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MUKAI-UMEMURA'S EXAMPLE OF THE FANO THREEFOLD WITH GENUS 12 AS A COMPACTIFICATION OF C^3

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§0. Introduction

Let (X, Y) be a smooth projective compactification with the non-normal irreducible boundary Y, namely, X is a smooth projective algebraic threefold and Y a non-normal irreducible divisor on X such that $X - Y$ is isomorphic to C^3. Then Y is ample and the canonical divisor K_X on X can be written as $K_X = -rY (1 \leqq r \leqq 4)$. Thus X is a Fano threefold. In particular, $\operatorname{Pic} X \cong \mathbf{Z}\mathcal{O}_X(Y)$. The non-normality of Y implies that $r \leqq 2$ (cf. [4]). In the case of $r = 2$, such a (X, Y) is uniquely determined up to isomorphism, in fact, $(X, Y) \cong (V_5, H_5^\infty)$, where $X = V_5$ is a Fano threefold of degree 5 in $\mathbf{P}^6$, and $Y = H_5^\infty$ is a ruled surface swept out by lines which intersect the line Σ with the normal bundle $N_{\Sigma|X} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$, in particular, Σ is the singular locus of Y. In the case of $r = 1$, there is an example of such a compactification of $\mathbf{C}^3$, in fact, let $X = V'_{22}$ be a Fano threefold of genus $g = 12$ constructed by Mukai-Umemura [11] and $Y = H'_{22}$ be the ruled surface swept out by conics which intersect the line ℓ in V'_{22} with the normal bundle $N_{\ell|X} \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(1)$, then H'_{22} is a non-normal hyperplane section of V'_{22} such that $V'_{22} - H'_{22}$ is isomorphic to $\mathbf{C}^3$, in particular, the line ℓ is the singular locus of H'_{22} (cf. [6]).

Now, in this paper, we will construct a birational map $\pi : V'_{22} \cdot\cdot\!\rightarrow V_5$ such that the restriction π_0 of π on $V'_{22} - H'_{22}$ gives an isomorphism $V'_{22} - H'_{22} \cong V_5 - H_5^\infty \cong \mathbf{C}^3$, via the resolution of indeterminancy of the double projection of V'_{22} from the singular locus $\operatorname{Sing} H'_{22}$ of H'_{22} which is a line on V'_{22} (see Theorem 1). Furthermore, we will study the detailed structure of the desingularization and the normalization of the boundary divisor H'_{22} (see Theorem 2).

Recently, Mukai ([11_a]) proved that there is a 4-dimensional family of Fano threefolds of first kind with index one, genus 12 which are the compactifications of $\mathbf{C}^3$ with non-normal boundaries, in particular, our example (V'_{22}, H'_{22}) belongs

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to this Mukai's family.

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Notation

K_X	Canonical divisor on a variety X
ω_X	Canonical sheaf on X
$N_{C\|X}$	Normal bundle of C in X
$\|H\|$	Complete linear system associated with a divisor H
$\mathrm{Bs}\,\|H\|$	Base locus of the linear system $\|H\|$
Sing X	Singular locus of X
$\rho(X)$	Picard number of X
E_{red}	Reduction of a scheme E
supp D	Support of a divisor D
(i)-curve	Smooth rational curve with self-intersection number $-i$
$b_i(X)$	$:= \dim H^i(X\,;\mathbf{R})$
$h^i(\mathscr{F})$	$:= \dim H^i(*\,;\mathscr{F})$
$\chi(\mathscr{F})$	$:= \sum_{i=0} (-1)^i h^i(\mathscr{F})$

§1. Mukai-Umemura's example

Let $\mathbf{C}[x, y]$ be the polynomial ring of two complex variables x and y. The special linear group $SL(2, \mathbf{C})$ acts $\mathbf{C}(x, y)$ as follows:

$$\begin{cases} x^\sigma = ax + by \\ y^\sigma = cx + dy \end{cases} \quad \text{for } \sigma = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in SL(2, \mathbf{C}).$$

Let us denote by R_n a vector space of homogeneous polynomials of degree n in $\mathbf{C}[x, y]$. Let $f(x, y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i}y^i \in R_n$ be a non-zero homogeneous polynomial of degree n. We take $(a_0 : a_1 : \ldots : a_n)$ as homogeneous coordinates on the projective space $\mathbf{P}(R_n) \cong \mathbf{P}^n$, on which $SL(2, \mathbf{C})$ acts. Let us denote by $X(f)$ the closure of $SL(2, \mathbf{C})$-orbit $SL(2, \mathbf{C}) \cdot f$ of f in $\mathbf{P}(R_n)$. Then $SL(2, \mathbf{C})$ acts on $X(f)$.

Now, we consider the following two polynomials:

$$f_6(x, y) = xy(x^4 - y^4), \text{ and}$$

$$h_{12}(x, y) = xy(x^{10} + 11x^5y^5 + y^{10}).$$

We put

$$V_5 := X(f_6) \hookrightarrow \mathbf{P}(R_6) \cong \mathbf{P}^6, \text{ and}$$

$$V'_{22} := X(h_{12}) \hookrightarrow \mathbf{P}(R_{12}) \cong \mathbf{P}^{12}.$$

Then we have

LEMMA 1 (Lemma 3.3 in [11]). (1) *$V_5 \hookrightarrow \mathbf{P}^6$ is a Fano threefold of index 2, genus 21 and the hyperplane section of V_5 is the positive generator of* Pic $V_5 \cong \mathbf{Z}$

(2) *V'_{22} is a Fano threefold of index 1, genus 12 and the hyperplane section of V'_{22} is the positive generator of* Pic $V'_{22} \cong \mathbf{Z}$.

The defining equations for V_5, V'_{22} are given as follows respectively:

$$(V_5) \quad \begin{cases} a_0a_4 - 4a_1a_3 + 3a_2^2 = 0 \\ a_0a_5 - 3a_1a_4 + 2a_2a_3 = 0 \\ a_0a_6 - 9a_2a_4 + 8a_3^2 = 0 \\ a_1a_6 - 3a_2a_5 + 2a_3a_4 = 0 \\ a_2a_6 - 4a_3a_5 + 3a_4^2 = 0 \end{cases}$$

$$(V'_{22}) \qquad \sum_{\lambda=0}^{\rho} \binom{8}{\lambda}\binom{8}{\rho-\lambda}(a_\lambda a_{\rho+4-\lambda} - 4a_{\lambda+1}a_{\rho+3-\lambda} + 3a_{\lambda+2}a_{q+2-\lambda}) = 0 \qquad (0 \leq \rho \leq 16)$$

Now, we put

$$H_5^\infty := V_5 \cap \{a_0 = 0\} \hookrightarrow \mathbf{P}^5,$$

$$H'_{22} := V'_{22} \cap \{a_0 = 0\} \hookrightarrow \mathbf{P}^{11}.$$

Let us denote by Sing H_5^∞ (resp. Sing H'_{22}) the singular locus of H_5^∞ (resp. H'_{22}). Then we have

PROPOSITION 1 ([5]). (1) $V_5 - H_5^\infty = V_5 \cap \{a_0 \neq 0\} \cong \mathbf{C}^3$,

(2) $\Sigma := \text{Sing } H_5^\infty = \{a_0 = a_1 = \cdots = a_4 = 0\} \cong \mathbf{P}^1(a_5 : a_6) \hookrightarrow \mathbf{P}^6$ *is a line on V_5. In particular, H_5^∞ is a non-normal hyperplane section of V_5 swept out by lines which intersect the line Σ.*

PROPOSITION 2 ([6]). *H'_{22} is a non-normal hyperplane section such that $V'_{22} - H'_{22} = V'_{22} \cap \{a_0 \neq 0\} \cong \mathbf{C}^3$.*

We will study the detailed structure of H'_{22} below.

