pi t}} \quad \text{for } \lambda_o \gg 1. \tag{3.170}$$

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Equation (3.169*a*) is the same result as Badr & Dennis (1985) and our previous study (Ueda & Kida 2021). Using the value of s_o which is calculated in (3.148), (3.169*b*) is obtained as

$$\begin{aligned} F_{by} &\sim 2\rho\pi\epsilon\left[-4\Omega\sqrt{\frac{\nu_o T}{\pi}}(\sqrt{2}-1)-4\Omega\sqrt{\frac{\nu T}{\pi}}s_o\right]+8\rho\Omega\sqrt{\pi\nu t}(1-\sqrt{2}) \\ &\sim -16\rho\Omega\sqrt{\pi\nu t}(\sqrt{2}-1)-8\rho\Omega\sqrt{\pi\nu t}s_o. \end{aligned} \tag{3.171}$$

The value of (3.171), which is the present result, is calculated as

$$\frac{F_{by}}{\rho\Omega\sqrt{\pi\nu t}} \sim -4.9012, \tag{3.172}$$

whereas the value of the asymptotic solution obtained by Badr & Dennis (1985) is given by

$$\frac{F_{by}}{\rho\Omega\sqrt{\pi\nu t}} \sim -4.2879. \tag{3.173}$$

It can therefore be found that the variance on the values of the lift force calculated by the two approaches (see § 3.4) results from the pressure jump on the body (cylinder) surface. The inner expansion of the pressure in the outer domain of the fluid cannot continuously connect to the inner expansion of the pressure in the outer domain of the solid body (circular cylinder). Therefore, the pressure jump is found to be caused by the matching procedure of the outer solution in the fluid domain.

4. Concluding remarks

This study aims to elucidate the variance of the drag force calculated by the integration of the penalization layer against the asymptotic solution for $t \ll 1$, which is demonstrated in our previous paper (see Ueda & Kida 2021). We first consider the problem that a circular cylinder impulsively rotates from a quiescent state. In this first problem, it is verified that the results of the moment calculated by the different two approaches (approaches (I) and (II) in § 3.2) are the same as the result calculated by the integration of the penalization layer. Note that the pressure on the cylinder surface is independent of the moment and, therefore, it can be deduced that the pressure plays a key role to the drag force on the cylinder.

We therefore consider the second problem in § 3.3 that a circular cylinder impulsively starts with rotating and translating velocities from a quiescent state. The drag and lift forces are calculated by two approaches: (i) based on the time derivative of the momentum of an entire fluid domain and the pressure on the control surface of which radius is sufficiently large, and (ii) based on the integration of the penalization layer. For $t \ll 1$ and $\lambda_o \gg 1$, the drag force calculated by approach (ii) gives a half-value of that calculated by approach (i) that yields the same result as the classical asymptotic solution of Bar-Lev & Yang (1975). This finding is the same as our previous study (Ueda & Kida 2021) for an impulsively started translating circular cylinder without rotation. Furthermore, the calculated lift force is completely different between the two approaches, as shown in figure 7(*a*). To resolve the variance, this study successfully derives an alternative formula to calculate the hydrodynamic force, assuming that there is a pressure jump between the outside and inside of the cylinder surface. It can be found that the variance is caused by the pressure jump on the cylinder surface. Furthermore, the comparison of the drag force

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with the previous result is discussed in Appendix F. In the penalization method the no-slip boundary condition on the cylinder surface is not imposed, although it is imposed in the previous analysis. This condition generates the additional terms on the time derivative of the slip velocity, which is related to the pressure jump.

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Author ORCIDs.
Y. Ueda https://orcid.org/0000-0001-5340-0671.

Appendix A

Using the formulae of Abramowitz & Stegun (1954), the following relations of the inverse Laplace transform are obtained;

$$\mathcal{L}^{-1}\left(\sqrt{\frac{s+2a}{s}}-1\right) = a\mathrm{e}^{-aT}\left[\mathrm{I}_1(aT)+\mathrm{I}_0(aT)\right], \tag{A1}$$

$$\mathcal{L}^{-1}\left(\mathrm{e}^{-k\sqrt{s}}\right) = \frac{k}{2\sqrt{\pi T^3}}\mathrm{e}^{-k^2/4T}, \tag{A2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s\sqrt{s+\lambda_o}}\right) = \frac{1}{\sqrt{\lambda_o}}\mathrm{erf}(\sqrt{\lambda_o T}), \tag{A3}$$

$$\mathcal{L}^{-1}\left(\exp(-a|R|)\right) = \frac{|R|}{2\sqrt{\pi\nu_o T^3}}\exp\left(-\frac{R^2}{4 \ni_o T}\right), \tag{A4}$$

Appendix B

We look at the following differential equation, in which U is a function of R and θ:

$$sU - \nu_o\frac{\partial^2 U}{\partial R^2} = F(R,\theta). \tag{B1}$$

Setting U as $U = A(R,\theta)\mathrm{e}^{-aR}$, (B1) is rewritten as, by the differential equation with respect to the function A,

$$\frac{\partial^2 A}{\partial R^2} - 2a\frac{\partial A}{\partial R} = -\frac{1}{\nu_o}F\mathrm{e}^{aR}. \tag{B2}$$

