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UNIPOTENT MATRIX GROUPS OVER DIVISION RINGS

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ABSTRACT. If G is a unipotent group of $n\times n$ matrices over a division ring of characteristic 0 or prime p greater than $(n-1)(n-[n/2])$ where $[n/2]$ is the greatest integer less than or equal to $n/2$, then it is proved that G can be simultaneously triangularized.

The study of linear groups (i.e. groups of matrices) over division rings is still in its infancy, and there are few if any significant results. In particular, it is well-known that a unipotent linear group over a commutative field can always be simultaneously triangularized but the situation for linear groups over division rings is apparently open ([3], p. 100 and p. 136, and [4]). Walter S. Sizer [5] recently proved that a solvable unipotent linear group over a division ring can be simultaneously triangularized. (An element a in a ring with 1 is called unipotent if $(a-1)$ is nilpotent.)

In this note we shall prove the following theorem.

THEOREM. *Let G be a unipotent group of $n\times n$ matrices over a division ring Δ of characteristic* 0 *or prime $p>(n-1)(n-[n/2])$ where $[n/2]$ is the greatest integer less than or equal to $n/2$. Then G can be simultaneously triangularized.*

In order to prove the Theorem we need a result essentially due to Heineken [2] with an improvement due to R. H. Bruck [1].

THEOREM (Heineken). *Let H be a finitely generated multiplicative group in an algebra A (with 1) over $\mathbb{Q}$, the field of rational numbers, or $\mathbb{Z}_p$, the field of integers modulo the prime p, such that $p>(n-1)(n-[n/2])$ and*

$$(h-1)^n=0 \quad \text{for all} \quad h\in H.$$

Then, there exists a positive integer $k(n)$ such that

$$(h_1-1)(h_2-1)\cdots(h_{k(n)}-1)=0$$

for all $h_1, h_2, \ldots, h_{k(n)}\in H$.

REMARKS. Heineken's original theorem has the hypothesis that A be a ring in which $((n-1)^2)!$ is invertible. However, there seems to be some confusion in Heineken's proof concerning the application of Kostrikin's Theorem on Lie

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rings satisfying an Engel condition (see Math. Rev. 26, #206). In order to eliminate this problem and to incorporate the improvement due to Bruck, we have made the above hypotheses on A. A proof of Heineken's Theorem is in Bruck's notes [1]. It involves a modification of Heineken's original proof in [2] in order to weaken the condition $p>(n-1)^2$ to $p>(n-1)(n-[n/2])$. Special arguments also have to be used for $n=3$ and $p=2$ or 3.

We also need a result due to Levitzki.

THEOREM (Levitzki). *Let S be a multiplicative semigroup consisting of nilpotent $n\times n$ matrices over a division ring. Then, S can be simultaneously (strictly upper) triangularized.*

Proof. See page 135 of [3].

Proof of theorem. Let G be a unipotent group of $n\times n$ matrices over a division ring Δ of characteristic 0 or prime $p>(n-1)(n-[n/2])$. If $g\in G$, we assert that $(g-1)^n=0$. For we may regard G as a group of linear transformations acting faithfully on the left of an n-dimensional right vector space V over Δ. We have

$$V>(g-1)V>(g-1)^2V>\cdots>(g-1)^nV$$

where the inclusions are strict since $(g-1)$ is nilpotent. Since V has dimension n, $(g-1)^nV=0$.

Let S be the multiplicative semigroup generated by the $(g-1)$, $g\in G$. A typical element in S is $a=(g_1-1)\cdots(g_s-1)$, $g_1,\ldots,g_s\in G$. Let H be the subgroup of G generated by $g_1,\ldots,g_s$. By Heineken's Theorem applied to H as a multiplicative group in the full ring of $n\times n$ matrices over Δ, $a^m=0$ for some positive integer m. Therefore, S consists of nilpotent matrices. According to Levitzki's Theorem, S can be simultaneously triangularized. But, this fact implies that G can be simultaneously triangularized.

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