LEMMA 2. (1) *$\ell := \operatorname{Sing} H'_{22} = \{a_0 = \cdots = a_{10} = 0\} \cong \mathbf{P}^1(a_{11} : a_{12}) \hookrightarrow \mathbf{P}^{12}$ is a line on V'_{22}.*

(2) *The normal bundle $N_{\ell|V'_{22}} \cong O_\ell(-2) \oplus O_\ell(1)$, and there is no other line in V'_{22} which intersects the line ℓ.*

(3) *H'_{22} is a unique member of the linear system $|O_{V'_{22}}(1) \otimes I^3_\ell|$, where I_ℓ is the ideal sheaf of ℓ in $O_{V'_{22}}$. In particular, H'_{22} is a ruled surface swept out by conics in V'_{22} which intersect the line ℓ.*

Proof. We shall rewrite the defining equation (V'_{22}) as follows:

$$
(V'_{22})^* \quad \begin{cases}
(e.0) & a_0a_4 - 4a_1a_3 + 3a_2^2 = 0 \\
(e.1) & a_0a_5 - 3a_1a_4 + 2a_2a_3 = 0 \\
(e.2) & 7a_0a_6 - 12a_1a_5 - 15a_2a_4 + 20a_3^2 = 0 \\
(e.3) & a_0a_7 - 6a_2a_5 + 5a_3a_4 = 0 \\
(e.4) & 5a_0a_8 + 12a_1a_7 - 42a_2a_6 - 20a_3a_5 + 45a_4^2 = 0 \\
(e.5) & a_0a_9 + 6a_1a_8 - 6a_2a_7 - 28a_3a_6 + 27a_4a_5 = 0 \\
(e.6) & a_0a_{10} + 12a_1a_9 + 12a_2a_8 - 76a_3a_7 - 21a_4a_6 \\
 & \quad + 72a_5^2 = 0 \\
(e.7) & a_0a_{11} + 24a_1a_{10} + 90a_2a_9 - 130a_3a_8 - 405a_4a_7 \\
 & \quad + 420a_5a_6 = 0 \\
(e.8) & a_0a_{12} + 60a_1a_{11} + 534a_2a_{10} - 380a_3a_9 - 3195a_4a_8 \\
 & \quad - 720a_5a_7 + 2940a_6^2 = 0 \\
(e.9) & a_1a_{12} + 24a_2a_{11} + 90a_3a_{10} - 130a_4a_9 - 405a_5a_8 \\
 & \quad + 420a_6a_7 = 0 \\
(e.10) & a_2a_{12} + 12a_3a_{11} + 12a_4a_{10} - 76a_5a_9 - 21a_6a_8 \\
 & \quad + 72a_7^2 = 0 \\
(e.11) & a_3a_{12} + 6a_4a_{11} - 6a_5a_{10} - 28a_6a_9 + 27a_7a_8 = 0 \\
(e.12) & 5a_4a_{12} + 12a_5a_{11} - 42a_6a_{10} - 20a_7a_9 + 45a_8^2 = 0 \\
(e.13) & a_5a_{12} - 6a_7a_{10} + 5a_8a_9 = 0 \\
(e.14) & 7a_6a_{12} - 12a_7a_{11} - 15a_8a_9 + 20a_9^2 = 0 \\
(e.15) & a_7a_{12} - 3a_8a_{11} + 2a_9a_{10} = 0 \\
(e.16) & a_8a_{12} - 4a_9a_{11} + 3a_{10}^2 = 0
\end{cases}
$$

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For simplicity, let us denote by $\{a_j = 1\}$ the affine part $\{a_j \neq 0\}$ of $\mathbf{P}^{12}(a_0:\ldots: a_j :\ldots: a_{12})$, namely, $\{a_j = 1\} \cong \mathbf{C}^{12}(a_0,\ldots,a_{j-1}, a_{j+1},\ldots,a_{12})$.

CLAIM 1. $H'_{22} \cap \{a_1 = 1\} \cong \mathbf{C}^{12}(a_2, a_6)$

In fact, setting $a_0 = 0$, $a_1 = 1$ in the equations (e.0) − (e.9) in $(V'_{22})^*$, one can easily see that the coordinate functions $a_3, a_4, a_7, a_8,\ldots, a_{12}$ are given by the polynomials of a_2 and a_6. This proves the claim.

Now, we have $H'_{22} \cap \{a_1 = 0\} = V'_{22} \cap \{a_0 = a_1 = 0\} = \{a_0 = a_1 = \ldots = a_{10} = 0\} \cong \mathbf{P}^1(a_{11} : a_{11})$ (a line in V'_{22}). Since $H'_{22} - H'_{22} \cap \{a_1 = 0\} \cong \mathbf{C}^2$ by the Claim 1, we have that H'_{22} is non-normal (cf. [5]) and hence $\mathrm{Sing}\, H'_{22} = H'_{22} \cap \{a_1 = 0\}$. This proves (1).

Next, let us consider the affine part $H'_{22} \cap \{a_{12} = 1\} \hookrightarrow \mathbf{C}^{12}(a_1,\ldots,a_{11})$ of H'_{22}. Setting $a_0 = 0$, $a_{12} = 1$ in the defining equation $(V'_{22})^*$, one can get the defining equation of $H'_{22} \cap \{a_{12} = 1\}$ in $\mathbf{C}^{11}$. More precisely, from (e.9) − (e.16) with $a_{12} = 1$, putting $x := a_9$, $y := a_{10}$, $z_{10} := a_{11}$, one can get the following:

$$\begin{cases} (\mathrm{e}.16)' & a_8 = 2^2xz - 3y^2 \\ (\mathrm{e}.15)' & a_7 = 2^2\cdot 3xz^2 - 3^2y^2z - 2xy \\ (\mathrm{e}.14)' & 7a_6 = 2^4\cdot 3^2xz^3 - 2^2\cdot 3^3y^2z^2 + 2^2\cdot 3^2xyz \\ & \qquad - 3^2\cdot 5y^3 - 2^2\cdot 5x^2 \\ (\mathrm{e}.13)' & a_5 = 2^3\cdot 3^2xyz^2 - 2\cdot 3^3y^3z + 3xy^2 - 2^2\cdot 5x^2z \\ (\mathrm{e}.12)' & a_4 = -2^4\cdot 3x^2z^2 - 2^5x^2y + 2^3\cdot 3^3xy^2z - 3^3\cdot 5y^4 \\ (\mathrm{e}.11)' & a_3 = -2^4\cdot 5x^3 - 2^4\cdot 3^3x^2z^3 + 2^4\cdot 3^3x^2yz \\ & \qquad + 2^3\cdot 3^4xy^2z^2 - 2^2\cdot 3^4xy^3 - 3^5y^4z \\ (\mathrm{e}.10)' & a_2 = -2^5\cdot 5^2x^3z - 2^7\cdot 3^3x^2z^4 + 2^4\cdot 3^3\cdot 11x^2yz^2 \\ & \qquad + 2^6\cdot 3^4xy^2z^3 - 2^3\cdot 3^3\cdot 29xy^3z - 2^3\cdot 3^5y^4z^2 \\ & \qquad + 2^3\cdot 3^2\cdot 7x^2y^2 + 3^4\cdot 5^2y_5 \\ (\mathrm{e}.9)' & a_1 = -2^4\cdot 3^2\cdot 5\cdot 7x^3z^2 - 2^8\cdot 3^4x^2z^5 + 2^6\cdot 3^3\cdot 19x^2yz^3 \\ & \qquad + 2^7\cdot 3^5xy^2z^4 - 2^5\cdot 3^4\cdot 17xy^3z^2 - 2^4\cdot 3^6y^4z^3 \\ & \qquad - 2^3\cdot 3^3x^2y^2z + 2^2\cdot 3^6\cdot 5y^5z + 2^7\cdot 5x^3y \\ & \qquad + 3^3\cdot 5\cdot 19xy^4 \end{cases}$$

CLAIM 2. $H'_{22} \cap \{a_{12} \neq 0\} \cong V(f) := \{(x, y, z) \in \mathbf{C}^3 ; f(x, y, z) = 0\}$, *where*

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$$(*)\quad f(x, y, z) = b_0x^4 + (b_1yz + b_2z^3)x^3 + (b_3y^3 + b_4y^2z^2 + b_5yz^4)x^2 + (b_6y^4z + b_7y^3z^3)x + b_8y^6 + b_9y^5z^2,$$

$$(b_0 = -2^8\cdot 5^2,\ b_1 = 2^9\cdot 3^3\cdot 5,\ b_2 = -2^6\cdot 3^4\cdot 5,$$
$$b_3 = -2^8\cdot 3^3\cdot 7,\ b_4 = -2^4\cdot 3^4\cdot 127,\ b_5 = 2^9\cdot 3^5,$$
$$b_6 = 2^2\cdot 3^6\cdot 89,\ b_7 = -2^8\cdot 3^6,\ b_8 = -3^6\cdot 5^3,\ b_9 = 2^5\cdot 3^7).$$

In fact, putting $a_1,\ldots,a_8$ in $(\mathrm{e}.k)'$ $(9 \leqq k \leqq 16)$ into (e.8) with $a_{12} = 1$, one can get the equation $f(x, y, z) = 0$. It is easy to see that the polynomial $f(x, y, z)$ is irreducible. Hence, $V(f)$ is the defining equation of $H'_{22} \cap \{a_{12} \neq 0\}$ in $\mathbf{C}^3$.