The solution to (B2) is readily obtained as

$$A = -\frac{\mathrm{e}^{2aR}}{2a\nu_o}\int^R F\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{2a\nu_o}\int^R F\mathrm{e}^{aR}\,\mathrm{d}R + C_1\mathrm{e}^{2aR} + C_0, \tag{B3}$$

where C_0 and C_1 are the integral constants. Therefore, we obtain the function U that is the solution to (B1):

$$U = -\frac{\mathrm{e}^{aR}}{2a\nu_o}\int^R F\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{\mathrm{e}^{-aR}}{2a\nu_o}\int^R F\mathrm{e}^{aR}\,\mathrm{d}R + C_1\mathrm{e}^{aR} + C_0\mathrm{e}^{-aR}. \tag{B4}$$

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When $U \to 0$ as $R \to \infty$, the integral constant C_0 is determined as $C_0 = 0$ and, therefore, the solution to (B1) is obtained as

$$U = \frac{e^{aR}}{2a\nu_o} \int_R^\infty F e^{-aR} \, dR + \frac{e^{-aR}}{2a\nu_o} \int_0^R F e^{aR} \, dR + C_0 e^{-aR}. \tag{B5}$$

Appendix C

Using the Laplace transform, (3.84*a*) is written as, for $R \geq 0$,

$$sU^0_{\theta 1} - \nu_o \frac{\partial^2 U^0_{\theta 1}}{\partial R^2} = \nu_o \frac{\partial U^0_{\theta 0}}{\partial R}, \tag{C1}$$

where $U^0_{\theta 1} = \mathcal{L}(\hat{u}^0_{\theta 1})$ and $U^0_{\theta 0} = \mathcal{L}(\hat{u}^0_{\theta 0})$. From (3.61), we find that $U^0_{\theta 0} = \{\bar{a}/[s(a + \bar{a})]\}\varOmega e^{-aR}$. Based on the boundary condition of (3.85*a–d*), the solution to (C1) is obtained as, using Appendix B,

$$U^0_{\theta 1} = -\frac{1}{2} \frac{\bar{a}}{s(a + \bar{a})} \varOmega R e^{-aR} + D e^{-aR}, \tag{C2}$$

where D is an integral constant that is a function of s and θ.

For $R < 0$, by virtue of (3.61) and (3.85*a–d*), the function $\hat{u}^0_{\theta 1}$ is found to be written as the form of $\hat{u}^0_{\theta 1} = R\varOmega H(T) + \bar{u}^0_{\theta 1}$. Then, (3.84*a*) reduces to

$$s\bar{U}^0_{\theta 1} - \nu_o \frac{\partial^2 \bar{U}^0_{\theta 1}}{\partial R^2} + \lambda_o \bar{U}^0_{\theta 1} = \nu_o \left(-\varOmega \frac{a\bar{a}}{s(a + \bar{a})} e^{\bar{a}R} \right), \tag{C3}$$

where $\bar{U}^0_{\theta 1} = \mathcal{L}(\bar{u}^0_{\theta 1})$. The solution to (C3) is obtained as, using Appendix B,

$$U^0_{\theta 1} = \frac{1}{2} \frac{a}{s(a + \bar{a})} \varOmega R e^{\bar{a}R} + \frac{\varOmega}{s} R + E e^{\bar{a}R}, \tag{C4}$$

where E is an integral constant that is a function of s and θ. These integral constants D and E can be determined as, from the enforcements of the continuity of the velocity and its radial derivative at $R = 0$,

$$D = E = -\frac{3}{2} \frac{\varOmega}{s} \frac{1}{a + \bar{a}}. \tag{C5}$$

The velocity $\hat{u}^0_{\theta 1}$ can be found to be the same as the solution of the pure rotation problem obtained in (3.32).

The Laplace transform to (3.84*c*) is written as, for $R \geq 0$,

$$sU^s_{\theta 1} - \nu_o \frac{\partial^2 U^s_{\theta 1}}{\partial R^2} = 2 \left(\frac{a - \bar{a}}{a\bar{a}} \right) - 2\nu_o \frac{a\bar{a}}{s(a + \bar{a})} e^{-aR}, \tag{C6}$$

where $U^s_{\theta 1} = \mathcal{L}(\hat{u}^s_{\theta 1})$. Here, we introduce $\bar{U}^s_{\theta 1}$ as $U^s_{\theta 1} = 2(R/s) + 2[(a - \bar{a})/(sa\bar{a})] + \bar{U}^s_{\theta 1}$. Then, (C6) reduces to

$$s\bar{U}^s_{\theta 1} - \nu_o \frac{\partial^2 \bar{U}^s_{\theta 1}}{\partial R^2} = -2\nu_o \frac{a\bar{a}}{s(a + \bar{a})} e^{-aR}. \tag{C7}$$

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By virtue of the boundary condition (3.85a–d), $\bar{U}^s_{\theta 1}$ is found to be not divergent as $R \rightarrow \infty$. Hence, we set $\bar{U}^s_{\theta 1}$ as

$$\bar{U}^s_{\theta 1} = -\frac{\bar{a}}{s(a+\bar{a})} R\mathrm{e}^{-aR} + D_s \mathrm{e}^{-aR}, \tag{C8}$$

where D_s is an integral constant that is a function of s and θ.

