ON SIMPLE ALTERNATIVE RINGS

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1. Introduction. The only known simple alternative rings which are not associative are the *Cayley algebras*. Every such algebra has a scalar extension which is isomorphic over its center F to the algebra $\mathsf{C} = e_{11}\mathsf{F} + e_{00}\mathsf{F} + \mathsf{C}_{10} + \mathsf{C}_{01}$, where $\mathsf{C}_{ij} = e_{ij}\mathsf{F} + f_{ij}\mathsf{F} + g_{ij}\mathsf{F}$ $(i, j = 0, 1;\ i \neq j)$. The elements e_{11} and e_{00} are orthogonal idempotents and $e_{ii}x_{ij} = x_{ij}e_{jj} = x_{ij}$, $e_{jj}x_{ij} = x_{ij}e_{ii} = 0$, $x_{ij}^2 = 0$ for every x_{ij} of C_{ij}. The multiplication table of C is then completed by the relations[1]

(1) $$f_{10}g_{10} = e_{01},\ g_{10}e_{10} = f_{01},\ e_{10}f_{10} = g_{01},$$

(2) $$g_{01}f_{01} = e_{10},\ e_{01}g_{01} = f_{10},\ f_{01}e_{01} = g_{10},$$

(3) $$e_{ij}e_{ji} = f_{ij}f_{ji} = g_{ij}e_{ji} = e_{ii},$$

(4) $$e_{ij}f_{ji} = e_{ij}g_{ji} = f_{ij}e_{ji} = f_{ij}g_{ji} = g_{ij}e_{ij} = g_{ij}f_{ij} = 0.$$

R. H. Bruck and E. Kleinfeld have recently shown[2] that *every alternative division ring of characteristic not two is either associative or a Cayley algebra.* Their methods do not seem to be readily applicable to the simple case but we shall use the machinery of idempotents to prove the following result.

THEOREM. *Every simple alternative ring which contains an idempotent not its unity quantity is either associative or is the Cayley algebra* C.

2. Elementary properties. Our results are based on properties which were given by Zorn.[1] He assumed that the characteristic was not 2 or 3 and did not give complete details of his computations. As we shall make no assumption about the characteristic of our rings it will be necessary for us to re-derive the properties of Zorn and so make our exposition quite self-contained.

We first note that an alternative ring C is a mathematical system having the usual properties of associative rings except that the associative law for products is replaced by the identities $x(xy) = (xx)y$, $(yx)x = y(xx)$. It is easy to see that the *associator*

$$(x, y, z) = (xy)z - x(yz)$$

is an alternating function of its arguments x, y, z, a result which implies that

(5) $$z(xy + yx) = (zx)y + (zy)x,\ (xy + yx)z = x(yz) + y(xz),$$
$$z(xy) + y(xz) = (zx)y + (yx)z,$$

for every x, y, z of C. We shall assume henceforth that C *contains an idempotent* u *not the unity quantity* of C.

Received January 18, 1951.

[1]The multiplication table of a Cayley algebra was given in this form by M. Zorn, *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ., vol. 8 (1930), 123–147.

[2]*The structure of alternative division rings*, Proc. Amer. Math. Soc., vol. 2 (1951), 878-890.

The ring $\mathfrak{C}$ may be expressed as the module direct sum $\mathfrak{C}=\mathfrak{C}_{11}+\mathfrak{C}_{10}+\mathfrak{C}_{01}+\mathfrak{C}_{00}$ of its submodules $\mathfrak{C}_{ij}$ where $\mathfrak{C}_{ij}$ consists of all x_{ij} of $\mathfrak{C}$ such that $ux_{ij} = ix_{ij}$, $x_{ij}u = jx_{ij}$ $(i, j = 0, 1)$. Indeed if $x = x_{11} + x_{10} + x_{01} + x_{00}$ then $x_{11} = u(xu)$, $x_{10} = ux - u(xu)$, $x_{01} = xu - u(xu)$, $x_{00} = x - xu - ux - u(xu)$. This decomposition is precisely that of the associative case and needs no additional argument. However the multipicative properties of the modules $\mathfrak{C}_{ij}$ need to be derived. We proceed as follows:

Let $ux = \lambda x$, $xu = \mu x$, $uy = \alpha y$, $yu = \beta y$. Then

$$\begin{aligned}(x, y, u) &= (xy)u - x(yu) = (xy)u - \beta xy = -(y, x, u) = y(xu) - (yx)u \\ &= \mu yx - (yx)u = -(x, u, y) = x(uy) - (xu)y = (\alpha - \mu)xy = (y, u, x) \\ &= (yu)x - y(ux) = (\beta - \lambda)yx = (u, x, y) = (ux)y - u(xy) \\ &= \lambda xy - u(xy) = -(u, y, x) = u(yx) - (uy)x = u(yx) - \alpha yx.\end{aligned}$$