By the defining equation of $V(f)$, one can see the singular locus Sing $V(f) = \{x = y = 0\}$ and the multiplicity of $V(f)$ at a general point of Sing $V(f)$ is equal to three.
Thus $H'_{22} \in |\mathcal{O}_{V'_{22}}(1) \otimes I_\ell^3|$. Since $h^0(\mathcal{O}_{V'_{22}}(1) \otimes I_\ell^3) \leqq 1$ by Iskovskih [7], H'_{22} is a unique member of $|\mathcal{O}_{V'_{22}}(1) \otimes I_\ell^3|$. This implies that any conics in V'_{22} intersecting the line ℓ is always contained in H'_{22}. By Iskovskih [7], for every point $p \in V'_{22}$, there is a finite number of conics passing through p. Thus we have the assertion (3). The assertion (2) is proved in Mukai-Umemura [11].

Q.E.D.

§2. Double projection

We will study the double projection of V'_{22} from the line ℓ, which is the singular locus of H'_{22}. For simplicity, we put $X := V'_{22}$, $Y := H'_{22}$.

First, let us consider the linear system $|\mathscr{H}| := |\mathcal{O}_X(1) \otimes I_\ell^2|$ on X. Let $\sigma_1 : X_1 \to X$ be the blowing up of X along the line ℓ in X. By Lemma 2-(2), we have $L_1 := \sigma_1^{-1}(\ell) \cong \mathbf{F}_3$ (Hirzebruch surface). We put $|\mathscr{H}_1| := |\sigma_1^* H - 2L_1|$, where $H \in |\mathcal{O}_X(1)|$. Let Y_1 be the proper transform of Y in X_1. By Lemma 2-(3), we have a linear equivalence $Y_1 \sim \sigma_1^* H - 3L_1$. By Lemma 5.4 in Iskovskih [7], we have

LEMMA 3. (1) $\dim|\mathscr{H}| = \dim|\mathscr{H}_1| = 6$,

(2) $\dim|\sigma_1^* H - 3L_1| = 0$, *namely, Y_1 is the unique member of the linear system* $|\sigma_1^* H - 3L_1|$,

(3) $(\sigma_1^* H - 2L_1)^3 = 2$,

(4) *$Y_1 \cdot L_1 \sim 3\ell_1 + 7f_1$ in L_1, where ℓ_1, f_1 is the negative section, a fiber of L_1 respectively.*

Let K_{X_1} be a canonical divisor on X_1. Then we have $K_{X_1} \sim -\sigma_1^* H + L_1$. Since $(L_1 \cdot \ell_1) = 1$, we have $(K_{X_1} \cdot \ell_1) = 0$. By the following exact sequence of normal bundles:

$$\begin{array}{ccccccccc} 0 & \rightarrow & N_{\ell_1|L_1} & \rightarrow & N_{\ell_1|X_1} & \rightarrow & N_{L_1|X_1|\ell_1} & \rightarrow & 0 \\ & & \wr\!\| & & \wr\!\| & & \wr\!\| & & \\ & & \mathcal{O}(-3) & & \mathcal{O}(a) \oplus \mathcal{O}(b) & & \mathcal{O}(1) & & \end{array}$$

where $a + b = 2$, we have

LEMMA 4.

$$N_{\ell_1|X_1} \cong \begin{cases} \text{(a)} & \mathcal{O}(-1) \oplus \mathcal{O}(-1), \\ \text{(b)} & \mathcal{O}(-2) \oplus \mathcal{O} \\ \text{(c)} & \mathcal{O}(-3) \oplus \mathcal{O}(1). \end{cases}, \text{ or}$$

LEMMA 5. Bs $|H_1| = \ell_1$, *where* Bs $|H_1|$ *is the base locus of* $|H_1|$.

Proof. Since $(\sigma_1^* H - 2L_1) \cdot \ell_1 = -1$, $\ell_1 \subseteqq \mathrm{Bs}\,|\mathcal{H}_1|$. By Lemma 2-(2), there is no other line in X which intersects ℓ. Thus, by the same argument as in the proof of Lemma 5.4-(ii) in [7], we have the claim.

Q.E.D.

Let us denote by $\pi_{2\ell}$ a rational map defined by the linear system $|\mathcal{O}_X(1) \otimes I_\ell^2|$, which is called the "double projection from ℓ". Then we have a diagram:

$$\begin{array}{ccc} X_1 & & \\ \sigma_1 \downarrow & \searrow^{\Phi_1} & \\ X & \overset{\pi_{2\ell}}{\cdots\cdots\rightarrow} & \mathbf{P}^6 \end{array}$$

where $\Phi_1 := \Phi_{|\mathcal{H}_1|}$ is a rational map defined by the linear system $|\mathcal{H}_1|$.

Next, we will resolve the indeterminancy of the rational map $\Phi_1 : X_1 \cdots\rightarrow \mathbf{P}^6$

LEMMA 6. (1) Sing $Y_1 = 2\ell_1$, *namely, ℓ_1 is the singular locus of Y_1 with the* *multiplicity* 2,

(2) $Y_1 \cap L_1 = A_1 + A_2 + A_3$, *where* A_i*'s are non-singular rational curves and* $A_1 \sim 2\ell_1$, $A_2 \sim \ell_1 + 4f_1$, $A_3 \sim 3f_1$ *in* L_1.

Proof. Looking at the blowing up $\sigma_1 : X_1 \to X$ locally, one may identify the Zariski open set $\sigma_1^{-1}(X_1 \cap \{a_{12} \neq 0\})$ with the blowing up $\mu : M \to \mathbf{C}^3(x, y, z)$ with the center Sing $V(f) = \{x = y = 0\}$. M is covered by two coodinate patches $U_0 = \mathbf{C}^3(r, s, t)$, $U_1 = \mathbf{C}^3(u, v, w)$, with $r \cdot v = 1$ on $U_0 \cap U_1$, and μ is given by

$$\mu : \begin{cases} x = rs = u \\ y = s = uv \\ z = t = w. \end{cases}$$

Let V_1 be the proper transform of $V(f)$ in M. Then we have

$$V_1 \cap U_0 = \{f_1^*(r, s, t) = 0\}, \text{where}$$

$$f_1^* := b_0 r^4 s + (b_1 st + b_2 t^3) r^3 + (b_3 s^2 + b_4 st^2 + b_5 t^4) r^2 + (b_6 s^2 t + b_7 st^3) r + b_8 s^3 + b_9 s^2 t^2, \text{ and}$$

$$V_1 \cap \{s = 0\} = \{r^2 t^3 (b_2 r + b_5 t) = 0\}.$$

This shows that $\{r = s = 0\}$ is the singular locus of V_1 with the multiplicity 2 and $V_1 \cap \{s = 0\}$ consists of three irreducible non-singular rational curves. Since $Y_1 \cdot L_1 \sim 2\ell_1 + 7f_1$, we have the assertions (1) and (2).

Q.E.D.

Let $\sigma_1 : X_i \to X_{i-1}$ be the blowing up of X_{i-1} along the section ℓ_{i-1} of L_{i-1} with $(\ell_{i-1}^2)_{L_{i-1}} \leqq 0$, and put $L_i := \sigma_i^{-1}(\ell_{i-1})$ $(i \geqq 2)$. Let f_i be a fiber of L_i, Y_i the proper transform of Y_{i-1} in X_i, and put $\mathscr{H}_i := \sigma_i^* \mathscr{H}_{i-1} - L_i$.

LEMMA 7.
(1) $Y_2 \cap L_2 = B_1 + B_2$, *where* $B_1 \sim 2\ell_2$, $B_2 \sim 2f_2$ *in* L_2,
(2) Sing $Y_2 = 2\ell_2$,
(3) Bs $|\mathscr{H}_2| = \ell_2$.

