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R-ISOMORPHISMS OF TRANSFORMATION GROUPS AND PROLONGATIONS

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1. Introduction. In [**8**] the notion of a reparameterizing isomorphism of transformation groups, henceforth called an R-isomorphism, is introduced generalizing Ura's type-2 isomorphism (see [**13**]). We have shown [**8**] that an R-isomorphism is weaker than a transformation group isomorphism. For example, although R-isomorphisms preserve pointwise almost periodicity and minimality they do not preserve the existence of slices [**7**] or almost periodicity. This suggests that R-isomorphisms might be a useful classification tool in topological dynamics.

An R-isomorphism is defined in terms of continuous and bi-continuous families of homeomorphisms and the following important question arises. When is a continuous family of homeomorphisms a bicontinuous family?

In the first part of the paper we show that if the continuous family consists of homeomorphisms of a locally compact metric space and if the family satisfies an equicontinuity condition then such a continuous family is bicontinuous. Using the terminology and notation found in [**8**] we show that if $(h, (f_x : T \to T|x \in X))$ is an R-homomorphism of the transformation group (X, T, π) onto the transformation group (Y, T, ρ) where T is a locally compact metric topological group and if $(f_t : X \to T|t \in T)$, where f_t is defined by $x \to tf_x$, is an equicontinuous collection of functions then

$$(h, (f_x : T \to T|x \in X))$$

is an R-isomorphism.

In order for R-isomorphisms to be of use in any sort of classification theory, it is necessary to determine those topological dynamics properties preserved by R-isomorphisms and those not preserved. Although some attempt to do this was made by us in [**8**], we were not able to identify any general family of properties that were R-isomorphism invariant.

For dynamical systems Markus [**10**] found that the properties of being unstable, completely unstable and unstable with no saddle at infinity are preserved by R-isomorphisms. Now we generalize the notions of limit set, prolongation and prolongational limit set so that they can be used in topological dynamics. Our definitions are equivalent to Hajek's generalizations (see [**5**]). Limit set, prolongation and prolongational limit set are R-isomorphism invariant and so any topological dynamics property characterizable in terms of these sets is preserved by R-isomorphisms. In particular, the property of a

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transformation group being proper is equivalent to the prolongational limit sets being empty generalizing the analogous situation occurring in dynamical systems and this result is very useful in the study and characterization of actions of locally compact semigroups with zero (see [**6**; **9**]).

2. Definitions, notation and a lemma. A general reference for the standard topological dynamics terms is [**4**]. To fix the notation at the outset, a transformation group, henceforth abbreviated *tg*, usually denoted as an ordered triple (X, T, π) where X is a topological space, T a topological group and π a map from $X \times T$ onto X will be denoted by (X, T) unless ambiguity is imminent. We shall let all functions act on the right and denote an element $t \in T$ acting on $x \in X$ by xt.

Let X, Y, Z be topological spaces. $(f_x : Y \to Z | x \in X)$ is said to be a *continuous family* if $f : X \times Y \to Z$, by $(x, y) \mapsto yf_x$, is continuous; $(f_x : Y \to Z | x \in X)$ is a *bicontinuous family* if the maps $f_x : Y \to Z$ are one-to-one for $x \in X$ and the maps $f : X \times Y \to Z$ by $(x, y) \mapsto yf_x$ and $\hat{f} : X \times Z \to Y$, by $(x, z) \mapsto zf_x^{-1}$ are continuous.

Let (X, T) and (Y, T) be *tgs*. An *R-homomorphism from* (X, T) into (Y, T) is an ordered pair $(h, (f_x : T \to T | x \in X))$ where h is a continuous map from X into Y and $(f_x : T \to T | x \in X)$ is a continuous family of homeomorphisms of T such that $xth = xhtf_x$ for $x \in X$ and $t \in T$. An *R-isomorphism from* (X, T) *onto* (Y, T) is an R-homomorphism $(h, (f_x : T \to T | x \in X))$ where h is a surjective homeomorphism and $(f_x | x \in X)$ is a bicontinuous family of homeomorphisms of T.