For $R < 0$, the Laplace transform to (3.84c) is written as, taking into account $\mathcal{L}(\partial \hat{u}^s_{\theta 0}/\partial R) = -2a\bar{a}/[s(a+\bar{a})]\mathrm{e}^{\bar{a}R}$ from (3.61) and $\hat{u}_{\theta 1} \rightarrow 0$ as $R \rightarrow -\infty$ from (3.86a–d),

$$sU^s_{\theta 1} - \nu_o \frac{\partial^2 U^s_{\theta 1}}{\partial R^2} - \lambda_o U^s_{\theta 1} = -2\nu_o \frac{a\bar{a}}{a+\bar{a}} \mathrm{e}^{\bar{a}R}. \tag{C9}$$

The solution to (C9) is therefore obtained as

$$U^s_{\theta 1} = \frac{a}{s(a+\bar{a})} R\mathrm{e}^{\bar{a}R} + E_s \mathrm{e}^{\bar{a}R}, \tag{C10}$$

where E_s is an integral constant that is a function of s and θ. These indeterminate constants can be determined as, by the enforcements of the continuity of the velocity and its radial derivative at $R = 0$,

$$D_s = \frac{2}{sa} - \frac{3}{s(a+\bar{a})}, \quad E_s = \frac{2}{s\bar{a}} - \frac{3}{s(a+\bar{a})}. \tag{C11a,b}$$

Thus, we obtain (3.88).

In contrast, $\hat{u}^c_{\theta 1}$ can be obtained for $R \geq 0$ by the Laplace transform to (3.84b),

$$sU^c_{\theta 1} - \nu_o \frac{\partial^2 U^c_{\theta 1}}{\partial R^2} = -F^c, \quad F^c = \mathcal{L}\left(\hat{u}_{r1} \frac{\partial \hat{u}^0_{\theta 0}}{\partial R} + \hat{u}^0_{\theta 0} \hat{u}^s_{\theta 0}\right), \tag{C12a,b}$$

where $U^c_{\theta 1} = \mathcal{L}(\hat{u}^c_{\theta 1})$. The solution to (C12$a$,$b$) is readily obtained as, based on the boundary condition of (3.85a–d) (i.e. $U^c_{\theta 1} \rightarrow 0$ as $R \rightarrow \infty$),

$$U^c_{\theta 1} = -\frac{1}{2\nu_o a}\left(\mathrm{e}^{aR} \int_R^\infty F^c \mathrm{e}^{-aR}\,\mathrm{d}R + \mathrm{e}^{-aR} \int_0^R F^c \mathrm{e}^{aR}\,\mathrm{d}R\right) + D_c \mathrm{e}^{-aR}, \tag{C13}$$

where D_c is an integral constant that is a function of s and θ.

Similarly, the governing equation for $R < 0$ is written as, taking the Laplace transform to (3.84b),

$$sU^c_{\theta 1} - \nu_o \frac{\partial^2 U^c_{\theta 1}}{\partial R^2} + \lambda_o U^c_{\theta 1} = -F^c. \tag{C14}$$

The solution to (C14) is obtained as, based on the boundary condition of (3.86a–d) (i.e. $U^c_{\theta 1} \rightarrow 0$ as $R \rightarrow -\infty$),

$$U^c_{\theta 1} = -\frac{1}{2\nu_o \bar{a}}\left(\mathrm{e}^{-\bar{a}R} \int_{-\infty}^R F^c \mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \mathrm{e}^{\bar{a}R} \int_0^R F^c \mathrm{e}^{-\bar{a}R}\,\mathrm{d}R\right) + E_c \mathrm{e}^{\bar{a}R}, \tag{C15}$$

where E_c is an integral constant that is a function of s and θ. These indeterminate constants, D_c and E_c, can be determined as, by the enforcements of the continuity of the velocity and

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its derivative with respect to R at $R=0$,

$$D_c = \frac{\bar{a}-a}{2\nu_o a(a+\bar{a})}\int_0^\infty F^c \mathrm{e}^{-aR}\,\mathrm{d}R - \frac{1}{\nu_o(a+\bar{a})}\int_{-\infty}^0 F^c \mathrm{e}^{\bar{a}R}\,\mathrm{d}R, \tag{C16a}$$

$$E_c = -\frac{1}{\nu_o(a+\bar{a})}\int_0^\infty F^c \mathrm{e}^{-aR}\,\mathrm{d}R + \frac{a-\bar{a}}{2\nu_o\bar{a}(a+\bar{a})}\int_{-\infty}^0 F^c \mathrm{e}^{\bar{a}R}\,\mathrm{d}R. \tag{C16b}$$