Proof. By Lemma 4, we have the following three cases:

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$$L_2 \cong \begin{cases} \text{(a)} & \mathbf{P}^1 \times \mathbf{P}^1, \\ \text{(b)} & \mathbf{F}_2, \\ \text{(c)} & \mathbf{F}_4. \end{cases}$$

Since $Y_2 \sim \sigma_2^* Y_1 - 2L_2$, we have

$$Y_2 \cdot L_2 \sim \begin{cases} \text{(a)} & 2\ell_2 \quad\quad\ \text{if } L_2 \cong \mathbf{P}^1 \times \mathbf{P}^1, \\ \text{(b)} & 2\ell_2 + 2f_2 \text{ if } L_2 \cong \mathbf{F}_2, \\ \text{(c)} & 2\ell_2 + 4f_2 \text{ if } L_2 \cong \mathbf{F}_4. \end{cases}$$

On the other hand, by blowing up $U_0 = \mathbf{C}^3(r, s, t)$ along $\{r = s = 0\}$, one can get the local equation for Y_2. From this, one can show that $\operatorname{Sing} Y_2 = 2\ell_2$, and $Y_2 \cap L_2 = B_1 + B_2$, where $B_1 \sim 2\ell_2$, $B_2 \sim 2f_2$ in L_2. Thus we have $L_2 \cong \mathbf{F}_2$. Since $(H_2, \ell_2) = -1$, $\ell_2 \subseteq \operatorname{Bs}|\mathscr{H}_2|$. On the other hand, since $|\mathscr{H}_2| \cap L_2 \subseteq |\mathscr{H}_{2|L_2}| = |\ell_2 + f_2|$, we have the claim.

Q.E.D.

COROLLARY 8. *$L_2 \cong \mathbf{F}_2$, namely, $N_{\ell_1|X_1} \cong \mathcal{O}(-2) \oplus \mathcal{O}$.*

Similarly, one can show the following

LEMMA 9.

(1) *$Y_3 \cap L_3 = C_1 + C_2$, where $C_1 \sim 2\ell_3$, $C_2 \sim 2f_3$ in L_3.*

(2) *$\operatorname{Sing} Y_3 = 2\ell_3 + 2f_3$,*

(3) *$\operatorname{Bs}|\mathscr{H}_3| = \ell_3$,*

(4) *$L_3 \cong \mathbf{F}_2$, namely, $N_{\ell_3|X_3} \cong \mathcal{O}(-2) \oplus \mathcal{O}$,*

(5) *$Y_4 \cap L_4 = D$, where $D \sim 2\ell_4$ in L_4,*

(6) *$L_4 \cong \mathbf{P}^1 \times \mathbf{P}^1$, namely, $N_{\ell_3|X_3} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Let $L_j^{(4)}$ $(1 \leqq j \leqq 3)$ be the proper transform of L_j in X_4 and $A_i^{(4)}$ $(1 \leqq i \leqq 3)$, $f_1^{(4)}$ be the proper transforms of A_i, a fiber f_1 in X_4 respectively. Then we have easily

(2.1) $$\mathscr{H}_4 = \sigma_4^* \mathscr{H}_3 - L_4 \sim Y_4 + L_1^{(1)} + 2L_2^{(4)} + 3L_3^{(4)} + 4L_4$$

(2.2) $$K_{X_4} \sim -(Y_4 + 2L_1^{(4)} + 3L_2^{(4)} + 4L_3^{(4)} + 5L_4)$$

(2.3) $$(L_4 \cdot \ell_4) = (L_4 \cdot f_4) = -1, \ (\ell_4 \cdot \ell_4) = 0$$

(2.4) $$(\mathscr{H}_4^3) = (\mathscr{H}_4 \cdot \mathscr{H}_4 \cdot \mathscr{H}_4) = 5$$

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$$|\mathscr{H}_4| \cap L_4 = |\ell_4| \tag{2.5}$$

$$(\mathscr{H}_4 \cdot \mathscr{H}_4 \cdot Y_4) = 0 \tag{2.6}$$

$$(\mathscr{H}_4 \cdot \mathscr{H}_4 \cdot L_1^{(4)}) = 5 \tag{2.7}$$

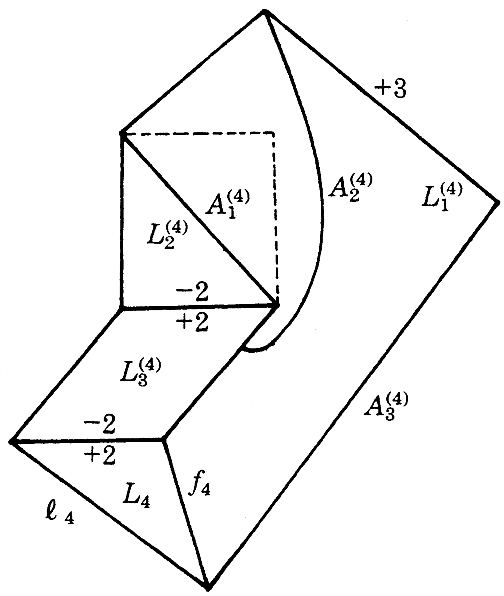
$$(\mathscr{H}_4 \cdot \mathscr{H}_4 \cdot L_4) = (\mathscr{H}_4 \cdot \mathscr{H}_4 \cdot L_j^{(4)}) = 0 \ (j = 2,3) \tag{2.8}$$

$$(\mathscr{H}_4 \cdot A_2^{(4)}) = 5, \ (\mathscr{H}_4 \cdot f_1^{(4)}) = 1. \tag{2.9}$$

By (2.5), we have

LEMMA 10. $\mathrm{Bs}\,|\mathscr{H}_4| = \phi$.

Let $\Phi : X_4 \to \mathbf{P}^6$ be a morphism defined by the linear system $|\mathscr{H}_4|$. We put $V := \Phi(X_4)$. By (2.4), $\deg V = 5$. By (2.6), (2.7), (2.8), $X - Y \cong X_4 - (Y_4 \cup L_4 \cup L_1^{(4)} \cup L_2^{(4)} \cup L_3^{(4)}) \cong V - \Phi(L_1^{(4)}) \cong \mathbf{C}^3$.



By (2.3), L_4 can be blown down along ℓ_4, and then blowing downs can be done step by step (cf. Reid [12]). Finally we have a smooth projective threefold V^+ with

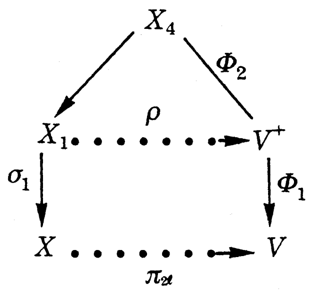
$b_2(V^+) = 2$, and morphisms $\Phi_2 : X_4 \to V^+$, $\Phi_1 : V^+ \to V$, a birational map $\rho : X \cdots\to V^+$ (which is called a flop) such that

(i) $\Phi = \Phi_1 \circ \Phi_2$

(ii) $X_1 - \ell_1 \overset{\rho}{\cong} V^+ - \Sigma_1$, where $\Sigma_1 := \Phi_2(L_4 \cap L_1^{(4)})$

(iii) $V^+ - \rho(Y_1) \cong V - \Phi(Y_4)$

(D-1)



Let Y_1^+, L_1^+ be the proper transforms of Y_1, L_1 in V^+ respectively. We put $\Gamma := \Phi(Y_4) = \Phi_1(Y_1^+)$ and $Z := \phi(L_1^{(4)}) = \Phi_1(L_1^+)$. Then, by (2.6), (2.7), (2.9), Γ is a smooth rational curve of degree 5 in $\mathbf{P}^6$ and Z is a ruled surface swept out by lines which intersect the line $\Sigma := \Phi_1(\Sigma_1)$ on V. In particular, $\Gamma \hookrightarrow Z$ and $\Gamma \cap \Sigma = \{\text{one point}\}$. Let γ be a conic in X which intersect the line ℓ. Then $\gamma \hookrightarrow Y$. Let γ_1 be the proper transform of γ in X_1 and $\gamma_1^+ := \rho(\gamma_1) \hookrightarrow Y_1^+$. Since $K_{V^+} = \rho_*(K_{X_1}) = -Y_1^+ - 2L_1^+$, we have $(K_{V^+} \cdot \gamma_1^+) = -1$. Thus, $\Phi_1 : V^+ \to V$ be the contraction of an extremal ray by K.M.M. [9]. Since Y_1^+ is contracted to the smooth curve Γ by Φ_1, V is smooth by Mori [10]. By (2.4), we have deg $V = 5$. Moreover, we have $K_V \sim -2Z$. Since $V - Z \cong \mathbf{C}^3$ by construction, Z is ample, thus, V is a Fano threefold of first kind with index 2, genus 21. Since Z is swept out by lines in V, Z is non-normal. In fact, the singular locus of Z is just the line $\Sigma := \Phi_1(\Sigma_1)$. Therefore we have $(V, Z) \cong (V_5, H_5^\infty)$ (see §1), namely

