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# A GENERALIZATION OF SMITH'S DETERMINANT

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ABSTRACT. We shall evaluate the determinants of $n \times n$ matrices of the form $[f(i,j)]$, where $f(m,r)$ is an even function of $m \pmod r$. Among the examples of determinants of this kind are H. J. S. Smith's determinant $\det[(i,j)]$, where $(m,r)$ is the greatest common divisor of $m$ and $r$, and a generalization of Smith's determinant due to T. M. Apostol.

Smith [11] showed that

$$\det[(i,j)] = \phi(1)\ldots\phi(n)$$

where $\phi$ is Euler's function. He also showed that if $g$ is an arithmetical function and if

$$f(m,r) = \sum_{d|(m,r)} g(d),$$

then $\det[f(i,j)] = g(1)\ldots g(n)$.

Apostol [1] extended Smith's result by showing that if $g$ and $h$ are arithmetical functions and if

$$f(m,r) = \sum_{d|(m,r)} g(d)h(r/d),$$

then $\det[f(i,j)] = g(1)\ldots g(n)h(1)^n$. He noted that as a consequence of this, $\det[c(i,j)] = n!$, where $c(m,r)$ is Ramanujan's sum.

(A) Because we want our main result to properly contain Apostol's, we shall give an independent proof that $\det[c(i,j)] = n!$. We have

$$\sum_{d|r} c(m,r) = \begin{cases} r \text{ if } r|m \\ 0 \text{ if } r\nmid m. \end{cases}$$

Thus, if we set $\beta(d,r) = 1$ or $0$ according as $d$ does or does not divide $r$, then $[c(i,j)][\beta(i,j)]$ is equal to a lower triangular matrix having diagonal elements $1, 2, \ldots, n$. Since $\det[\beta(i,j)] = 1$, the result follows.

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(B) Suppose that for each $r, f(m,r)$ is an even function of $m$ (mod $r$), i.e., $f((m,r),r) = f(m,r)$ for all $m$. Then, E. Cohen [3] showed that $f(m,r)$ can be written uniquely in the form

$$f(m,r) = \sum_{d|r} c(m,d)\alpha(d,r).$$

If we set $\alpha(d,r) = 0$ whenever $d \nmid r$, then $[f(i,j)] = [c(i,j)][\alpha(i,j)]$. The matrix $[\alpha(i,j)]$ is an upper triangular matrix, and therefore,

$$\det\,[f(i,j)] = n!\alpha(1,1)\ldots\alpha(n,n).$$

This is the generalization of Smith's determinant alluded to in the title of the paper. The functions considered by Smith and by Apostol are even functions of $m$ (mod $r$).

(C) Cohen showed in [3] that $f(m,r)$ is an even function of $m$ (mod $r$) if and only if there is a function $F$ of two positive integer variables such that

$$f(m,r) = \sum_{d|(m,r)} F(d, r/d) \text{ for all } m.$$

In terms of the function $F$,

$$\alpha(d,r) = \frac{1}{r} \sum_{e|(r/d)} F(r/e, e)e$$

for every divisor $d$ of $r$. Thus, $\alpha(r,r) = F(r,1)/r$, and

$$\det\,[f(i,j)] = F(1,1)\ldots F(n,1).$$

For the functions considered by Apostol, $F(m,r) = g(m)h(r)$.

(D) The determinant $\det\,[(i,j)^s]$, where $s$ is a real number, was evaluated by Smith. Of course, it can be evaluated by using the result of (B) directly. For each $r$, $(m,r)^s$ is an even function of $m$ (mod $r$) and by ([4], Corollary 11),

$$\alpha(d,r) = r^{s-1} \sum_{e|r} c(r/d, e)/e^s$$

for every divisor $d$ or $r$. Thus,

$$\alpha(r,r) = \frac{1}{r} \sum_{e|r} (r/e)^s \mu(e) = \frac{1}{r}\,\phi_s(r),$$

and $\det\,[(i,j)^s] = \phi_s(1)\ldots\phi_s(n)$.

(E) Let $N(m,r,s)$ denote the number of solutions $x_1, \ldots, x_s$ of the linear congruence

$$m \equiv X_1 + \ldots + X_s \pmod r$$

such that $(x_i, r) = 1$ for $i = 1, \ldots, s$. Two solutions are considered to be distinct if and only if they are distinct (mod $r$). If $s$ is a positive integer then ([4], Corollary 12)

$$c(m,r)^s = \sum_{d|r} N(r/d, r, s)c(m,r).$$

Thus, $\alpha(r, r) = N(1, r, s)$ and

$$\det [c(i,j)^s] = n!N(1,1,s)\ldots N(1,n,s).$$

$N(1,1,s) = 1$, and using H. Rademacher's formula for $N(n, r, s)$ (see [9], and the reference given there), if $r > 1$ then

$$N(1, r, s) = r^{s-1} \prod_{p|r} \frac{(p-1)^s - (-1)^s}{p^s},$$

where the product is over the distinct prime divisors of $r$. From this we see immediately that