Similar to the above, the Laplace transform to (3.84d) is also written as

$$sU^{cs}_{\theta 1} - \nu_o\frac{\partial^2 U^{cs}_{\theta 1}}{\partial R^2} + \lambda_o \chi U^{cs}_{\theta 1} = -F^{cs}, \quad F^{cs} = \mathcal{L}\left(\hat{u}_{r1}\frac{\partial \hat{u}^s_{\theta 0}}{\partial R} + \begin{Bmatrix} \hat{u}^{s2}_{\theta 0} - 4 & \text{for } R \geq 0 \\ \hat{u}^{s2}_{\theta 0} & \text{for } R < 0 \end{Bmatrix}\right), \tag{C17a,b}$$

where $U^{cs}_{\theta 1} = \mathcal{L}(\hat{u}^{cs}_{\theta 1})$. The solution to (C17$a,b$) is obtained as, based on the boundary condition of (3.85a–d) and (3.86a–d) (i.e. $U^{cs}_{\theta 1} \to 0$ as $R \to \pm\infty$),

$$U^{cs}_{\theta 1} = \begin{cases} -\dfrac{1}{2\nu_o a}\left(\mathrm{e}^{aR}\displaystyle\int_R^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \mathrm{e}^{-aR}\int_0^R F^{cs}\mathrm{e}^{aR}\,\mathrm{d}R\right) + D_{cs}\mathrm{e}^{-aR} & \text{for } R \geq 0, \\ -\dfrac{1}{2\nu_o \bar{a}}\left(\mathrm{e}^{-\bar{a}R}\displaystyle\int_{-\infty}^R F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \mathrm{e}^{\bar{a}R}\int_0^R F^{cs}\mathrm{e}^{-\bar{a}R}\,\mathrm{d}R\right) + E_{cs}\mathrm{e}^{\bar{a}R} & \text{for } R < 0, \end{cases} \tag{C18}$$

in which the indeterminate constants D_{cs} and E_{cs} are determined by

$$D_{cs} = \frac{\bar{a}-a}{2\mu_o a(a+\bar{a})}\int_0^\infty F^{cs} \mathrm{e}^{-aR}\,\mathrm{d}R - \frac{1}{\nu_o(a+\bar{a})}\int_{-\infty}^0 F^{cs} \mathrm{e}^{\bar{a}R}\,\mathrm{d}R, \tag{C19a}$$

$$E_{cs} = -\frac{1}{\nu_o(a+\bar{a})}\int_0^\infty F^{cs} \mathrm{e}^{-aR}\,\mathrm{d}R + \frac{a-\bar{a}}{2\nu_o\bar{a}(a+\bar{a})}\int_{-\infty}^0 F^{cs} \mathrm{e}^{\bar{a}R}\,\mathrm{d}R. \tag{C19b}$$

Appendix D

In order to obtain the pressure p^o_1 of the outer flow, it is necessary to obtain the outer flow of the order of ϵ^2, because $p^o_1/\rho \sim -\partial\phi^o_2/\partial T$, where ϕ^o_2 is the velocity potential of the order of ϵ^2. Therefore, let us attempt to derive the solution of the order of ϵ^2 in the outer fluid domain where the solution exhibits the inviscid one (i.e. potential flow). To do so, we need to obtain u^i_{r2} in the inner fluid domain that is governed by the continuity equation (3.10c):

$$\frac{\partial u^i_{r2}}{\partial R} + u^i_{r1} + \frac{\partial u^i_{\theta 1}}{\partial \theta} - R\frac{\partial u^i_{\theta 0}}{\partial \theta} = 0. \tag{D1}$$

Since the matching procedure yields $u^i_{r2} \to 0$ as $R \to -\infty$, (D1) is rewritten as

$$u^i_{r2} = -\int_{-\infty}^R \left(u^i_{r1} + \frac{\partial u^i_{\theta 1}}{\partial \theta} - R\frac{\partial u^i_{\theta 0}}{\partial \theta}\right)\mathrm{d}R. \tag{D2}$$

Taking the Laplace transform to (D2) and using (3.62), (3.67) and (3.83), we have, for $R \geq 0$,

$$U_{r2} = -\mathrm{I}^o_r\cos\theta - \mathrm{I}^c_{\theta 1}\cos\theta - \mathrm{I}^s_{\theta 1}\sin\theta - \mathrm{I}^c_{\theta 0}\cos\theta - \mathrm{I}^{cs}_{\theta 1}\cos 2\theta, \tag{D3}$$