For the remainder of this section let T be a locally compact metric space with metric d. Also, if (t_a) is a net in T then by the notation $t_a \to \infty$ we mean that no subnet is contained in a compact set.

LEMMA 1. *Let $f : T \to T$ be a homeomorphism. Let $x_0 \in T$ and $\epsilon > 0$. Then there is a positive integer M such that if $d(x_0, y) > M$ then $d(x_0 f, yf) > \epsilon$ for $y \in T$.*

Proof. Suppose no such M exists for some $\epsilon_0 > 0$. Then there exists a sequence (x_n) such that $x_n \to \infty$ and $d(x_0 f, x_n f) \leqq \epsilon_0$. T is locally compact so that we may assume (by taking a subsequence if necessary) that $x_n f \to y_0$ where $d(x_0 f, y_0) \leqq \epsilon_0$. But then $(x_n f)f^{-1} \to \infty$ which is a contradiction.

3. Continuous and bicontinuous families. Let T be a locally compact metric space, X a Hausdorff space and $(f_x : T \to T | x \in X)$ a family of homeomorphisms of T. For $t \in T$ define $f_t : X \to T$ by $x \mapsto tf_x$.

THEOREM 1. *If $(f_x : T \to T | x \in X)$ is a continuous family of homeomorphisms of T and $(f_t | t \in T)$ is an equicontinuous collection then*

$$(f_x | x \in X)$$

is bicontinuous.

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Proof. Let $x \in X$. We need only show (see [**3**, p. 261]) that if $K \subset Wf_x$ for K compact and W open subsets of T then there is a neighbourhood U of x in X with $K \subset Wf_y$ for all $y \in U$.

If no such U exists then we have sequences $(t_n) \subset K$, $(x_n) \subset X$ and $(t_n') \subset T - W$ such that $t_n \to t$ for some $t \in K$, $x_n \to x$ and $t_n = t_n'f_{xn}$. Let $t = t_0 f_x$ where t_0 is some element of W. If any subsequence of (t_n') converges then it follows that f_x is not one-to-one which is a contradiction. Therefore, we shall assume that $t_n' \to \infty$.

Let $\epsilon > 0$. We shall show that there is a positive integer N such that if $m, n \geqq N$ then

$$(*) \qquad d(t_0 f_{xn'} t_m' f_{xn}) < \epsilon.$$

In fact, the triangle inequality implies that

$$d(t_0 f_{xn}, t_m' f_{xn}) \leqq d(t_0 f_{xn}, t_n' f_{xn}) + d(t_n' f_{xn}, t_m' f_{xm}) + d(t_m' f_{xm}, t_m' f_x) + d(t_m' f_x, t_m' f_{xn}).$$

Continuity of the function $(x, t) \mapsto tf_x$ implies that there is an integer N_1 such that if $n \geqq N_1$ then $d(t_0 f_{xn}, t_n' f_{xn}) < \epsilon/4$. Similarly there is an integer N_2 such that if $n, m \geqq N_2$ then $d(t_n' f_{xn}, t_m' f_{xm}) < \epsilon/4$. Equicontinuity of $(f_t | t \in T)$ implies that there is an integer N_3 such that if $m, n \geqq N_3$ then

$$d(tf_{xm}, tf_x) < \epsilon/4 \quad \text{and} \quad d(tf_x, tf_{xn}) < \epsilon/4 \text{ for all } t \in T.$$

Hence $(*)$ follows.

We claim now that for any $\epsilon' > 0$ there is a positive integer N' such that $d(t_0, y) > N'$ implies that $d(t_0 f_x, yf_x) > \epsilon'$ and $d(t_0 f_{xn}, yf_{xn}) > \epsilon'$ for an infinite number of n and for $y \in T$ where the particular infinite subset of integers n depends on y. It is clear that once the claim is established, then it, the fact that $t_n' \to \infty$ together with $(*)$ give a contradiction and so a proof of the theorem.