We thus obtain the identities

$$(xy)u = (\alpha + \beta - \mu)xy,\ u(xy) = (\lambda + \mu - \alpha)xy, \tag{6}$$

$$(yx)u = (\lambda + \mu - \beta)yx,\ u(yx) = (\alpha + \beta - \lambda)yx, \tag{7}$$

$$(\alpha - \mu)xy = (\beta - \lambda)yx, \tag{8}$$

where (7) is obviously derivable from (6) by the interchange of x and y and the consequent interchanges of λ, μ with α, β. If $\lambda = \mu = \alpha = \beta = 1$ we have $(xy)u = u(xy) = xy$ and so $\mathfrak{C}_{11}$ is a subring of $\mathfrak{C}$. Similarly the values $\lambda = \mu = \alpha = \beta = 0$ yield $u(xy) = (xy)u = 0$ and so $\mathfrak{C}_{00}$ is a subring of $\mathfrak{C}$. We now put $\lambda = \mu = 1$ and $\alpha = \beta = 0$ to obtain $xy = yx$, $(xy)u = -xy$, $(yx)u = 2yx$, and so $(xy)u = 2xy$, $3xy = 0$. But $[(xy)u]u = -(xy)u = xy = (xy)u^2 = (xy)u = -xy$ and $2xy = 0$, $xy = 0$. This proves[3] that $\mathfrak{C}_{11}$ *and and* $\mathfrak{C}_{00}$ *are orthogonal subrings of* $\mathfrak{C}$.

We next put $\lambda = \mu = 1 = \alpha$ and $\beta = 0$. Then $(xy)u = 0$ and $u(xy) = xy$, $yx = 0$, and so $\mathfrak{C}_{11}\mathfrak{C}_{10} \subseteq \mathfrak{C}_{10}$, $\mathfrak{C}_{10}\mathfrak{C}_{11} = 0$. By symmetry $\mathfrak{C}_{01}\mathfrak{C}_{11} \subseteq \mathfrak{C}_{01}$, $\mathfrak{C}_{11}\mathfrak{C}_{01} = 0$. Similarly, the values $\lambda = \mu = \beta = 0$ and $\alpha = 1$ yield $xy = 0$, $u(yx) = yx$, $(yx)u = 0$, and so $\mathfrak{C}_{00}\mathfrak{C}_{10} = 0$, $\mathfrak{C}_{10}\mathfrak{C}_{00} \subseteq \mathfrak{C}_{10}$, and $\mathfrak{C}_{01}\mathfrak{C}_{00} = 0$, $\mathfrak{C}_{00}\mathfrak{C}_{01} \subseteq \mathfrak{C}_{01}$ by symmetry. The relations $\mathfrak{C}_{10}\mathfrak{C}_{01} \subseteq \mathfrak{C}_{11}$, $\mathfrak{C}_{01}\mathfrak{C}_{10} \subseteq \mathfrak{C}_{00}$ follow from (6) ,(7) by taking $\lambda = \beta = 1$, $\alpha = \mu = 0$.

The properties derived so far for the component modules $\mathfrak{C}_{ij}$ are properties satisfied by all associative rings. In the associative case $\mathfrak{C}_{10}{}^2 = \mathfrak{C}_{01}{}^2 = 0$. However, this last result need not hold in the alternative case, and we now put $\alpha = \lambda = 1$, $\mu = \beta = 0$ and obtain $(xy)u = xy$, $u(xy) = 0$, $xy = -yx$. Thus we have the property

$$x_{10}y_{10} = -y_{10}x_{10} = z_{01}, \tag{9}$$

for every x_{10} and y_{10} of $\mathfrak{C}_{10}$, where z_{01} is in $\mathfrak{C}_{01}$. Similarly

$$x_{01}y_{01} = -y_{01}x_{01} = z_{10}. \tag{10}$$

[3]This seems to be one of the few places in our development where an assumption about the characteristic would make any difference.