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in which the coefficients are defined by

$$\left.\begin{aligned}\mathrm{I}_r^o &= \int_{-\infty}^{0} U_{r1}\,\mathrm{d}R + \int_0^R U_{r1}\,\mathrm{d}R, \quad \mathrm{I}_{\theta 1}^c = \int_{-\infty}^{0} U_{\theta 1}^s\,\mathrm{d}R + \int_0^R U_{\theta 1}^s\,\mathrm{d}R, \\ \mathrm{I}_{\theta 1}^s &= -\int_{-\infty}^{0} U_{\theta 1}^c\,\mathrm{d}R - \int_0^R U_{\theta 1}^c\,\mathrm{d}R, \quad \mathrm{I}_{\theta 0}^c = -\int_{-\infty}^{0} RU_{\theta 0}^s\,\mathrm{d}R - \int_0^R RU_{\theta 0}^s\,\mathrm{d}R, \\ \mathrm{I}_{\theta 1}^{cs} &= \int_{-\infty}^{0} U_{\theta 1}^{cs}\,\mathrm{d}R + \int_0^R U_{\theta 1}^{cs}\,\mathrm{d}R.\end{aligned}\right\} \tag{D4}$$

Here, $U_{\theta 0}^s = \mathcal{L}(\hat{u}_{\theta 0}^s)$, $U_{r1} = \mathcal{L}(\hat{u}_{r1})$, $U_{\theta 1}^s = \mathcal{L}(\hat{u}_{\theta 1}^s)$, $U_{\theta 1}^c = \mathcal{L}(\hat{u}_{\theta 1}^c)$ and $U_{\theta 1}^{cs} = \mathcal{L}(\hat{u}_{\theta 1}^{cs})$. These coefficients can be calculated as, using the asymptotic solutions that are obtained above,

$$\mathrm{I}_r^o = \frac{2a}{s\bar{a}^2(a+\bar{a})} + 2\frac{a-\bar{a}}{sa\bar{a}}R + \frac{2\bar{a}}{sa^2(a+\bar{a})}(1-\mathrm{e}^{-aR}) + \frac{1}{s}R^2, \tag{D5a}$$

$$\begin{aligned}\mathrm{I}_{\theta 1}^c = &-\frac{a}{s\bar{a}^2(a+\bar{a})} + \frac{E_s}{\bar{a}} + \frac{D_s}{a}(1-\mathrm{e}^{-aR}) + \frac{R^2}{s} + 2\frac{a-\bar{a}}{sa\bar{a}}R + \frac{\bar{a}}{sa(a+\bar{a})}R\mathrm{e}^{-aR} \\ &- \frac{\bar{a}}{sa^2(a+\bar{a})}(1-\mathrm{e}^{-aR}),\end{aligned} \tag{D5b}$$

$$\begin{aligned}\mathrm{I}_{\theta 1}^s = &-\frac{E_c}{\bar{a}} - \frac{D_c}{a}(1-\mathrm{e}^{-aR}) - \frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\mathrm{e}^{\bar{a}R}\,\mathrm{d}R + \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\,\mathrm{d}R \\ &+ \frac{\mathrm{e}^{aR}}{2\nu_o a^2}\int_R^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R - \frac{\mathrm{e}^{-aR}}{2\nu_o a^2}\int_0^R F^c\mathrm{e}^{aR}\,\mathrm{d}R - \frac{1}{2\nu_o a^2}\int_0^\infty F^c\mathrm{e}^{-aR}\,\mathrm{d}R \\ &+ \frac{1}{\nu_o a^2}\int_0^R F^c\,\mathrm{d}R,\end{aligned} \tag{D5c}$$

$$\mathrm{I}_{\theta 0}^c = -\frac{2a}{s\bar{a}^2(a+\bar{a})} + \frac{2\bar{a}}{sa(a+\bar{a})}R\mathrm{e}^{-aR} - \frac{2\bar{a}}{sa^2(a+\bar{a})}(1-\mathrm{e}^{aR}) + \frac{1}{s}R^2, \tag{D5d}$$

$$\begin{aligned}\mathrm{I}_{\theta 1}^{cs} = &\frac{E_{cs}}{\bar{a}} + \frac{D_{cs}}{a}(1-\mathrm{e}^{-aR}) + \frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^{0} F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R - \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^{0} F^{cs}\,\mathrm{d}R \\ &- \frac{\mathrm{e}^{aR}}{2\nu_o a^2}\int_R^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{\mathrm{e}^{-aR}}{2\nu_o a^2}\int_0^R F^{cs}\mathrm{e}^{aR}\,\mathrm{d}R \\ &+ \frac{1}{2\nu_o a^2}\int_0^\infty F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R - \frac{1}{\nu_o a^2}\int_0^R F^{cs}\,\mathrm{d}R.\end{aligned} \tag{D5e}$$

The outer expansion of U_{r2} (i.e. U_{r2}^{io}) is written as, using (D3) with (D5a)–(D5e),

$$\begin{aligned}U_{r2}^{io} = &-\cos\theta\left[4\frac{a-\bar{a}}{sa\bar{a}}R + \frac{3}{s}R^2 + \frac{E_s}{\bar{a}} + \frac{D_s}{a} - \frac{\bar{a}}{sa^2(a+\bar{a})} - \frac{a}{s\bar{a}^2(a+\bar{a})}\right] \\ &- \sin\theta\left[-\frac{E_c}{\bar{a}} - \frac{D_c}{a} - \frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\mathrm{e}^{\bar{a}R}\,\mathrm{d}R + \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\,\mathrm{d}R\right.\end{aligned}$$