It remains, therefore, only to prove the claim. Let $N(\epsilon)$ denote the positive integer guaranteed by Lemma 1 such that $d(t_0, y) > N(\epsilon)$ implies that $d(t_0 f_x, yf_x) > \epsilon$ for $y \in T$. If the claim is false then there is an $\epsilon_0 > 0$ such that for each positive integer $N \geqq N(\epsilon_0)$ there is $y_N \in T$ with $d(t_0, y_N) > N$ and $d(t_0 f_{xn}, y_N f_{xn}) \leqq \epsilon_0$ for all but a finite number of n. Applying Lemma 1 for $\epsilon = 3\epsilon_0$ we obtain a positive integer $N(3\epsilon_0)$ with the property that if $d(t_0, t) > N(3\epsilon_0)$, $y \in T$ then $d(t_0 f_x, yf_x) > 3\epsilon_0$. On the other hand, we have

$$(**) \qquad d(t_0 f_x, yf_x) \leqq d(t_0 f_x, t_0 f_{xn}) + d(t_0 f_{xn}, yf_{xn}) + d(yf_{xn}, yf_x)$$

for all n and $y \in T$. Select $y_{N(3\epsilon_0)} \in T$ such that $d(t_0, y_{N(3\epsilon_0)}) > N(3\epsilon_0)$ and $d(t_0 f_{xn}, y_{N(3\epsilon_0)} f_{xn}) \leqq \epsilon_0$ for all but a finite number of n. Continuity of $f : X \times T \to T$ and $(**)$ imply that $d(t_0 f_x, y_{N(3\epsilon_0)} f_x) < 3\epsilon_0$ which is a contradiction.

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If T is a locally compact topological space let $K = T \cup \{p\}, p \notin T$ be its one-point compactification. If $f : T \to T$ is a homeomorphism of T onto T then there is a unique homeomorphism $\hat{f} : K \to K$ which extends f (by fixing p).

The following theorem is found in [**8**]. We include its statement here for completeness.

THEOREM 2. *Let* $(f_x : T \to T | x \in X)$ *be a continuous family of homeomorphisms of* T *onto* T. *Then* $(f_x | x \in X)$ *is bicontinuous if and only if* $(\hat{f}_x : K \to K | x \in X)$ *is a continuous family.*

THEOREM 3. *Let* $f : X \times T \to T$ *be continuous where* X *is a Hausdorff space and* T *is a locally compact metric space. If the maps* $f_x : T \to T$ *defined by* $t \mapsto (x, t)f$ *are homeomorphisms of* T *for* $x \in X$ *and if* $(f_t : X \to T | t \in T)$ *is an equicontinuous collection, then there is a unique continuous extension* $\hat{f} : X \times K \to K$ *of* f *such that the maps* $\hat{f}_x : K \to K$ *by* $t \mapsto (x, t)\hat{f}$ *are homeomorphisms of* K *for each* x.

Proof. By hypothesis $(f_x : T \to T | x \in X)$ is a continuous family of homeomorphisms of T and $(f_t | t \in T)$ has the equicontinuity condition called for in Theorem 1. It follows that $(f_x | x \in X)$ is a bicontinuous family. Theorem 2 applies so that $(\hat{f}_x : K \to K | x \in X)$ is a continuous family and hence $\hat{f} : X \times K \to K$ is the continuous extension of f.

We remark that an important application of Theorem 3 occurs when $T = \mathbf{R}^n$, Euclidean n-space, and $K = S^n$, the n-sphere.

4. Prolongations in a *tg* (X, T). Henceforth T is a locally compact topological group, X a Hausdorff space and (X, T) a *tg*. We define

$L(x) = \{y \in X | xt_a \to y$ for some net $(t_a) \subset T$ with $t_a \to \infty\}$;

$D(x) = \{y \in X | x_a t_a \to y$ for some net $(x_a) \subset X$ with $x_a \to x$ and some net $(t_a) \subset T\}$;

$J(x) = \{y \in X | x_a t_a \to y$ for some net $(x_a) \subset X$ with $x_a \to x$ and some net $(t_a) \subset T$ with $t_a \to \infty\}$.

$L(x)$ is called the *limit set of* x. We observe that for $T = \mathbf{R}$ or $\mathbf{Z}$, $L(x)$ is equal to the union of the omega and alpha limit sets as defined in [**2**, 2.2.5]. $D(x)$ is called the *prolongation of* x and $J(x)$ is called the *prolongational limit set* of x. The definitions of these sets are equivalent to those found in [**5**] and so we remark that $L(x)$ $D(x)$, $J(x)$ are closed T-invariant sets.

