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ANALYTIC MAPPINGS WITH NEGATIVE COEFFICIENTS IN THE UNIT DISC

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Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $U(|z| < 1)$ and let $F(z) = (1-\lambda)f(z) + \lambda\big(f(z)*h(z)\big)$ where $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ with c_n's are known and are nonnegative, $\lambda \geq 0$. In the present paper, using convolution methods we investigate the mapping properties of $F(z)$ when $f(z)$ belongs respectively to several subclasses of analytic functions with negative coefficients.

1.

Let A denote the class of functions $f(z)$ analytic in $U = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. A function $f \in A$ is said to be starlike of order α, $0 \leq \alpha < 1$, denoted by S_α, if $\text{Re}\{z\big(f'(z)/f(z)\big)\} > \alpha$ for $z \in U$ and is said to be convex of order α, $0 \leq \alpha < 1$, denoted by K_α, if $\text{Re}\{1+z\big(f''(z)/f'(z)\big)\} > \alpha$ for $z \in U$. Further, let P_α denote the class of functions $f \in A$ such that $\text{Re}\big(f'(z)\big) > \alpha$, $0 \leq \alpha < 1$, $z \in U$.

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Given two functions $f, g \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution or Hadamard product $f(z) * g(z)$ is defined by $f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. Ruscheweyh [3] using convolution techniques, introduced and studied an important subclass of A, the class of pre-starlike functions of order α, which is denoted by R_α. Thus $f \in A$ is said to be pre-starlike of order α, $0 \le \alpha < 1$, if $f(z) * z(1-z)^{2\alpha-2} \in S_\alpha$. It may be noted that $R_0 = K_0$ and $R_{\frac{1}{2}} = S_{\frac{1}{2}}$.

Let T be the class of functions of the form $z - \sum_{n=2}^{\infty} |a_n| z^n$, which are analytic and univalent in U, and set $T[\alpha] = S_\alpha \cap T$, $K[\alpha] = K_\alpha \cap T$ and $P[\alpha] = P_\alpha \cap T$. Silverman [6] studied the classes $T[\alpha]$ and $K[\alpha]$, and Gupta and Jain [2] studied $P[\alpha]$ by obtaining several results for these sub-classes including the characterization of extreme points of their closed convex hulls. An analogous study for $R_\alpha \cap T$ $(= R[\alpha])$ was made by Silverman and Silvia [7].

It is well known [6] that $f \in T$ if and only if

$$(1.1) \qquad \sum_{n=2}^{\infty} n|a_n| \le 1$$

and a necessary and sufficient condition for $f(z)$ to be in $T[\alpha]$ is that

$$(1.2) \qquad \sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha}\right) |a_n| \le 1 .$$

It is also known [4] that a necessary and sufficient condition for $f(z)$ to be in $P[\alpha]$ is that

$$(1.3) \qquad \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha}\right) |a_n| \le 1 ,$$

and $f \in R[\alpha]$ if and only if

$$\sum_{n=2}^{\infty} \frac{(n-\alpha)B(\alpha,n)}{1-\alpha} |a_n| \le 1 \tag{1.4}$$

where $B(\alpha, n) = \left[\prod_{k=2}^{n} (k-2\alpha)\right]/(n-1)!$, $n = 2, 3, 4, \ldots$ (see [7]).

Sarangi and Uralegaddi [5] have studied the mapping properties of the function $F(z)$ defined by

$$2F(z) = \left(zf(z)\right)' \tag{1.5}$$

where $f(z)$ is in $T[\alpha]$, $K[\alpha]$ or $P[\alpha]$. Recently, Bhoonsurmath and Swami [1] extended these results by studying the mapping properties of the function $F(z)$ defined by

$$F(z) = (1-\lambda)f(z) + \lambda\left(zf'(z)\right) , \quad \lambda \ge 0 , \tag{1.6}$$

where $f(z)$ is in T, $T[\alpha]$, $K[\alpha]$ or $P[\alpha]$.