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$$- \frac{1}{2\nu_o a^2}\int_0^{\infty} F^c \mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{\nu_o a^2}\int_0^{\infty} F^c\,\mathrm{d}R\bigg]$$
$$- \cos 2\theta\bigg[- \frac{E_{cs}}{\bar{a}} - \frac{D_{cs}}{a} - \frac{1}{2\nu_o \bar{a}^2}\int_{-\infty}^{0} F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R$$
$$+ \frac{1}{\nu_o \bar{a}^2}\int_{-\infty}^{0} F^{cs}\,\mathrm{d}R - \frac{1}{2\nu_o a^2}\int_0^{\infty} F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{\nu_o a^2}\int_0^{\infty} F^{cs}\,\mathrm{d}R\bigg]. \tag{D6}$$

On the other hand, since the outer solution in the fluid domain becomes potential, the velocity potential ϕ^o of the outer solution in the fluid domain can be expressed as

$$\phi^o = \left(r + \frac{1}{r}\right)\cos\theta H(T) - 2\epsilon\frac{A}{r}\cos\theta + \epsilon^2\sum_{n=1}\frac{1}{r^n}(a_n^c\cos n\theta + a_n^s \sin n\theta) + O(\epsilon^3). \tag{D7}$$

The inner expansion of ϕ^o is then written as

$$u_r^{oi} = (2\epsilon R - 3\epsilon^2 R^2)\cos\theta H(T) + 2A\epsilon(1 - 2\epsilon R)\cos\theta$$
$$- \epsilon^2\sum_{n=1} n(a_n^c\cos n\theta + a_n^s\sin n\theta) + O(\epsilon^3). \tag{D8}$$

The matching procedure to the inner solution in the fluid domain (i.e. $\mathcal{L}(u_r^{oi}) = U_r^{io}$) yields

$$\mathcal{L}(a_1^c) = \frac{E_s}{\bar{a}} + \frac{D_s}{a} - \frac{\bar{a}}{sa^2(a+\bar{a})} - \frac{a}{s\bar{a}^2(a+\bar{a})}, \tag{D9a}$$
$$\mathcal{L}(a_1^s) = -\frac{E_c}{\bar{a}} - \frac{D_c}{a} - \frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\mathrm{e}^{\bar{a}R}\,\mathrm{d}R + \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^{0} F^c\,\mathrm{d}R$$
$$- \frac{1}{2\nu_o a^2}\int_0^{\infty} F^c\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{\nu_o a^2}\int_0^{\infty} F^c\,\mathrm{d}R, \tag{D9b}$$
$$\mathcal{L}(a_2^s) = 0, \tag{D9c}$$
$$\mathcal{L}(a_2^c) = -\frac{E_{cs}}{\bar{a}} - \frac{D_{cs}}{a} - \frac{1}{2\nu_o\bar{a}^2}\int_{-\infty}^{0} F^{cs}\mathrm{e}^{\bar{a}R}\,\mathrm{d}R + \frac{1}{\nu_o\bar{a}^2}\int_{-\infty}^{0} F^{cs}\,\mathrm{d}R$$
$$- \frac{1}{2\nu_o a^2}\int_0^{\infty} F^{cs}\mathrm{e}^{-aR}\,\mathrm{d}R + \frac{1}{\nu_o a^2}\int_0^{\infty} F^{cs}\,\mathrm{d}R, \tag{D9d}$$
$$\mathcal{L}(a_n^{c,s}) = 0 \quad \text{for } n \geq 3. \tag{D9e}$$

The velocity potential ϕ^o in the outer fluid domain is therefore written as

$$\phi^o = \left(r + \frac{1}{r}\right)\cos\theta H(T) - 2\epsilon\frac{A}{r}\cos\theta + \epsilon^2\frac{a_1^c\cos\theta + a_1^s\sin\theta}{r} + \epsilon^2\frac{a_2^c}{r^2}\cos 2\theta + O(\epsilon^3), \tag{D10}$$

which also gives

$$\frac{\partial\phi^o}{\partial T} = -2\frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta + \epsilon\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}T}(a_1^c\cos\theta + a_1^s\sin\theta) + \epsilon\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}T}(a_2^c\cos 2\theta) + O(\epsilon^2). \tag{D11}$$