It is known that these sets are preserved under *tg* isomorphisms. If (X, T) is a dynamical system (i.e., $T = \mathbf{R}$) then it is known that these sets are preserved by R-isomorphisms (see [**13**] and [**10**]). We show now that these distinguished sets are preserved by R-isomorphisms in any *tg* (X, T).

THEOREM 4. *Let* $(h, (f_x : T \to T | x \in X))$ *be an* R*-isomorphism from tg* (X, T) *onto* (Y, T). *Then for* $x \in X$,

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(i) $L(x)h = L(xh)$;
(ii) $D(x)h = D(xh)$;
(iii) $J(x)h = J(xh)$.

Proof. (i) Suppose $y \in L(x)$. Then there is a net $(t_a) \subset T$ with $t_a \to \infty$ and $xt_a \to y$. It follows that $xt_a h = xht_a f_x \to yh$ so that $yh \in L(xh)$ since f_x is a homeomorphism of T. For the other inclusion we observe that if $(h, (f_x|x \in X))$ is an R-isomorphism then $(h^{-1}, (\psi_y : T \to T|y \in Y))$ where ψ_y is a homeomorphism defined by $t \mapsto tf_{yh^{-1}}{}^{-1}$ is an R-isomorphism from (Y, T) onto (X, T). Hence if $y \in L(xh)$, by the first inclusion $yh^{-1} \in L(xhh^{-1}) = L(x)$. Therefore it follows that $y \in L(x)h$.

For (ii), suppose $y \in D(x)$. Then there exist nets $(x_a) \subset X$, $(t_a) \subset T$ with $x_a \to x$ and $x_a t_a \to y$. This implies that $x_a t_a h = x_a h t_a f_{x_a} \to yh$. Since $x_a h \to xh$ we conclude that $yh \in D(xh)$. The other inclusion follows as in the proof of (i) using h^{-1}.

To prove (iii), let $y \in J(x)$. We shall show that $yh \in J(xh)$. There are nets $(x_a) \subset X$, $(t_a) \subset T$ with $x_a \to x$, $t_a \to \infty$ and $x_a t_a \to y$. Therefore $(x_a t_a)h = x_a h t_a f_{x_a} \to yh$ and $x_a h \to xh$. We show that $t_a f_{x_a} \to \infty$. If not, there is a subnet of $(t_a f_{x_a})$ which converges to some $t \in T$. For notation purposes, assume $t_a f_{x_a} \to t$. Hence $(x_a, t_a f_{x_a}) \to (x, t)$ and since $(f_x : T \to T|x \in X)$ is a bicontinuous family it follows that $t_a \to tf_x{}^{-1} \in T$ which gives a contradiction.

As in (i) and (ii) above, the other inclusion follows by using the R-isomorphism $(h^{-1}, (\psi_y|y \in Y))$ and thus completes the proof of the theorem.

It follows now that any *tg* property that can be characterized in terms of $D(x)$, $L(x)$ and/or $J(x)$ is invariant under R-isomorphisms. The remainder of this paper deals with just such properties.

5. Cartan and proper *tg*s. The notion of slice [**7**; **11**; **12**] is important in the study of *tg*s (X, T) since existence of slices gives some sort of local product structure on X in terms of T or some factor group and X/T. Palais' study of Cartan and proper *tg*s [**11**] arose precisely because such *tg*s possessed slices at each point (if X is completely regular and G is a Lie group) and so generalized (from the slice point of view at least) *tg*s (X, T) where T is a compact Lie group.

In this section, for T an arbitrary locally compact topological group and X a first countable locally compact Hausdorff space we characterize the properties proper and Cartan in terms of $J(x)$. For dynamical systems such characterizations are known [**2**; **13**]. In particular a Cartan *tg* (proper *tg*) for dynamical systems is usually called wandering (dispersive). A dynamical system is wandering (dispersive) if $x \notin J(x)$ $(J(x) = \emptyset)$ for all x in X. In dynamical systems theory such *tg*s are studied because of their parallelizability properties (see for example [**13**]).