In the present paper, using convolution methods, we study the mapping properties of the function $F(z)$ defined by

$$F(z) = (1-\lambda)f(z) + \lambda\left(f(z)*h(z)\right) \tag{1.7}$$

where $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ with c_n's known and nonnegative, $\lambda \ge 0$ and when $f(z)$ is respectively in T, $T[\alpha]$, $K[\alpha]$, $P[\alpha]$ or $R[\alpha]$. The proposed study not only gives as a particular case, the results of Sarangi and Uralegaddi [5] and Bhoonsurmath and Swami [1] but also lead to the mapping properties for the functions of the form

$$F(z) = (1-\lambda)f(z) + \lambda z , \quad 0 \le \lambda \le 1 , \tag{1.8}$$

where $f(z)$ is in T, $T[\alpha]$, $K[\alpha]$, $P[\alpha]$ or $R[\alpha]$.

It may be noted that choosing $h(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} nz^n$ in (1.7) we get the mapping properties of the functions defined in (1.6) while taking $h(z) = z$ will lead the mapping properties for the functions defined by (1.8) where $f(z)$ belongs respectively to the classes T, $T[\alpha]$, $K[\alpha]$, $P[\alpha]$ or $R[\alpha]$.

2.

THEOREM 1. *Let* $f(z)$ *be in* T *and* $F(z) = (1-\lambda)f(z) + \lambda(f(z)*h(z))$ *with* $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$. *If* λ *and the* c_n*'s are real, nonnegative and satisfy the condition* $1 - \lambda + \lambda c_n \geq 0$, *then* $\text{Re}\left(zF'(z)/F(z)\right) > \beta$, $0 \leq \beta < 1$, *for* $|z| < r(\lambda, \beta; c_n)$ *where*

$$r(\lambda, \beta; c_n) = \inf_n \left(\frac{n(1-\beta)}{(n-\beta)(1-\lambda+\lambda c_n)}\right)^{1/(n-1)} \quad (n = 2, 3, \ldots) .$$

The result is sharp.

Proof. Since $f(z)$ is in T we have

$$F(z) = (1-\lambda)f(z) + \lambda(f(z)*h(z))$$

$$= z - \sum_{n=2}^{\infty} (1-\lambda+\lambda c_n)|a_n|z^n .$$

It is sufficient to show that $|(zF'(z)/F(z))-1| < 1 - \beta$ for $|z| < r(\lambda, \beta; c_n)$. Suppose λ and the c_n's are real, nonnegative and satisfy the condition $1 - \lambda + \lambda c_n \geq 0$. Then

$$\left|z\frac{F'(z)}{F(z)} - 1\right|$$

$$= \left|\left(\sum_{n=2}^{\infty} (n-1)(1-\lambda+\lambda c_n)|a_n|z^{n-1}\right) \Big/ \left(1 - \sum_{n=2}^{\infty} (1-\lambda+\lambda c_n)|a_n|z^{n-1}\right)\right|$$

$$\leq \left(\sum_{n=2}^{\infty} (n-1)(1-\lambda+\lambda c_n)|a_n||z|^{n-1}\right) \Big/ \left(1 - \sum_{n=2}^{\infty} (1-\lambda+\lambda c_n)|a_n||z|^{n-1}\right)$$

Thus $|(zF'(z)/F(z))-1| \leq 1 - \beta$ if

$$\sum_{n=2}^{\infty} (n-1)(1-\lambda+\lambda c_n)|a_n||z|^{n-1} \leq (1-\beta)\left(1 - \sum_{n=2}^{\infty} (1-\lambda+\lambda c_n)|a_n||z|^{n-1}\right)$$

which is equivalent to

$$\sum_{n=2}^{\infty} \frac{(n-\beta)(1-\lambda+\lambda c_n)}{1-\beta} |a_n||z|^{n-1} \leq 1 . \qquad (2.1)$$

Since $f(z)$ is in T , from (1.1) it follows that the inequality (2.1)

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will be true if

$$(2.2) \qquad \frac{(n-\beta)\left(1-\lambda+\lambda c_n\right)}{1-\beta}\left|a_n\right||z|^{n-1} \le n\left|a_n\right| \quad (n = 2, 3, 4, \ldots) .$$

Solving (2.2) for $|z|$, we obtain

$$(2.3) \qquad |z| \le \left(\frac{n(1-\beta)}{(n-\beta)\left(1-\lambda+\lambda c_n\right)}\right)^{1/(n-1)}, \quad n = 2, 3, 4, \ldots .$$

Setting $|z| = r\left(\lambda, \beta; c_n\right)$ in (2.3), the result follows.