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Differentiating (D10) with respect to either r or θ, the velocities, u_r^o and u_θ^o, in the outer fluid domain are calculated as, for $r > 1$,

$$u_r^o = \left(1 - \frac{1}{r^2}\right)\cos\theta + 2\epsilon A \frac{1}{r^2}\cos\theta - \epsilon^2 (a_1^c \cos\theta + a_1^s \sin\theta)\frac{1}{r^2} - \epsilon^2 \frac{2}{r^3} a_2^c \cos 2\theta + O(\epsilon^3), \tag{D12a}$$

$$u_\theta^o = -\left(1 + \frac{1}{r^2}\right)\sin\theta + 2\epsilon A \frac{1}{r^2}\sin\theta - \epsilon^2 (a_1^c \sin\theta - a_1^s \cos\theta)\frac{1}{r^2} - \epsilon^2 \frac{2}{r^3} a_2^c \sin 2\theta + O(\epsilon^3). \tag{D12b}$$

The inner expansion of u_r^o and u_θ^o are then written as, respectively,

$$u_r^{oi} = 2\epsilon R\cos\theta + 2\epsilon A\cos\theta + O(\epsilon^2), \tag{D13a}$$

$$u_\theta^{oi} = -2(1-\epsilon R)\sin\theta + 2\epsilon A\sin\theta + O(\epsilon^2). \tag{D13b}$$

The inner expansion of the pressure in the outer fluid domain, p^{oi}, is then obtained as, using the unsteady Bernoulli's equation,

$$\frac{p^{oi}}{\rho} = 2\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta - 2\sin^2\theta + \frac{1}{2} - \epsilon\left[2R\frac{\mathrm{d}A}{\mathrm{d}T}\cos\theta - 4A\sin^2\theta + \frac{\mathrm{d}}{\mathrm{d}T}(a_1^c\cos\theta + a_1^s\sin\theta) + \frac{\mathrm{d}}{\mathrm{d}T}(a_2^c\cos 2\theta)\right] + O(\epsilon^2). \tag{D14}$$

From the matching condition $p^{io} = p^{o1}$, we can obtain (3.109).

Appendix E

Let us estimate the order of the following integral for $\lambda_o \gg 1$:

$$I = \mathcal{L}^{-1}\left(\int_{-\infty}^{0} F^c \exp(\bar{a}R)\,\mathrm{d}R\right) = \int_{-\infty}^{0}\mathrm{d}R\int_0^T f^c(T-\xi, R)\frac{|R|}{2\sqrt{\pi\nu_o\xi^3}}\exp\left(-\lambda_o\xi - \frac{R^2}{4\nu_o\xi}\right)\mathrm{d}\xi. \tag{E1}$$

Assuming that f^c is continuous in the integral domain, f^c has a finite maximum value $|f^c|_{max}$ in the domain. Therefore, we have

$$|I| \le |f^c|_{max}\int_{-\infty}^{0}\mathrm{d}R\int_0^T \frac{|R|}{2\sqrt{\pi\nu_o\xi^3}}\exp\left(-\lambda_o\xi - \frac{R^2}{4\nu_o\xi}\right)\mathrm{d}\xi \le |f^c|_{max}\sqrt{\frac{\nu_o}{\pi}}\int_0^T\frac{\exp(-\lambda_o\xi)}{\sqrt{\xi}}\mathrm{d}\xi = |f^c|_{max}\sqrt{\frac{\nu_o}{\pi}}\frac{2}{\sqrt{\lambda_o}}\mathrm{Erf}(\sqrt{\lambda_o T}) \sim O(1/\sqrt{\lambda_o}), \tag{E2}$$

where $\mathrm{Erf}(x) = \int_0^x \mathrm{e}^{-x^2}\,\mathrm{d}x$.

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Appendix F

The hydrodynamic forces (F_{bx}, F_{by}) are given by (3.153*a*) and (3.153*b*). Therefore, we have, for a circular cylinder,

$$\begin{bmatrix} F_{bx}/\rho \\ F_{by}/\rho \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} \dfrac{\partial}{\partial\theta}\left(\dfrac{p}{\rho}\right)\sin\theta - \nu\omega\sin\theta - u_r u \\ -\dfrac{\partial}{\partial\theta}\left(\dfrac{p}{\rho}\right)\cos\theta + \nu\omega\cos\theta - u_r v \end{bmatrix}_{r=1} \mathrm{d}\theta. \tag{F1}$$

Since $\chi = 0$ in the fluid domain, the governing equation of the motion becomes, with respect to θ,

$$\frac{\mathrm{D}u_\theta}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{r\partial\theta} + \nu\frac{\partial\omega}{\partial r}. \tag{F2}$$

Therefore, we have

$$\begin{aligned} \begin{bmatrix} F_{bx}/\rho \\ F_{by}/\rho \end{bmatrix} = {} & \nu\int_0^{2\pi}\left(\frac{\partial\omega}{\partial r} - \omega\right)\Bigg|_{r=1}\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}\mathrm{d}\theta - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} u_\theta|_{r=1}\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}\mathrm{d}\theta \\ & - \int_0^{2\pi} u_r|_{r=1}\begin{bmatrix} u \\ v \end{bmatrix}_{r=1}\mathrm{d}\theta. \end{aligned} \tag{F3}$$