Now $x_{10}^2 = (ux_{10})x_{10} = ux_{10}^2 = uz_{01} = 0$, and by symmetry we have the relation

$$x_{ij}^2 = 0 \qquad (i, j = 0, 1; i \neq j). \tag{11}$$

Zorn also gave the following result:

Lemma 1. *Let x, y, z be elements of the component modules of C not all in the same subring C_{ii}. Then $(x, y, z) = 0$ except possibly when at least two of the elements are in the same module C_{ij} $(i \neq j)$.*

We also have the identities

$$z_{jj}(x_{ij}y_{ij}) = (x_{ij}z_{jj})y_{ij} = x_{ij}(y_{ij}z_{jj}), \tag{12}$$

$$(x_{ij}y_{ij})z_{ii} = (z_{ii}x_{ij})y_{ij} = x_{ij}(z_{ii}y_{ij}), \tag{13}$$

$$x_{ij}(y_{ji}z_{ji}) = z_{ji}(x_{ij}y_{ji}) = (z_{ji}x_{ij})y_{ji}, \tag{14}$$

$$x_{ij}(y_{ij}z_{ij}) = z_{ij}(x_{ij}y_{ij}) = y_{ij}(z_{ij}x_{ij}), \tag{15}$$

$$(x_{ij}y_{ij})z_{ij} = (z_{ij}x_{ij})y_{ij} = (y_{ij}z_{ij})x_{ij}. \tag{16}$$

We use (5) to write

$$z_{ji}(x_{ij}y_{ji}) - x_{ij}(y_{ji}z_{ji}) = z_{ji}(x_{ij}y_{ji}) + x_{ij}(z_{ji}y_{ji}) = (z_{ji}x_{ij} + x_{ij}z_{ji})y_{ji} = (z_{ji}x_{ij})y_{ji}$$

since $(x_{ij}z_{ji})y_{ji} = 0$. This proves (14). Also

$$(x_{ij}y_{ij})z_{ii} + (z_{ii}y_{ij})x_{ij} = x_{ij}(y_{ij}z_{ii}) + z_{ii}(y_{ij}x_{ij}) = 0$$

and so $(x_{ij}y_{ij})z_{ii} = -(z_{ii}y_{ij})x_{ij} = x_{ij}(z_{ii}y_{ij})$. Interchange x and y to obtain $(y_{ij}x_{ij})z_{ii} = -(z_{ii}x_{ij})y_{ij} = -(x_{ij}y_{ij})z_{ii}$ and we have proved (13). Formula (12) follows by symmetry. Now $(x_{ii}, y_{ii}, z_{jj}) = 0$ trivially,

$$(x_{ii}, y_{ii}, z_{ij}) = -(x_{ii}, z_{ij}, y_{ii}) = x_{ii}(z_{ij}y_{ii}) - (x_{ii}z_{ij})y_{ii} = 0,$$
$$(x_{ii}, y_{ij}, z_{jj}) = -(y_{ij}, x_{ii}, z_{jj}) = y_{ij}(x_{ii}z_{jj}) - (y_{ij}x_{ii})z_{jj} = 0,$$
$$(x_{ii}, y_{ij}, z_{ji}) = -(y_{ij}, x_{ii}, z_{ji}) = y_{ij}(x_{ii}z_{ji}) - (y_{ij}x_{ii})z_{ji} = 0.$$

The remaining properties of the associator follow by symmetry. Formula (15) states that the factors in $x_{ij}(y_{ij}z_{ij})$ may be permuted cyclically. To prove this result we use the final relation in (5) to write

$$z_{ij}(x_{ij}y_{ij}) + y_{ij}(x_{ij}z_{ij}) = (z_{ij}x_{ij})y_{ij} + (y_{ij}x_{ij})z_{ij}.$$

The left member is in $\mathsf{C}_{ij}\mathsf{C}_{ij}^2 \subseteq \mathsf{C}_{ij}\mathsf{C}_{ji} \subseteq \mathsf{C}_{ii}$ and the right member is in $\mathsf{C}_{ij}^2\mathsf{C}_{ij} \subseteq \mathsf{C}_{ji}\mathsf{C}_{ij} \subseteq \mathsf{C}_{jj}$. Since $i \neq j$ both members vanish and we have

$$-y_{ij}(x_{ij}z_{ij}) = y_{ij}(z_{ij}x_{ij}), \quad (z_{ij}x_{ij})y_{ij} = -(y_{ij}x_{ij})z_{ij} = (x_{ij}y_{ij})z_{ij}$$

from which we have both (15) and (16).

Corollary. *The ring C is associative if and only if both C_{11} and C_{00} are associative and $\mathsf{C}_{10}^2 = \mathsf{C}_{01}^2 = 0$.*

https://doi.org/10.4153/CJM-1952-013-x Published online by Cambridge University Press