THEOREM 1. *Let (V_{22}', H_{22}'), $\ell := \mathrm{Sing}\, H_{22}'$, (V_5, H_5^∞) be as before. Then the double projection $\pi_{2\ell} : V_{22}' \to V_5$ of V_{22}' from the line ℓ gives an isomorphism $V_{22}' - \mathscr{H}_{22}' \xrightarrow{\sim} V_5 - H_5^\infty (\cong \mathbf{C}^3)$.*

Remark 1. Let $\Sigma := \mathrm{Sing}\, H_5^\infty$ be the singular locus of H_5^∞. Then, Σ is a line on V_5 with the normal bundle $N_{\Sigma|V_5} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. The set $\{x \in \Sigma$; there is a unique line passing through the point $x\}$ consists of the only point p (cf. [5]). One can easily see that there is a smooth rational curve Γ of degree 5 in V_5 such that

$\Gamma \cap \Sigma \{p\}$ and $\Gamma \hookrightarrow H_5^\infty$. Then the linear system $|\mathcal{O}_{V_5}(3) \otimes I_\Gamma^2|$ defines the inverse birational map $\pi_{2\ell}^{-1} : V_5 \cdots\to V'_{22}$ with $V_5 - H_5^\infty \xrightarrow{\sim} V'_{22} - H'_{22}$ (cf. [7]).

§3. Normalization and resolution of the boundary divisor

First, we will prepare some general results on a non-normal hyperplane section of a Fano threefold of special series.

Let X be a Fano threefold of special series, namely, X is a smooth threefold $V_{2g-2} \hookrightarrow \mathbf{P}^{g+1}$ of degree $2g-2$. Then the anticanonical line bundle $-K_X$ is an ample generator of $\operatorname{Pic} X \cong \mathbf{Z}$. Let Y be a non-normal member of the linear system $|-K_X|$. Since $\operatorname{Pic} X \cong \mathbf{Z}[Y]$, Y is irreducible. Let $\sigma : S \to Y$ be the normalization, and let $I \hookrightarrow \mathcal{O}_Y$ be the conductor of σ. We put $E := \operatorname{loc} I$ (the locus of I) and $D := \sigma^{-1}(E)$. Since Y is Cohen-Macaulay, E and D are Cohen-Macaulay. Since $Y \sim -K_X$, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and $H^i(X, \mathcal{O}_X(-Y)) = 0$ for $i < 3$, we have

(3.1) $$\omega_Y \cong \mathcal{O}_Y$$

(3.2) $$H^1(Y, \mathcal{O}_Y) = 0,\ H^2(Y, \mathcal{O}_Y) \cong \mathbf{C}$$

(3.3) $$\omega_S \cong I \otimes \sigma^* \omega_Y \cong I \text{ (i.e. } K_S \sim -D \text{ as a Weil divisor).}$$

By (3.34.2), (3.34.3) in Mori [10], we have exact sequences:

(3.4) $$0 \to \mathcal{O}_Y \to \sigma_* \mathcal{O}_S \to \omega_E \to 0$$

(3.5) $$0 \to \sigma_* \omega_S \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$$

Taking σ^* in (3.5), we have

(3.6) $$0 \to \omega_S \to \mathcal{O}_S \to \sigma^* \mathcal{O}_E \cong \mathcal{O}_D \to 0.$$

By (3.2), (3.3), (3.3), we have

LEMMA 11 ([14]). *$h^0(\mathcal{O}_E) = 1$ and $h^1(\mathcal{O}_E) = 0$, namely E_{red} is connected and each irreducible component E_i of E_{red} is a smooth rational curve.*

Take a general hyperplane section H of X. From (3.4), we get

(3.7) $$0 \to \mathcal{O}_Y(H) \to \sigma_* \mathcal{O}_s \otimes \mathcal{O}_Y(H) \to \omega_E(H) \to 0.$$

Since $H^1(Y, \mathcal{O}_Y(H)) = 0$, we have

(3.8) $$h^0(\sigma_* \mathcal{O}_s \otimes \mathcal{O}_Y(H)) = h^0(\mathcal{O}_Y(H)) + h^0(\omega_E(H)).$$

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We put $\delta := (H \cdot E)_X$.

CLAIM (3.9). $h^0(S, \sigma^*H) = g + \delta$.

In fact, since E is Cohen-Macaulay, $h^0(\omega_Y(H)) = h^1(\mathcal{O}_E(-H))$. By the following exact sequence:

$$0 \to \mathcal{O}_E(-H) \to \mathcal{O}_E \to \mathcal{O}_{E \cap H} \to 0,$$

we have $h^1(\mathcal{O}_E(H)) = h^0(\mathcal{O}_{E \cap H}) - h^0(\mathcal{O}_E) = \delta - 1$. Since $h^0(\sigma^*H) = h^0(\sigma_*\mathcal{O}_S(\sigma^*H)) = h^0(\sigma_*\mathcal{O}_S \otimes \mathcal{O}_Y(H))$ and $h^0(\mathcal{O}_Y(H)) = g + 1$, we have $h^0(S, \sigma^*H) = g + \delta$.

Let $\Delta(S, \sigma^*H) := \dim S + \deg \sigma^*H - h^0(S, \sigma^*H)$ be the Δ-genus of the polarized variety (S, σ^*H) (cf. [3]). Since $\dim S = 2$ and $\deg \sigma^*H = (H^3)_X = 2g - 2$, we have

LEMMA 12. $\Delta(S, \sigma^*H) = g - \delta$.

LEMMA 13. $(D \cdot \sigma^*H) = 2(E \cdot H) = 2\delta$.

Proof. By (3.36.2) in Mori [10], we have

$$0 \to \mathcal{O}_E \to \sigma_*\mathcal{O}_D \to \omega_E \to 0.$$

Thus we have $\chi(\sigma_*\mathcal{O}_D \otimes H) = \chi(\mathcal{O}_E(H)) + \chi(\omega_E(H)) = 2\delta + \chi(\mathcal{O}_E) + \chi(\omega_E) = 2$. On the other hand, $\chi(\sigma_*\mathcal{O}_D \otimes H) = \chi(\mathcal{O}_D \otimes \sigma^*H) = (D \cdot \sigma^*H) + \chi(\mathcal{O}_D)$. Since $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S) - \chi(\omega_S) = 0$, we have $(D \cdot \sigma^*H) = 2\delta$.

Q.E.D.

Let $C \in |\sigma^*H|$ be a smooth member. By Bertini's theorem, such a member C exists. Let us denote by $g(C)$ the genus of C.

LEMMA 14. $g(C) = g - \delta$.

Proof. By the adjunction theorem, $2g(C) - 2 = C(\omega_S + C)$. Since $(C^2) = 2g - 2$ and $(C \cdot \omega_S) = 2\delta$ by Lemma 13, we have $g(C) = g - \delta$.

Q.E.D

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Let $\mu : M \to S$ be the minimal resolution, and put $\psi := \mu \circ \sigma : M \to Y$. Since $K_S \sim -D$ (as a Weil divisor), we have $K_M \sim -\widehat{D} - \sum_i m_i \Delta_i$ ($m_i > 0$, $m_i \in \mathbf{Z}$), where $\widehat{D}$ is the proper transform of D in M and $\cup_i \Delta_i$ is the exceptional set of μ.

LEMMA 15. *M is rational or ruled.*

Proof. Since $H^0(M, \mathcal{O}_M(mK_M)) = 0$ for $m > 0$, by the classification of surfaces, we have the lemma.

Q.E.D.

LEMMA 16. *If $h^1(\mathcal{O}_M) = 0$, then* Sing S *consists of at worst rational singularities, in particular, S is rational.*

Proof. Let us consider the following exact sequence:

$$0 \to H^1(S, \mathcal{O}_S) \to H^1(M, \mathcal{O}_M) \to H^0(S, R^1\mu_*\mathcal{O}_M) \to H^2(S, \mathcal{O}_S) \to .$$

By assumption, we have $H^1(M, \mathcal{O}_M) = 0$. Since $H^2(S, \mathcal{O}_S) \cong H^0(S, \omega_S) = 0$, we have the claim.