The functions given by

$$f_n(z) = z - \frac{1}{n} z^n \quad (n = 2, 3, \ldots)$$

show that the result obtained in the theorem is sharp.

As stated in Section 1, choosing $h(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} nz^n$ so that $c_n = n$ in the theorem, we get the following result found in [1].

COROLLARY 1. *Let $f(z)$ be in T and $F(z) = (1-\lambda)f(z) + \lambda zf'(z)$ for $z \in U$ where $\lambda \ge 0$. Then* $\operatorname{Re}\left(zF'(z)/F(z)\right) > \beta$, $0 \le \beta < 1$ *for* $|z| < r(\lambda, \beta)$ *where*

$$r(\lambda, \beta) = \inf_n \left(\frac{n(1-\beta)}{(n-\beta)(1-\lambda+n\lambda)}\right)^{1/(n-1)} \quad (n = 2, 3, 4, \ldots) .$$

The result is sharp, with the extremal function being of the form

$$f_n(z) = z - \frac{z^n}{n} \quad (n = 2, 3, \ldots) .$$

COROLLARY 2. *Let $f(z)$ be in T and $F(z) = (1-\lambda)f(z) + \lambda z$ for $z \in U$ and $0 \le \lambda \le 1$. Then* $\operatorname{Re}\left(zF'(z)/F(z)\right) > \beta$, $0 \le \beta < 1$ *for* $|z| < r(\lambda, \beta)$ *where*

$$r(\lambda, \beta) = \inf_n \left(\frac{n(1-\beta)}{(1-\lambda)(n-\beta)}\right)^{1/(n-1)} \quad (n = 2, 3, 4, \ldots) .$$

The result is sharp with the extremal function being of the form

$$f_n(z) = z - \frac{z^n}{n}, \quad n = 2, 3, \ldots .$$

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The result is obtained by choosing $h(z) = z$ so that $c_n = 0$ in the above theorem.

It may be noted that for $\lambda = \frac{1}{2}$ our Corollary 1 gives the corresponding result due to Sarangi and Uralegaddi [5].

THEOREM 2. *Let* $f(z)$ *be in* $T[\alpha]$ *and*

$$F(z) = (1-\lambda)f(z) + \lambda\big(f(z)*h(z)\big)$$

with $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$. *If* λ *and the* c_n*'s are real, nonnegative and satisfy the condition* $1 - \lambda + \lambda c_n \geq 0$, *then* $F(z)$ *is starlike of order* β , $0 \leq \beta < 1$ *for* $|z| < r(\lambda, \alpha, \beta; c_n)$ *where*

$$r(\lambda, \alpha, \beta; c_n) = \inf_n \left\{\frac{(n-\alpha)(1-\beta)}{(1-\alpha)(n-\beta)(1-\lambda+\lambda c_n)}\right\}^{1/(n-1)} , \quad n = 2, 3, \ldots .$$

The result is sharp.

Proof. The proof is similar to that of Theorem 1. The only difference is that the estimate (1.2) is to be used in place of (1.1).

The result is sharp with the extremal function being of the form

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha} z^n \quad (n = 2, 3, \ldots) .$$

THEOREM 3. *Let* $f(z)$ *be in* $K[\alpha]$ *and*

$$F(z) = (1-\lambda)f(z) + \lambda\big(f(z)*h(z)\big)$$

with $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$. *If* λ *and the* c_n*'s are real, nonnegative and satisfy the condition* $1 - \lambda + \lambda c_n \geq 0$ *then* $F(z)$ *is convex of order* β , $0 \leq \beta < 1$ *for* $|z| < r(\lambda, \alpha, \beta; c_n)$ *where* $r(\lambda, \alpha, \beta; c_n)$ *is as stated in Theorem* 2. *The result is sharp with the extremal function being of the form*

$$f_n(z) = z - \frac{1-\alpha}{n(n-\alpha)} z^n \quad (n = 2, 3, \ldots) .$$

Proof. We have

$$(2.4) \qquad zF'(z) = (1-\lambda)zf'(z) + \lambda z\big(f(z)*h(z)\big)' = (1-\lambda)zf'(z) + \lambda\big(zf'(z)*h(z)\big) .$$