3. Construction of ideals. We first consider the product $p_{ii} = x_{ij}(y_{ij}z_{ij})$ which is an element of $\mathbf{C}_{ij}\mathbf{C}_{ij}^2 \subseteq \mathbf{C}_{ii}$ and let a_{ii} be any element of C_{ii}. Then $a_{ii}p_{ii} = (a_{ii}x_{ij})(y_{ij}z_{ij})$ by Lemma 1. But then (15) and (13) imply that

$$a_{ii}p_{ii} = z_{ij}[(a_{ii}x_{ij})y_{ij}] = z_{ij}[(x_{ij}y_{ij})a_{ii}] = [z_{ij}(x_{ij}y_{ij})]a_{ii} = p_{ii}a_{ii}$$

for every a_{ii} of $\mathbf{C}_{ii}$ and p_{ii} of $\mathbf{C}_{ij}\mathbf{C}_{ij}^2$, $\mathbf{C}_{ii}(\mathbf{C}_{ij}\mathbf{C}_{ij}^2) = (\mathbf{C}_{ij}\mathbf{C}_{ij}^2)\mathbf{C}_{ii} \subseteq \mathbf{C}_{ij}\mathbf{C}_{ij}^2$. If b_{ii} is in C_{ii} then

$$\begin{aligned} b_{ii}(a_{ii}p_{ii}) &= b_{ii}[(a_{ii}x_{ij})(y_{ij}z_{ij})] = [b_{ii}(a_{ii}x_{ij})](y_{ij}z_{ij}) \\ &= [(b_{ii}a_{ii})x_{ij}](y_{ij}z_{ij}) = (b_{ii}a_{ii})p_{ii}, \end{aligned}$$

and $\mathbf{C}_{ij}\mathbf{C}_{ij}^2$ is contained in the centre of $\mathbf{C}_{ii}$. By symmetry we have the following result:

LEMMA 2. *The modules $\mathbf{C}_{ij}\mathbf{C}_{ij}^2$ and $\mathbf{C}_{ji}^2\mathbf{C}_{ji}$ are ideals of $\mathbf{C}_{ii}$ which are contained in the centre of $\mathbf{C}_{ii}$ $(i, j = 0, 1; i \neq j)$.*

We next prove the following result:

LEMMA 3. *Let B_i be an ideal of $\mathbf{C}_{ii}$. Then*

$$\mathbf{D}_i = \mathbf{B}_i + \mathbf{B}_i\mathbf{C}_{ij} + \mathbf{C}_{ji}\mathbf{B}_i + (\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij} \qquad (i, j = 0, 1; i \neq j) \tag{17}$$

is an ideal of $\mathbf{C}$.

We have $(\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij} = \mathbf{C}_{ji}(\mathbf{B}_i\mathbf{C}_{ij})$ by Lemma 1. We now compute

$$\mathbf{C}_{ii}\mathbf{D}_i = \mathbf{C}_{ii}\mathbf{B}_i + (\mathbf{C}_{ii}\mathbf{B}_i)\mathbf{C}_{ij} \subseteq \mathbf{D}_i, \quad \mathbf{D}_i\mathbf{C}_{ii} = \mathbf{B}_i\mathbf{C}_i + \mathbf{C}_{ji}(\mathbf{B}_i\mathbf{C}_{ii}) \subseteq \mathbf{D}_i,$$
$$\mathbf{C}_{ij}\mathbf{D}_i = \mathbf{C}_{ij}(\mathbf{C}_{ji}B_i) + \mathbf{C}_{ij}[(\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij}] \subseteq \mathbf{C}_{ii}\mathbf{B}_i + [\mathbf{C}_{ij}(\mathbf{C}_{ji}\mathbf{B}_i)]\mathbf{C}_{ij} + (\mathbf{C}_{ji}\mathbf{B}_i)(\mathbf{C}_{ij}^2)$$

by (14). Then $\mathbf{C}_{ij}\mathbf{D}_i \subseteq \mathbf{B}_i + \mathbf{B}_i\mathbf{C}_{ij} + (\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ji} \subseteq \mathbf{B}_i + \mathbf{B}_i\mathbf{C}_{ij} + \mathbf{B}_i\mathbf{C}_{ji}^2 \subseteq \mathbf{D}_i$ since $\mathbf{C}_{ji}^2 \subseteq \mathbf{C}_{ij}$. Also

$$\mathbf{D}_i\mathbf{C}_{ij} = \mathbf{B}_i\mathbf{C}_{ij} + (\mathbf{B}_i\mathbf{C}_{ij})\mathbf{C}_{ij} + (\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij} \subseteq \mathbf{B}_i\mathbf{C}_{ij} + C_{ij}^2\mathbf{B}_i + \mathbf{C}_{ji}(\mathbf{B}_i C_{ij}) \subseteq \mathbf{D}_i.$$