Q.E.D.

Now, Mukai-Umemura's example V'_{22} is a special class of Fano threefolds of special series with the genus $g = 12$, and H'_{22} is a non-normal hyperplane section of V'_{22} such that $V'_{22} - H'_{22} \cong \mathbf{C}^3$. We can apply the above lemmas to these $X := V'_{22}$ and $Y := H'_{22}$.

LEMMA 17. *Assume that $(X, Y) = (V'_{22}, H'_{22})$. Then we have*

(1) $E_{\mathrm{red}} \cong \mathbf{P}^1$,

(2) $Y - E_{\mathrm{red}} \cong \mathbf{C}^2$,

(3) $H^1(Y; \mathbf{Z}) = 0$, $H^2(Y; \mathbf{Z}) \cong \mathbf{Z}$, $H^3(Y; \mathbf{Z}) = 0$,

(4) *S is a rational surface and* Sing S *consists of at worst rational singularities.*

(5) $g(C) = 12 - \delta$ *for a general smooth member* $C \in |\sigma^* H|$.

Proof. By Lemma 2 and its proof, we have (1) and (2). Since $X - Y \cong \mathbf{C}^3$, we have $H^i(X; \mathbf{Z}) \cong H^i(Y; \mathbf{Z})$ for $i \geqq 0$. It is known that $H^i(V'_{22}; \mathbf{Z}) = H^i(\mathbf{P}^3; \mathbf{Z})$ for $i \geqq 0$, that is, V'_{22} has the same cohomology as $\mathbf{P}^3$. This proves (3). Let us consider the following exact sequence (cf. [1]):

$$(*) \quad 0 \to H^2(Y; \mathbf{Z}) \to H^2(S; \mathbf{Z}) \oplus H^2(E; \mathbf{Z}) \to H^2(D; \mathbf{Z}) \to$$

$$\to H^3(Y\,;\mathbf{Z}) \to H^3(S\,;\mathbf{Z}) \to 0$$

Since $H^3(Y\,;\mathbf{Z}) = 0$, we have $H^3(S\,;\mathbf{Z}) = 0$. Since $b_3(M) = b_3(S) = 0$ (cf. [2]), $b_1(M) = 0$, hence, $h^1(\mathcal{O}_M) = h^1(\mathcal{O}_S) = 0$. By Lemma 16, we have (4). Since $g = 12$, by Lemma 14, we have (5).

Q.E.D.

LEMMA 18. *$K_M + \phi^*H$ is nef.*

Proof. Assume that $K_M + \phi^*H$ is not nef. Then, by Cone theorem and Contraction theorem in [8] (cf. [9]), there is a contraction $\pi : M \to Z$ of the extremal ray, where Z is normal and $\pi^{-1}(z)$ is connected for any $z \in Z$.

Case (a). dim $Z = 2$. Then there is a curve R such that $\pi(R)$ is a point and $R^2 < 0$, $(K_M + \phi^*H) \cdot R < 0$. Since $(\phi^*H \cdot R) \geqq 0$ and $R^2 < 0$, we have $R \cong \mathbf{P}^1$ and $R^2 = -1$, hence, $(\phi^*H \cdot R) = 0$. Thus R is an exceptional curve of μ. Since $\mu : M \to S$ is the minimal resolution, this is a contradiction.

Case (b). dim $Z = 1$. Since M is rational, we have $Z \cong \mathbf{P}^1$. Since $\rho(M) = \rho(Z) + 1 = 2$, M is isomorphic to $\mathbf{F}_n$ (Hirzebruch surface), namely, $\pi : M \to Z \cong \mathbf{P}^1$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. For a fiber f, we have $(K_M + \phi^*H) \cdot f < 0$. Hence, $(\phi^*H \cdot f) = (H \cdot \phi(f)) = 1$ since $(K_M \cdot f) = -2$. Thus, Y is a ruled surface swept out by lines on X. By Lemma 2-(2), E_{red} is a line on X and $E_{\mathrm{red}} \cap \phi(f) = \emptyset$ for a general fiber f. This shows that $\phi(f) \subset Y - E_{\mathrm{red}} \cong \mathbf{C}^2$. This is a contradiction.

Case (c). dim $Z = 0$. In this case, $M \cong \mathbf{P}^2$. For a smooth member $C \in |\phi^*H|$, we put deg $C = d$. Then, $C^2 = d^2 = 22$, this is a contradiction.

Q.E.D.

By Lemma 2-(3), $Y := H'_{22}$ is a ruled surface swept out by conics which intersect the line $\ell :=$ Sing Y in $X := V'_{22}$, where $\ell = E_{\mathrm{red}}$. Take a general conic γ in Y. Then, $\gamma \cap E_{\mathrm{red}} \neq \emptyset$. Let $\hat{\gamma}$ be the proper transform of γ in M. Then we have $(\phi^*H \cdot \hat{\gamma}) = (H \cdot \gamma) = 2$. Since $K_M + \phi^*H$ is nef by Lemma 18, we have $(K_M + \phi^*H) \cdot \hat{\gamma} \geqq 0$, hence, $(K_M \cdot \hat{\gamma}) \geqq -2$. On the other hand, since $K_M \sim -\widehat{D} - \sum_i m_i \Delta_i$ $(m_i \geqq 0,\ m_i \in \mathbf{Z})$, we have $(K_M \cdot \hat{\gamma}) \leqq 0$.

CLAIM (1). $(K_M \cdot \hat{\gamma}) \neq 0$.

In fact, if $(K_M \cdot \hat{\gamma}) = 0$, then $(\widehat{D} \cdot \hat{\gamma}) = 0$, $(\Delta_i \cdot \hat{\gamma}) = 0$ for each i. We take a general γ. Thus $\mathbf{P}^1 \cong \hat{\gamma} \hookrightarrow M - \widehat{D} - \cup \Delta_i\ Y - E_{\mathrm{red}} \cong C^2$. This is a contradiction.

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CLAIM (2). There is an irreducible conic γ_0 in Y such that $(K_M \cdot \hat{\gamma}_0) = -2$ (that is, $\hat{\gamma}_0 \cong \mathbf{P}^1$ with the self-intersection number $\hat{\gamma}_0^2 = 0$).

In fact, by Claim (1), we have $(K_M \cdot \hat{\gamma}) = -1$ or -2 for any conic γ in Y. If $(K_M \cdot \hat{\gamma}) = -1$, then $\hat{\gamma}$ is a (-1)-curve. Thus, M contains a continuous family of (-1)-curves. This is a contradiction.

Let $\tau : M \to \mathbf{P}^1$ be a morphism defined by the linear system $|\hat{\gamma}_0|$. For a general p in $\mathbf{P}^1$, $\tau^{-1}(p) \sim \hat{\gamma}_0$.

LEMMA 19. $K_M + \phi^*H \sim (11 - \delta)\hat{\gamma}_0$.

Proof. By Basepoint-free Theorem of Kawamata [7], we have $\mathrm{Bs}\,|m(K_M + \phi^*H)| = \emptyset$ for $m \gg 0$. We put $\hat{f} := \tau^{-1}(p)$ (a general fiber of τ). By Claim (2), $(K_M + \phi^*H)\hat{f} = 0$. Let $\tau_m : M \to Z_0$ be a morphism defined by the linear system $|m(K_M + \phi^*H)|$. Since M is rational and since $\tau_m(\hat{f})$ is a point, we have $Z_0 \cong \mathbf{P}^1$, in particular, we have $m(K_M + \phi^*H) \sim k\hat{\gamma}_0$. Since $(\phi^*H \cdot K_M) = -2\delta$, $(\phi^*H \cdot \phi^*H) = 22$ and $(\phi^*H \cdot \hat{\gamma}_0) = 2$, we have $(22 - 2\delta)m = 2k$, hence, $k = (11 - \delta)m$. Since Pic M has no torsion, we have $(K_M + \phi^*H) \sim (11 - \delta)\,\hat{\gamma}_0$.

Q.E.D.

COROLLARY 20. $\mathrm{Bs}\,|K_M + \phi^*H| = \emptyset$.

Let $\hat{f}$ be a regular fiber of τ. Then $\phi(\hat{f}) = \gamma \hookrightarrow Y \hookrightarrow X$ is a conic in X.