$$\det [c(i,j)^s] = 0 \text{ if } s \text{ is even and } n \geqslant 2.$$

(F) For fixed $r$ and $s$, $N(m, r, s)$ is an even function of $m$ (mod $r$), and $\alpha(d, r) = c(r/d, r)^s/r$ ([3], Theorem 6). Thus, $\alpha(r, r) = c(1, r)^s/r = \mu(r)^s/r$, and

$$\det [N(i,j,s)] = (\mu(1)\ldots\mu(n))^s.$$

Therefore,

$$\det [N(i,j,s)] = \begin{cases} 1 & \text{if } n = 1, \text{ or } n = 2 \text{ and } s \text{ is even, or } n = 3 \\ -1 & \text{if } n = 2 \text{ and } s \text{ is odd} \\ 0 & \text{if } n \geqslant 4. \end{cases}$$

(G) Let $(m, r)_*$ be the largest divisor of $m$ that is a unitary divisor of $r$ (see [5] for the terms used in this paragraph). If $N^*(m, r, s)$ is the number of solutions $x_1, \ldots, x_s$ of the congruence in (E) such that $(x_i, r)_* = 1$ for $i = 1, \ldots, s$, then $N^*(m, r, s)$ is an even function of $m$ (mod $r$) and $\alpha(d, r) = c^*(r/d, r)^s/r$, where $c^*(m, r)$ is the unitary analogue of Ramanujan's sum ([9], Example 7). Thus, $\alpha(r, r) = c^*(1, r)^s/r = \mu^*(r)^s/r$ and

$$\det [N^*(i,j,s)] = (\mu^*(1)\ldots\mu^*(n))^s.$$

By ([5], Theorem 2.5), $\mu^*(r) = 1$ or $-1$ according as $r$ has an even or an odd number of distinct prime divisors. Therefore, $\det [N^*(i,j,s)] = 1$ if $s$ is even.

(H) We can evaluate $\det [f(i,j)]$ when $f(m, r)$ is any one of several generalizations of Ramanujan's sum. For example, consider the sum $c_k(m, r)$ introduced by Cohen in [2]. For all $r$,

$$\sum_{d|r} c_k(m, r) = \begin{cases} r^k & \text{if } r^k | m \\ 0 & \text{if } r^k \nmid m. \end{cases}$$

Thus, if $\beta(d, r)$ is defined as in (A), then $[c_k(i,j)][\beta(i,j)]$ is a lower triangular matrix having all of its diagonal elements except the first equal to zero when $k \geqslant 2$. Therefore,

$$\det [c_k(i,j)] = 0 \text{ if } n \geqslant 2 \text{ and } k \geqslant 2.$$

If we argue as in (B), it follows that if $f(m, r)$ is a $k$-even function of $m$ (mod $r$), as defined in [7], then

$$\det [f(i,j)] = 0 \text{ if } n \geqslant 2 \text{ and } k \geqslant 2.$$

(I) Let $A$ be a regular arithmetical convolution, defined by W. Narkiewicz in [10], and let $c_A(m, r)$ be the corresponding generalized Ramanujan sum defined in [8]. Then $c_A(m, r)$ is an even function of $m$ (mod $r$), and $\alpha(r, r) = 1$ ([8], Theorem 2). Therefore, $\det [c_A(i,j)] = n!$. An analogue of the even functions (mod $r$) corresponding to $A$ was developed in [8]. For these functions a result exactly similar to the one in (B) holds, and it contains in turn the unitary analogues of Smith's results obtained by H. Jager in [6].

(J) For $r = 1, \ldots, n$ let $D(r)$ be a nonempty set of positive divisors of $r$, and let $T(r) = \{x : 1 \leqslant x \leqslant r \text{ and } (x, r) \in D(r)\}$. If

$$g(m, r) = \sum_{x \in T(r)} e^{2\pi i m x/r},$$

then $g(m, r)$ is an even function of $m$ (mod $r$). In fact ([9], p. 138),

$$g(m, r) = \sum_{d \in D(r)} c(m, r/d).$$

Thus, $\alpha(r, r) = 1$ or 0 according as $1 \in D(r)$, or $1 \notin D(r)$, and consequently

$$\det [g(i,j)] = \begin{cases} n! & \text{if } 1 \in D(r) \text{ for } r = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

The sum $g(m, r)$ can be considered to be a generalized Ramanujan sum. If $D(r) = \{1\}$ then $g(m, r) = c(m, r)$. Let $k \geqslant 2$ and $0 < q < k$, and let $D(r)$ be the set of all divisors $d$ of $r$ such that if $p^t$ is the highest power of a prime $p$ dividing $d$ then $t \equiv 0, 1, \ldots,$ or $q - 1$ (mod $k$). Then $g(m, r) = D_{k,q}(m, r)$, the generalization of the Ramanujan sum defined in [12]. Since $1 \in D(r)$ for all $r$, $\det [D_{k,q}(i,j)] = n!$.

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