If we pass to a ring anti-isomorphic to $\mathbf{C}$ the module $\mathbf{D}_i$ is unchanged but $\mathbf{C}_{ji}$ is replaced by $\mathbf{C}_{ij}$. Hence $\mathbf{C}_{ji}\mathbf{D}_i \subseteq \mathbf{D}_i$, $\mathbf{D}_i\mathbf{C}_{ji} \subseteq \mathbf{D}_i$. Finally

$$\begin{aligned} \mathbf{C}_{jj}\mathbf{D}_i &= \mathbf{C}_{jj}(\mathbf{C}_{ji}\mathbf{B}_i) + \mathbf{C}_{jj}[(\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij}] \\ &= (\mathbf{C}_{jj}\mathbf{C}_{ji})\mathbf{B}_i + [\mathbf{C}_{jj}(\mathbf{C}_{ji}\mathbf{B}_i)]\mathbf{C}_{ij} \subseteq \mathbf{C}_{ji}\mathbf{B}_i + (\mathbf{C}_{ji}\mathbf{B}_i)\mathbf{C}_{ij} \subseteq \mathbf{D}_i, \end{aligned}$$

and $\mathbf{D}_i\mathbf{C}_{jj} = \mathbf{B}_i(\mathbf{C}_{ij}\mathbf{C}_{jj}) + (\mathbf{C}_{ji}\mathbf{B}_i)(\mathbf{C}_{ij}\mathbf{C}_{jj}) \subseteq \mathbf{D}_i$. This completes our proof.

LEMMA 4. *Let*

$$\mathbf{C}_{10}^2\mathbf{C}_{10} = \mathbf{C}_{10}\mathbf{C}_{10}^2 = \mathbf{C}_{01}^2\mathbf{C}_{01} = \mathbf{C}_{01}\mathbf{C}_{01}^2 = 0.$$

Then $\mathbf{G} = \mathbf{C}_{10}^2 + \mathbf{C}_{01}^2$ is a proper ideal of $\mathbf{C}$.

We have

$$\mathbf{C}_{ii}\mathbf{G} = \mathbf{C}_{ii}\mathbf{C}_{ji}^2 = (\mathbf{C}_{ji}\mathbf{C}_{ii})\mathbf{C}_{ji} \subseteq \mathbf{C}_{ji}^2 \subseteq \mathbf{G}, \; \mathbf{G}\mathbf{C}_{ii} = \mathbf{C}_{ij}^2\mathbf{C}_{ii} = \mathbf{C}_{ij}(\mathbf{C}_{ii}\mathbf{C}_{ij}) \subseteq \mathbf{G}.$$

https://doi.org/10.4153/CJM-1952-013-x Published online by Cambridge University Press

Also $\mathbf{C}_{ij}\mathbf{G} = \mathbf{C}_{ij}\mathbf{C}_{ji}^2 \subseteq \mathbf{C}_{ij}^2 \subseteq \mathbf{G}$, $\mathbf{G}\mathbf{C}_{ij} \subseteq \mathbf{C}_{ji}^2\mathbf{C}_{ij} \subseteq \mathbf{C}_{ij}^2 \subseteq \mathbf{G}$, as desired. Now $\mathbf{C}_{11} \neq 0$, $\mathbf{C}_{11}$ is not contained in $\mathbf{G}$, and $\mathbf{G}$ is a proper ideal of $\mathbf{C}$.

The constructions just given are sufficient for our needs and we proceed now to the simple case.

4. Simple rings. Lemma 1 implies that

$$x_{ii}[y_{ii}(z_{ij}w_{ji})] = x_{ii}[(y_{ii}z_{ij})w_{ji}] = [x_{ii}(y_{ii}z_{ij})]w_{ji}$$
$$= [(x_{ii}y_{ii})z_{ij}]w_{ji} = (x_{ii}y_{ii})(z_{ij}w_{ji}).$$

Since $x_{ii}(z_{ij}w_{ji}) = (x_{ii}z_{ij})w_{ji}$ and $(z_{ij}w_{ji})x_{ii} = z_{ij}(w_{ji}x_{ii})$ we see that $\mathbf{C}_{ij}\mathbf{C}_{ji}$ is an associative ideal of $\mathbf{C}_{ii}$. It follows immediately that $\mathbf{B} = \mathbf{C}_{10}\mathbf{C}_{01} + \mathbf{C}_{10} + \mathbf{C}_{01} + \mathbf{C}_{01}\mathbf{C}_{10}$ is an ideal of $\mathbf{C}$. If $\mathbf{C}$ is simple and $\mathbf{B} = 0$ then $\mathbf{C}_{00}$ is a proper ideal of $\mathbf{C}$, and $\mathbf{C} = \mathbf{C}_{11}$ has u as unity quantity contrary to hypothesis. Hence $\mathbf{B} = \mathbf{C}$, $\mathbf{C}_{ij}\mathbf{C}_{ji} = \mathbf{C}_{ii}$ is associative. If $\mathbf{B}_i$ were a non-zero proper ideal of $\mathbf{C}_{ii}$ the ideal $\mathbf{D}_i$ of Lemma 3 would be a non-zero proper ideal of $\mathbf{C}$. Thus we have