An alternative expression of the force imposed by the no-slip condition on the cylinder surface is defined as $F_{\omega x}$ and $F_{\omega y}$ and, then,

$$\begin{bmatrix} F_{\omega x}/\rho \\ F_{\omega y}/\rho \end{bmatrix} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}} \omega \begin{bmatrix} y \\ -x \end{bmatrix}\mathrm{d}v. \tag{F4}$$

Applying Green's theorem and the definition of vorticity ω, we easily have

$$\begin{bmatrix} F_{\omega x}/\rho \\ F_{\omega y}/\rho \end{bmatrix} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{C}_\infty - \mathscr{C}_o} u_s \begin{bmatrix} y \\ -x \end{bmatrix}\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}}\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}v, \tag{F5}$$

where $\mathscr{C}_\infty$ and $\mathscr{C}_o$ denote the contours of the control surfaces in the far field and on the cylinder surface, respectively. The tangential velocity is then described by u_s on $\mathscr{C}_\infty$ or $\mathscr{C}_o$. In addition, $\mathrm{d}v$ and $\mathrm{d}s$ indicate the small elements of $\mathscr{F}$ and the contour $\mathscr{C}_\infty$ or $\mathscr{C}_o$. From the equations of motion, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}}\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}v = \int_{\mathscr{F}}\begin{bmatrix} -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\Delta u \\ -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\Delta v \end{bmatrix}\mathrm{d}v. \tag{F6}$$

Since $\Delta u = -\partial\omega/\partial y$ and $\Delta v = \partial\omega/\partial x$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{F}}\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}v = \int_{\mathscr{C}_o}\begin{bmatrix} (p/\rho)\,\mathrm{d}y + \nu\omega\,\mathrm{d}x \\ -p/\rho - \nu\omega\,\mathrm{d}y \end{bmatrix}. \tag{F7}$$

Here, the conditions that p is constant and $\omega = 0$ on $\mathscr{C}_\infty$ has been used. Therefore, we finally arrive at

$$\begin{bmatrix} F_{bx}/\rho \\ F_{by}/\rho \end{bmatrix} = \begin{bmatrix} F_{\omega x}/\rho \\ F_{\omega y}/\rho \end{bmatrix} - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{C}_o} u_s\begin{bmatrix} y \\ -x \end{bmatrix}\mathrm{d}s - \int_{\mathscr{C}_o} u_n\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}s, \tag{F8}$$

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where u_n is the normal component of the velocity from the cylinder to the fluid flow domain at $\mathscr{C}_o$. Therefore, in the penalization method the two terms are needed in addition to the formula that is derived from the no-slip boundary condition.

Using the present results, we have, at $R = 0$,

$$\frac{\mathrm{D}u_\theta}{\mathrm{D}t} = \frac{1}{\epsilon}\mathcal{L}^{-1}\left(\frac{\bar{a}}{a+\bar{a}}\varOmega\right) + \frac{3}{2}\mathcal{L}^{-1}\left(\frac{\varOmega}{a+\bar{a}}\right) - \frac{2}{\epsilon}\mathcal{L}^{-1}\left(\frac{a}{a+\bar{a}}\right)\sin\theta + \hat{u}_{r1}\left(\frac{\partial \hat{u}^o_{\theta 0}}{\partial\theta}\cos\theta + \frac{\hat{u}^s_{\theta 0}}{\partial\theta}\sin\theta\cos\theta\right) + \hat{u}^o_{\theta 0}\hat{u}^s_{\theta 0}\cos\theta + \hat{u}^{s2}_{\theta 0}\sin\theta\cos\theta + O(\epsilon). \tag{F9}$$

Hence, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} u_\theta \sin\theta\,\mathrm{d}\theta &= -\frac{2\pi}{\epsilon}\mathcal{L}^{-1}\left(\frac{a}{a+\bar{a}}\right) + O(\epsilon)\\ &= -\frac{\pi}{\epsilon}\frac{\mathrm{d}}{\mathrm{d}T}\left[\mathrm{e}^{-\lambda_o T/2}\left(\mathrm{I}_0(\lambda_o T/2) + \mathrm{I}_1(\lambda_o T/2)\right)\right] + O(\epsilon)\\ &= \frac{\pi}{4}\frac{\lambda_o}{\epsilon}\mathrm{e}^{-\lambda_o T/2}\left[\mathrm{I}_0(\lambda_o T/2) - \mathrm{I}_2(\lambda_o T/2)\right] + O(\epsilon).\end{aligned} \tag{F10}$$

It can therefore be found that the present result of the drag force is the same as our previous result (see Ueda & Kida 2021) by adding the above result to the previous one. For the lift force, the additional term is written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} u_\theta \cos\theta\,\mathrm{d}\theta &= \pi\left(\hat{u}_{r1}\frac{\partial \hat{u}^o_{\theta 0}}{\partial\theta} + \hat{u}^o_{\theta 0}\hat{u}^s_{\theta 0}\right) + O(\epsilon)\\ &\approx O(1/\sqrt{\lambda_o}) + O(\epsilon).\end{aligned} \tag{F11}$$

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