Since $f(z) \in K[\alpha]$, it follows that $zf'(z) \in T[\alpha]$. So, applying Theorem 2 with $zf'(z)$ instead of $f(z)$, it follows from (2.4) that $zF'(z)$ is starlike of order β, $0 \le \beta < 1$ in $|z| < r(\lambda, \alpha, \beta; c_n)$ or equivalently $F(z)$ is convex of order β in $|z| < r(\lambda, \alpha, \beta; c_n)$ where $r(\lambda, \alpha, \beta; c_n)$ is as stated in Theorem 2.

The following theorem can be proved on similar lines as that of Theorem 2. Hence we omit the proof.

THEOREM 4. *Let* $f(z)$ *be a function in* $R[\alpha]$ *and*

$$F(z) = (1-\lambda)f(z) + \lambda\big(f(z)*h(z)\big)$$

with $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$. *If* λ *and the* c_n*'s are real, nonnegative and satisfy the condition* $1 - \lambda + \lambda c_n \ge 0$, *then* $F(z)$ *is starlike of order* β, $0 \le \beta < 1$ *for* $|z| < r(\lambda, \alpha, \beta; c_n)$ *where*

$$r(\lambda, \alpha, \beta; c_n) = \inf_n \left\{\frac{(n-\alpha)(1-\beta)B(\alpha,n)}{(1-\alpha)(n-\beta)\left(1-\lambda+\lambda c_n\right)}\right\}^{1/(n-1)} , \quad n = 2, 3, \ldots ,$$

where $B(\alpha, n) = \prod_{k=2}^{n} (k-2\alpha)/(n-1)!$, $n = 2, 3, \ldots$. *The result is sharp, the extremal function being of the form*

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha)B(\alpha,n)} z^n , \quad n = 2, 3, \ldots .$$

THEOREM 5. *Let* $f(z)$ *be in* $P[\alpha]$ *and*

$$F(z) = (1-\lambda)f(z) + \lambda\big(f(z)*h(z)\big)$$

with $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$, $z \in U$. *If* λ *and the* c_n*'s are real, nonnegative and satisfy the condition* $1 - \lambda + \lambda c_n \ge 0$, *then* $\operatorname{Re}\big(F'(z)\big) > \beta$, $0 \le \beta < 1$ *for* $|z| < r(\lambda, \alpha, \beta; c_n)$ *where*

$$r(\lambda,\ \alpha,\ \beta;\ c_n) = \inf_n \left\{\frac{1-\beta}{(1-\alpha)(1-\lambda+\lambda c_n)}\right\}^{1/(n-1)} ,\quad n = 2, 3, \ldots .$$

The result is sharp.

Proof. It is sufficient to prove that the values for $F'(z)$ lie in a circle centred at 1 whose radius is $1-\beta$ for $|z| < r(\lambda,\ \alpha,\ \beta;\ c_n)$.

We have

$$|F'(z)-1| = \left|\sum_{n=2}^{\infty} n(1-\lambda+\lambda c_n)|a_n|z^{n-1}\right| .$$

Thus, if λ and the c_n's satisfy the condition $1-\lambda+\lambda c_n \geq 0$, then $|F'(z)-1| \leq 1-\beta$ if

$$\sum_{n=2}^{\infty} \frac{n(1-\lambda+\lambda c_n)}{1-\beta} |a_n||z|^{n-1} \leq 1 . \tag{2.5}$$

Since $f(z)$ is in $P[\alpha]$, we have (1.3). Thus (2.5) will be true if

$$\frac{n(1-\lambda+\lambda c_n)}{1-\beta} |a_n||z|^{n-1} \leq \frac{n}{1-\alpha} |a_n| ,\quad n = 2, 3, \ldots . \tag{2.6}$$

Now the remaining part of the proof is similar to that of Theorem 1.

The functions of the form

$$f_n(z) = z - \frac{1-\alpha}{n} z^n \quad (n = 2, 3, \ldots)$$

show that the estimate obtained in the theorem is sharp.

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