LEMMA 5. *Let $\mathbf{C}$ be simple. Then $\mathbf{C}_{11}$ is a simple associative ring and $\mathbf{C}_{00}$ is either zero or a simple associative ring.*

When $\mathbf{C}$ is simple the set $\mathbf{G}$ of Lemma 4 cannot be a proper ideal of $\mathbf{C}$. Hence $\mathbf{C}$ is either associative or $\mathbf{G} = \mathbf{C}_{10}^2 + \mathbf{C}_{01}^2 \neq 0$, one of the modules $\mathbf{C}_{10}\mathbf{C}_{10}^2$, $\mathbf{C}_{01}^2\mathbf{C}_{01}$, $\mathbf{C}_{10}^2\mathbf{C}_{10}$, $\mathbf{C}_{01}\mathbf{C}_{01}^2$ must not be zero. Let $\mathbf{C}_{ij}\mathbf{C}_{ij}^2 \neq 0$. By Lemma 2 we know that $\mathbf{B}_i = \mathbf{C}_{ij}\mathbf{C}_{ij}^2$ is a non-zero ideal of $\mathbf{C}_{ii}$, by Lemma 5 that $\mathbf{B}_i = \mathbf{C}_{ii}$, $\mathbf{C}_{ii}$ coincides with its centre and must be a field. If $a_i = x_{ij}h_{ji} \neq 0$ where x_{ij} is in $\mathbf{C}_{ij}$ and y_{ji} is in $\mathbf{C}_{ij}^2$ then

$$a_i^2 = a_i(x_{ij}y_{ji}) = (a_ix_{ij})y_{ji} = [x_{ij}(y_{ji}x_{ij})]y_{ji} \neq 0$$

and so $y_{ji}x_{ij} \neq 0$, $\mathbf{C}_{ij}^2\mathbf{C}_{ij} \neq 0$. The converse is obvious and so $\mathbf{C}_{ij}\mathbf{C}_{ij}^2 \neq 0$ if and only if $\mathbf{C}_{ij}^2\mathbf{C}_{ij} \neq 0$. It follows that both $\mathbf{C}_{11}$ and $\mathbf{C}_{00}$ are fields. Moreover, since we may pass to an anti-isomorphic ring if necessary, we may assume that $\mathbf{C}_{10}\mathbf{C}_{10}^2 \neq 0$. We now prove

LEMMA 6. *The rings $\mathbf{C}_{11}$ and $\mathbf{C}_{00}$ are isomorphic fields with unity quantities $u = e_{11}$ and e_{00} respectively, $e = e_{11} + e_{00}$ is the unity quantity of $\mathbf{C}$, $e_{11} = e_{10}e_{01}$, $e_{00} = e_{01}e_{10}$ for quantities e_{ij} in $\mathbf{C}_{ij}$ such that $e_{01} = f_{10}g_{10}$ and f_{10}, g_{10} are in $\mathbf{C}_{10}$.*

We select f_{11} and g_{10} so that $x_{10}e_{01} = a_1 \neq 0$ in the field $\mathbf{C}_{11}$. Then a_1 has an inverse b_1 in $\mathbf{C}_{11}$ and $b_1(x_{10}e_{01}) = e_{11} = (b_1x_{10})e_{01} = e_{10}e_{01}$. Thus

$$e_{11}^2 = e_{11}(e_{10}e_{01}) = (e_{11}e_{10})e_{01} = [e_{10}(e_{01}e_{10})]e_{01} = e_{11}$$

and so $e_{01}e_{10} = e_{00} \neq 0$. But

$$e_{00}^2 = (e_{00}e_{01})e_{10} = [(e_{01}e_{10})e_{01}]e_{10} = (e_{01}e_{11})e_{10} = e_{01}e_{10} = e_{00}$$

is an idempotent of $\mathbf{C}_{00}$ and must be its unity quantity.

We now use Lemma 3 with $\mathbf{B}_i = \mathbf{C}_{ii} \neq 0$ and see that $\mathbf{C}_{11}\mathbf{C}_{10} = \mathbf{C}_{10}\mathbf{C}_{00} = \mathbf{C}_{10}$, $\mathbf{C}_{01}\mathbf{C}_{11} = \mathbf{C}_{00}\mathbf{C}_{01} = \mathbf{C}_{01}$. The fact that $\mathbf{C}_{01} = \mathbf{C}_{00}\mathbf{C}_{01}$ implies that $e_{00}x_{01} = x_{01}$

for every x_{01} of $\mathbf{C}_{01}$. Similarly $x_{10}e_{00} = x_{10}$ for every x_{10} of $\mathbf{C}_{10}$. It is now trivial to see that $e = e_{11} + e_{00}$ is the unity quantity of $\mathbf{C}$.