LEMMA 20. *Each Δ_i is contained in a singular fiber of τ.*

Proof. Assume that Δ_1 not contained in any singular fiber of τ. Then $\tau_{|\Delta_1} : \Delta_1 \to \mathbf{P}^1$ is a surjective morphism, hence, $(\Delta_1 \cdot \hat{f}) \neq 0$ for a regular fiber $\hat{f}$. Since $\phi(\Delta_1)$ is a point and since $\phi(\hat{f}) =: \gamma$ is a conic in $Y \hookrightarrow X$, we have an infinite number of conics in X passing through the point $\phi(\Delta_1) \in X$. On the other hand, for each point $x \in X$, the number of conics passing through the point x is finite by Iskovskih [7]. Thus we have an contradiction.

Q.E.D.

LEMMA 21. *Let B be an irreducible component of a singular fiber of $\tau : M \to \mathbf{P}^1$. Then $B^2 = -1$ or -2. Furthermore,*

(i) $B^2 = -1 \Leftrightarrow \phi(B) = E_{\mathrm{red}} \cong \mathbf{P}^1$

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(ii) $B^2 = -2 \Leftrightarrow B = \Delta_i$ for some i.

Proof. Since $(K_M + \phi^*H)\, B = (11 - \delta) \cdot (\hat{\gamma} \cdot B) = 0$, we get $(K_M\, B) = -(\phi^*H \cdot B) \leqq 0$. Since $B \cong \mathbf{P}^1$ and $B^2 < 0$, we have $B^2 = -1$ or $B^2 = -2$. (i): $B^2 = -1 \Leftrightarrow (K_M \cdot B) = -1 \Leftrightarrow (\phi^*H \cdot B) = 1 \Leftrightarrow (H \cdot \phi(B)) = 1 \Leftrightarrow \phi(B)$ is a line in $Y \Leftrightarrow \phi(B) = E_{\text{red}}$ (because $E_{\text{red}} = \operatorname{Sing} Y$ is a unique line in Y by Lemma 2-(2)). (ii): $B^2 = -2 \Leftrightarrow \phi(B)$ is a point of $Y \Leftrightarrow B$ is a component of the exceptional set of $\mu \Leftrightarrow B = \Delta_i$ for some i.

Q.E.D.

COROLLARY 22. *Sing S consists of (at worst) rational double points.*

Proof. For each Δ_i, one has $(\Delta_i \cdot \Delta_i) = -2$. This proves the corollary.

LEMMA 23. $\delta = 4$.

Proof. Let $C \in |\sigma^*H|$ be a smooth member. By Bertini theorem, such a member C exists. We put $C_0 := \sigma(C)$. Then $\sigma : C \to C_0$ is the normalization. We may assume that C_0 is contained in a $K3$ surface H_0, which is a hyperplane section of $X := V'_{22}$. Since $\operatorname{Sing} Y =: E_{\text{red}}$ is a line in X, Sing C_0 consists of only one point p_0. On the other hand, from the defining equation $(*)$ in Lemma 2, the local equation of C_0 around p_0 in H_0 can be written as $u_0x^3 + u_1x^2y + u_2xy^3 + u_3y^5 = 0$, where $p_0 = (0, 0)$. Thus C_0 has two singular points p_0 and p'_0 (infinitely near singular point lying over p_0) with the multiplicity three and two respectively. Since H_0 is a $K3$ surface, the arithmetic genus $p_a(C_0) = \frac{1}{2}(C_0\, C_0) + 1 = 12$, hence, the genus $g(C) = p_a(C_0) - 4 = 8$. Since $g(C) = 12 - \delta$ by Lemma 12, we have $\delta = 4$.

Q.E.D.

LEMMA 24. $K_M^2 = -6$ *and* $b_2(M) = 16$.

Proof. Since $(K_M + \phi^*H)^2 = K_M^2 - 4\delta + 22 = 0$ and $\delta = 4$, we have $K_M^2 = -6$. By Noether formula, we have $b_2(M) = 16$.

Q.E.D.

LEMMA 24. *The number of the singular fiber of $\tau : M \to \mathbf{P}^1$ is equal to one.*

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Proof. Let F_i $(1 \leqq i \leqq t)$ be a singular fiber of τ, $1 + \alpha_i$ the number of the irreducible components of F_i, and e_i the number of the irreducible components of $\overline{F_i - \Delta}$, where $\Delta := \cup \Delta_i$. By Lemma 21, $e_i =$ the number of irreducible components of $\hat{D} \cap F_i =$ the number of (-1)-curves in F_i. Since M is rational, we have $b_2(M) = 2 + \sum_i \alpha_i$. Since $b_2(M) = b_2(S) + b_2(\Delta)$ and $b_2(\Delta) = \sum_i (1 + \alpha_i - e_i)$, we have $b_2(S) = 2 - \sum (1 - e_i)$. On the other hand by the following exact sequence (cf. [1]):

$$\begin{array}{ccccccccc} 0 \to & H^2(Y\,;\mathbf{Z}) & \to & H^2(S\,;\mathbf{Z}) \oplus & H^2(E\,;\mathbf{Z}) & \to & H^2(D\,;\mathbf{Z}) & \to 0, \\ & \wr\!\!\parallel & & & \wr\!\!\parallel & & & \\ & \mathbf{Z} & & & \mathbf{Z} & & & \end{array}$$

we have $b_2(S) = b_2(D)$. Since $K_M \sim -\hat{D} - \sum m_i \Delta_i$ and $(K_M \cdot \hat{f}) = -2$ for a regular fiber $\hat{f}$ of τ, we have $(\hat{D} \cdot \hat{f}) = 2$. This shows that $b_2(\hat{D}) > \sum e_i$. Thus we have $2 - \sum(1 - e_i) = b_2(S) = b_2(D) = b_2(\hat{D}) > \sum e_i$, that is, $2 > t \geqq 1$. Therefore we have $t = 1$.

Q.E.D.

LEMMA 25. *$\hat{D} = 2\hat{D}_1 + 3\hat{D}_2 + 3\hat{D}_3$, where $\hat{D}_1$ is a section of $\tau : M \to \mathbf{P}^1$ and $\hat{D}_i$'s are the (-1)-curves in the singular fiber of τ for $i = 2,3$.*

Proof. Let $\sigma_1 : X_1 \to X$, Y_1, L_1, A_i $(1 \leqq i \leqq 3)$, ℓ_1, f_1 be as in Lemma 6. Since $Y_1 \sim \sigma^* H - 3L_1$, by the adjunction formula, we have $K_{Y_1} \sim -2L_1|_{Y_1} \sim -2(A_1 + A_2 + A_3)$ as a Weil divisor. Let $\nu : \bar{S}_1 \to Y_1$ be the normalization and A_1' (resp. $\bar{A}_1'$) be the closed subscheme in Y_1 (resp. $\bar{S}_1$) defined by the conductor of ν. Since $\mathrm{Sing}\, Y_1 = A_1$, $\mathrm{supp}\, A_1 = \mathrm{supp}\, A_1'$.

CLAIM (1). *There is a morphism $\eta : \bar{S}_1 \to S$ such that $\sigma \circ \eta = \sigma_1 \circ \nu$ (see D-2).*

$$\text{(D-2)} \qquad \begin{array}{ccccc} & & \bar{S}_1 & \xrightarrow{\ \nu\ } & Y_1 \\ & & {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\sigma_1} \\ M & \xrightarrow{\ \mu\ } & S & \xrightarrow{\ \sigma\ } & Y \end{array} \qquad \psi = \sigma \circ \mu : M \to Y$$

In fact, let $\bar{A}_i$ be the proper transform of A_1 in $\bar{S}_1$. Since $\bar{S}_1 - \cup\, \mathrm{supp}\, \bar{A}_i \cong Y_1 - \cup\, \mathrm{supp}\, A_i \cong Y - \mathrm{supp}\, E$, we have the claim. In particular, $\eta(\mathrm{supp}\, \bar{A}_3)$ is a point on S, $\bar{S}_1 - \mathrm{supp}\, \bar{A}_3 \cong S - \eta(\mathrm{supp}\, \bar{A}_3)$ and $\eta(\mathrm{supp}\, \bar{A}_1 \cup \mathrm{supp}\, \bar{A}_2) = \mathrm{supp}\, D$.

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We put $D_1 := \eta_*(\bar{A}_2)$. Since $A_2 \sim \ell_1 + 4f_1$ in L_1, A_2 is reduced, hence, D_1 is reduced. Let $\hat{D}_1$ is the proper transform of D_1 in M.