The importance of this characterization for arbitrary *tg*s lies with the fact that it bridges the work of Palais [**11**] with that of other mathematicians. For

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example, parts of Theorem 2 and its Corollary in [**10**] are a special case of Palais' more general Proposition 1.1.4, Theorem 1.2.9 in [**11**]. Moreover the fiber bundle structure of unstable dynamical systems discussed by Markus [**10**] is not unexpected since H-slices exist at each point of such *tg*s where $H = \{1\}$ and Palais [**12**] discusses at some length (but not with differentiability considerations) the fiber bundle structure of *tg*s which possess slices.

A *tg* (X, T) is Cartan if for each $x \in X$ there is a neighbourhood U containing x such that the closure of $\{t \in T | Ut \cap U \neq \emptyset\}$ is compact; (X, T) is proper if for any $x, y \in X$ there are neighbourhoods U of x and V of y such that the closure of $\{t \in T | Ut \cap V \neq \emptyset\}$ is compact.

THEOREM 5. *Let (X, T) be a tg where X is a locally compact first countable space and T is a locally compact topological group. Then*

(i) *(X, T) is a Cartan tg if and only if $x \notin J(x)$ for each $x \in X$.*

(ii) *(X, T) is proper if and only if $J(x) = \emptyset$ for all $x \in X$.*

Proof. (i) and (ii) follow easily from the definitions and the fact that if $x, y \in X$ and there do not exist neighbourhoods U of x and V of y such that the closure of $\{t \in T | Ut \cap V \neq \emptyset\}$ is compact, then $y \in J(x)$. To prove this fact, let $\{U_n\}$ be a base for the topology at x and $\{V_n\}$ a base at y. Let N be any fixed compact neighbourhood of the identity of T. Select $x_1 \in U_1$, $y_1 \in V_1$ and $t_1 \in T$ such that $x_1 t_1 = y_1$. Find $x_2 \in U_2$, $y_2 \in V_2$, $t_2 \in T$ such that $x_2 t_2 = y_2$ and $t_2 \notin N t_1$. We continue inductively and obtain sequences $(x_n) \subset X$, $(y_n) \subset X$, $(t_n) \subset T$ with $x_n \to x$, $x_n t_n = y_n \to y$ with $x_n \in U_n$, $y_n \in V_n$ and $t_n \notin \bigcup_{j<n} N t_j$. We claim that $t_n \to \infty$. If not, then we may assume $t_n \to t$ for some $t \in T$. Select a symmetric neighbourhood V of the identity of T with $VV \subset N$. There is a positive integer m such that if $n \geqq m$ then $t_n \in Vt$. Consequently $t \in V t_n$ for $n \geqq m$. Let p and q be integers greater than m with $p > q$. By construction

$$t_p \notin \bigcup_{i<p} N t_i.$$

But $t_p \in Vt$, $t \in V t_q$ and so $t_p \in V V t_q \subset N t_q$ which is a contradiction.

We remark in closing that there are many topological dynamics properties that may be described in terms of the sets $L(x)$, $D(x)$ and $J(x)$ in a natural way and so invariant under R-isomorphisms. For example, a *tg* (X, T) is minimal if and only if $L(x) = X$ for all x in X or (X, T) is transitive; (X, T) is ergodic if and only if $D(x) = X$ for all x in X.

Ahmad [**1**] introduced the notion of characteristic 0 for dynamical systems which generalizes naturally to *tg*s (X, T) in terms of the set $D(x)$. (X, T) is of characteristic 0 if and only if the closure of $xT = D(x)$ for all x in X. This property seems to embody the difference between Cartan and proper *tg*s and ergodic and minimal *tg*s. Specifically one can show that if X is locally compact and first countable then a Cartan *tg* (X, T) is proper if and only if it has characteristic 0. Similarly it can be shown that (X, T) is ergodic and has

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characteristic 0 if and only if it is minimal. These observations seem to indicate that further investigations of *tg*s having characteristic 0 would be most interesting.

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