The mapping

$$x_{11} \to x_{11}T = e_{01}(x_{11}e_{10}) = (e_{01}x_{11})e_{10}$$

is an isomorphism of $\mathbf{C}_{11}$ *onto* $\mathbf{C}_{00}$ such that

$$y_{10}(x_{11}T) = x_{11}y_{10}, \quad (x_{11}T)z_{01} = z_{01}x_{11}$$

for every x_{11} of $\mathbf{C}_{11}$, y_{10} of $\mathbf{C}_{10}$ and z_{01} of $\mathbf{C}_{01}$. Indeed we compute

$$\begin{aligned} y_{10}[e_{01}(x_{11}e_{10})] &= (y_{10}e_{01})(x_{11}e_{10}) + [y_{10}(x_{11}e_{10})]e_{01} = x_{11}[(y_{10}e_{01})e_{10} + (y_{10}e_{10})e_{01}] \\ &= x_{11}[y_{10}(e_{01}e_{10} + e_{10}e_{01})] = x_{11}y_{10}. \end{aligned}$$

Similarly $w_{01}x_{11} = (x_{11}T)w_{01}$. Also

$$\begin{aligned} (x_{11}T)(y_{11}T) &= [e_{01}(x_{11}e_{10})](y_{11}T) = e_{01}[(x_{11}e_{10})(y_{11}T)] \\ &= e_{01}[y_{11}(x_{11}e_{10})] = e_{01}[(x_{11}y_{11})e_{10}] = (x_{11}y_{11})T. \end{aligned}$$

Since $\mathbf{C}_{11}$ and $\mathbf{C}_{00}$ are fields, this proves that T is an isomorphism of $\mathbf{C}_{11}$ onto $\mathbf{C}_{00}$. Actually T has an inverse given by $x_{11} = e_{10}(x_{00}e_{01}) = e_{10}y_{01}$ since then

$$x_{11}T = [e_{01}(e_{10}y_{01})]e_{10} = [(e_{01}e_{10})y_{01} + (e_{01}y_{01})e_{10}]e_{10} = (x_{00}e_{01})e_{10} = x_{00},$$

a result following from $(z_{10}e_{10})e_{10} = z_{10}e_{10}{}^2 = 0$ and

$$(x_{00}e_{01})e_{10} + (x_{00}e_{10})e_{01} = (x_{00}e_{01})e_{10} = x_{00}e_{00} = x_{00}.$$

We now show that the set $\mathbf{Z}$ of all elements $z = z_{11} + z_{11}T$ is contained[4] in the centre of $\mathbf{C}$. Indeed $zy_{ii} = y_{ii}z$ for every y_{ii} of $\mathbf{C}_{ii}$ trivially. Also

$$zy_{10} = z_{11}y_{10} = y_{10}(z_{11}T) = y_{10}z, \quad zy = yz$$

for every y of $\mathbf{C}$. Since $\mathbf{Z} = \mathbf{C}_{11} + \mathbf{C}_{00}$ we know that the associators (z, x, y) with x and y in components $\mathbf{C}_{ij}$ are zero unless possibly when $x = x_{ij}$ and $y = y_{ij}$ are in the same $\mathbf{C}_{ij}$ $(i \neq j)$. But

$$\begin{aligned} [(z_{11} + z_{11}T)x_{10}]y_{10} &= (z_{11}x_{10})y_{10}, \\ (z_{11} + z_{11}T)(x_{10}y_{10}) = (z_{11}T)(x_{10}y_{10}) &= [x_{10}(z_{11}T)]y_{10} = (z_{11}x_{10})y_{10} \end{aligned}$$

as desired.

By our construction, $\mathbf{C}_{11} = e_{11}\mathbf{Z}$ and $\mathbf{C}_{00} = e_{00}\mathbf{Z}$ are one-dimensional algebras over $\mathbf{Z}$. We also note that since $e_{10}e_{01} = e_{10}(f_{10}g_{10}) = e_{11}$ we may use (15) to obtain $g_{10}(e_{10}f_{10}) = f_{10}(g_{10}e_{10}) = e_{11}$. Put

$$e_{10}f_{10} = g_{01}, \quad g_{10}e_{10} = f_{01}$$

and obtain (1). Then

$$g_{01}g_{10} = (e_{10}f_{10})g_{10} = (f_{10}g_{10})e_{10} = e_{00} = (g_{10}e_{10})f_{10} = f_{01}f_{10}$$

and we have (3). Now $e_{10}g_{01} = e_{10}(e_{10}f_{10}) = 0$, $e_{10}f_{01} = e_{10}(g_{10}e_{10}) = -e_{10}(e_{10}g_{10}) = 0$ since $e_{10}{}^2 = 0$. Similarly