CLAIM (2). *$\hat{D}_1$ is a section of $\tau : M \to \mathbf{P}^1$, and $(\psi^*H \cdot \hat{D}_1) = 1$.*

In fact, let γ be a general conic $Y \hookrightarrow X$, and $\bar{\gamma}$ the proper transform of γ in $Y_1 \hookrightarrow X_1$. Then we have $(L_1\, \bar{\gamma}) = 1$. Since $Y_1 \cdot L_1 \sim (2\ell_1) + (\ell_1 + 4f_1) + (3f_1)$ and $\bar{\gamma} \hookrightarrow Y_1$, we have $(A_2 \cdot \bar{\gamma}) = 1$, hence, $(\hat{D}_1 \cdot \bar{\gamma}) = 1$, where $\bar{\gamma}$ is the proper transform of γ in M. Thus $\hat{D}_1$ is a section of $\tau : M \to \mathbf{P}^1$. Since $(\sigma_1^*H \cdot A_2) = (\sigma_1^*H \cdot \ell_1 + 4f_1) = 1$, we have $(\psi^*H \cdot \hat{D}_1) = 1$.

CLAIM (3). *$\hat{D} \sim 2\hat{D}_1 + 3\hat{D}_2 + 3\hat{D}_3$, where $\hat{D}_2$, $\hat{D}_3$ are the (-1)-curves in the singular fiber of τ.*

In fact, since $-K_M \sim \hat{D} + \sum m_i \Delta_i$, we have $2 = (\hat{D} \cdot \hat{f}) + \sum m_i(\Delta_i \cdot \hat{f})$ for a regular fiber $\hat{f}$ of τ. Since Δ_i's are contained in the singular fiber of τ, we have $(\hat{D} \cdot \hat{f}) = 2$. Since $\eta(\text{supp}\ \bar{A}_1 \cup \bar{A}_2) = \text{supp}\ D$ and $(A_2 \cdot \bar{\gamma}) = 1$, we have $\hat{D} = 2\hat{D}_1 + \sum n_i \hat{D}_i$ $(i \geqq 2,\ n_i \in \mathbf{Z},\ n_i > 0)$. We note that the proper transform of γ in M is linearly equivalent to a regular fiber $\hat{f}$ of $\tau : M \to \mathbf{P}^1$. Since $\hat{D}_i$'s $(i \geqq 2)$ are contained in the singular fiber of τ, by Lemma 21, $\hat{D}_i$'s are the (-1)-curves in the singular fiber of τ. Hence $(\psi^*H \cdot \hat{D}_i) = 1$ $(i \geqq 2)$.

Let us recall the normalization $\sigma : C \to C_0$ (see the proof of Lemma 6). From the local defining equation of C_0 in H_0 there, one can see that $\sigma^{-1}(p_0)$ consists of three distinct points, where $p_0 := \text{Sing}\ C_0$. This shows that $\hat{D} = 2\hat{D}_1 + a\hat{D}_2 + b\hat{D}_3$, where $a + b = 6$, since $(\psi^*H \cdot \hat{D}) = 8$. On the other hand, since $\bar{K}_{S_1} \sim -2\nu^*(A_1 + A_2 + A_3) - \bar{A}_1'$ as a Weil divisor, we have $D \sim -K_S \sim 2\eta_*\nu^*A_2 + (2\eta_*\nu^*A_1 + \eta_*\bar{A}_1')$. Since $\text{supp}\ D_1 = \text{supp}\ \eta_*\nu^*A_2$ and $\text{supp}\ \eta_*\bar{A}_1' = \text{supp}\ \eta_*\bar{A}_1 \hookrightarrow \text{supp}\ D$, we have $a = b = 3$. This completes the proof.

Q.E.D.

THEOREM 2. *Let $(X, Y) := (V_{22}', H_{22}')$ be as in §1. Let $\sigma : S \to Y := H_{22}'$ be the normalization, and E the non-normal locus defined by the conductor of σ, and D the analytic inverse image of E. Let $\mu : M \to S$ be the minimal resolution and $\mu^{-1}(\text{Sing}\ S) = \cup \Delta_i$, where Δ_i's are irreducible. Then,*

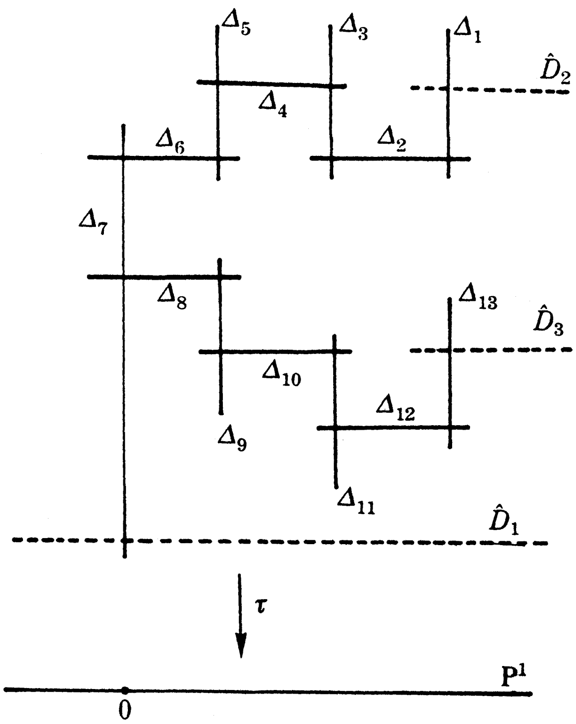
(1) *E is non-reduced and $E_{\text{red}} \cong \mathbf{P}^1$,*

(2) *$\text{Sing}\ S = p_0$, p_0 is a rational double point of A_{13}-type,*

(3) *$D \sim 2D_1 + 3D_2 + 3D_3$ as a Weil divisor on S, where D_i's are irreducible reduced Weil divisors on S such that $D_i \cong \mathbf{P}^1$ and $D_1 \cap D_2 \cap D_3 = \{p_0\}$,*

(4) *there is a fibering $\tau : M \to \mathbf{P}^1$ with exactly one singular fiber $\tau^{-1}(0)$ such that $\tau^{-1}(0) = \cup \Delta_i \cup \widehat{D}_2 \cup \widehat{D}_3$, $(\widehat{D}_i \cdot \widehat{D}_i) = -1$, $(\Delta_j \cdot \Delta_j) = -2$ for $i \geqq 2$, $j \geqq 1$, in particular, $\widehat{D}_1$ is a section of τ (see Figure 2 below), where $\widehat{D}_i$ is the proper transform of $\widehat{D}_i$ in M, and*

(5) *$K_M \sim -2\widehat{D}_1 - 3\widehat{D}_2 - 3\widehat{D}_3 - \sum_{i=1}^{7} (3+i)\,\Delta_i - \sum_{i=1}^{6} (3+i)\,\Delta_{14-i}$, where $(\widehat{D}_1 \cdot \Delta_7) = (\widehat{D}_2 \cdot \Delta_1) = (\widehat{D}_3 \cdot \Delta_{13}) = 1$, $(\widehat{D}_i \cdot \widehat{D}_j) = 0$ $(i \neq j)$, $(\Delta_i \cdot \Delta_{i+1}) = 1$, $(\Delta_i \cdot \Delta_j) = 0$ $(|i-j| > 1)$.*



Proof. By Lemma 2-(1), $E_{\text{red}} \cong \mathbf{P}^1$. By Lemma 23, $(E \cdot H) = 4$ for a hyperplane section H of $X := V'_{22}$. This proves (1). By Lemma 24, $\tau : M \to \mathbf{P}^1$ has exactly one singular fiber and the self-intersection number of each irreducible component the singular fiber is equal to -1 or -2. By Lemma 21 and Lemma 25, $\widehat{D}_2$ and $\widehat{D}_3$ are the (-1)-curves in the singular fiber of τ, and other components of the singular fiber are the exceptional divisor of μ. This enables us to determine the type of the singular fiber of τ (see Figure 2). This proves (2), (3), (4).

Since $K_M \sim -\widehat{D} - \sum m_i \Delta_i$, by the adjunction formula, we have (5)

Q.E.D.

Remark 2. Our example (V'_{22}, H'_{22}) of a compactification of $\mathbf{C}^3$ gives a counter example to Theorem (3.16) in the paper of Peternell [14].

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