[4]If $\mathbf{C}$ has characteristic not two or three the property $zy = yz$ implies that $(z, x, y) = 0$. However our proof is so arranged that $(z, x, y) = 0$ is quite trivial.

https://doi.org/10.4153/CJM-1952-013-x Published online by Cambridge University Press

$$f_{10}g_{01} = f_{10}e_{01} = g_{10}e_{01} = g_{10}f_{01} = 0,$$
$$g_{01}e_{10} = f_{01}e_{10} = g_{01}f_{10} = e_{01}f_{10} = e_{01}g_{10} = f_{01}g_{10} = 0$$

and we have completed a proof which shows that (4) holds. The computation

$$e_{01}(g_{10}e_{10}) + g_{10}(e_{01}e_{10}) = e_{01}f_{01} + g_{10} = (e_{01}g_{10} + g_{10}e_{01})e_{10} = 0$$

yields $g_{10} = f_{01}e_{01}$. The remaining formulae of (2) are derived similarly.

We have now shown that $\mathbf{C}$ contains an algebra $\mathbf{D}$ over $\mathbf{Z}$ with the multiplication table given by (1)-(4). It remains only to show that e_{ij}, f_{ij}, g_{ij} are linearly independent over $\mathbf{F}$ and that these elements form a basis of $\mathbf{C}_{ij}$ over $\mathbf{Z}$ in order to prove that $\mathbf{D}$ is the eight-dimensional Cayley algebra over $\mathbf{Z}$ and that $\mathbf{C} = \mathbf{D}$.

LEMMA 7. *Let $h_{ij}h_{ji} = e_{ij}$ so that $h_{ji}h_{ij} = e_{ji}$. Then $x_{ij}h_{ij} = 0$ if and only if $x_{ij} = \alpha h_{ij}$ for α in $\mathbf{Z}$.*

We have $x_{ij}(e_{ii} + e_{jj}) = x_{ij} = (x_{ij}h_{ij})h_{ji} + (x_{ij}h_{ji})h_{ij}$. If $x_{ij}h_{ij} = 0$ then $x_{ij} = \alpha h_{ij}$ with $x_{ij}h_{ji} = \alpha e_{ii}$ and α in $\mathbf{Z}$. The converse follows from $h_{ij}^2 = 0$.

LEMMA 8. *Let $x_{ij}e_{ji} = x_{ij}f_{ji} = x_{ij}g_{ji} = 0$. Then $x_{ij} = 0$.*

If $x_{ij}h_{ji} = \alpha e_{ii}$ and $h_{ji}x_{ij} = \beta e_{jj}$ then

$$h_{ji}(x_{ij}h_{ji}) = \alpha h_{ji} = (h_{ji}x_{ij})h_{ij} = \beta h_{ji}.$$

If $h_{ji} \neq 0$ then $\alpha = \beta$. Now $x_{ij}e_{ij} = \pm x_{ij}(f_{ji}g_{ji})$ by (1) and (2) and so

$$x_{ij}e_{ij} = \pm [g_{ji}(x_{ij}f_{ji}) - (g_{ji}x_{ij})f_{ji}]$$

by (14). It follows that $x_{ij}e_{ij} = 0$ and that $x_{ij} = \alpha e_{ij}$. Similarly $x_{ij} = \beta f_{ij}$. But if $\alpha \neq 0$ we have

$$\alpha e_{ij}f_{ij} = \pm \alpha g_{ij} = \beta f_{ij}^2 = 0$$

contrary to hypothesis. Hence $\alpha = 0$, $x_{ij} = 0$.

It is evident that the proof above implies that $f_{ij} \neq \alpha e_{ij}$ for α in $\mathbf{Z}$. If $g_{ij} = \alpha e_{ij} + \beta f_{ij}$ then

$$g_{ij}e_{ij} = \pm f_{ij} = \beta f_{ij}e_{ij} = \pm \beta g_{ij}$$

which has been shown to be impossible. We have shown that $\mathbf{D}$ is an eight-dimensional algebra.

We now let $x_{ij}e_{ji} = \alpha e_{ii}$, $x_{ij}f_{ji} = \beta e_{ii}$, $x_{ij}g_{ji} = \gamma e_{ii}$ for α, β, γ in $\mathbf{Z}$. Then $y_{ij} = x_{ij} - (\alpha e_{ij} + \beta f_{ij} + \gamma g_{ij})$ has the property that

$$y_{ij}e_{ji} = (\alpha - \alpha)e_{ii} = 0,$$
$$y_{ij}f_{ji} = (\beta - \beta)e_{ii} = 0,$$
$$y_{ij}g_{ji} = (\gamma - \gamma)e_{ii} = 0$$

and so $y_{ij} = 0$ by Lemma 8. This completes our proof.

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