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Integral equivariant cohomology of affine Grassmannians

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Abstract. We give explicit presentations of the integral equivariant cohomology of the affine Grassmannians and flag varieties in type A, arising from their natural embeddings in the corresponding infinite (Sato) Grassmannian and flag variety. These presentations are compared with results obtained by Lam and Shimozono, for rational equivariant cohomology of the affine Grassmannian, and by Larson, for the integral cohomology of the moduli stack of vector bundles on $\mathbb{P}^1$.

1 Introduction

The main aim of this note is to provide a simple presentation, in terms of generators and relations, of the torus-equivariant cohomology of the affine Grassmannian and flag variety, $\widetilde{\mathrm{Gr}}_n$ and $\widetilde{\mathrm{Fl}}_n$. In particular, we obtain these rings as quotients of polynomial rings, with the quotient map arising geometrically as the pullback via embeddings in the Sato Grassmannian and flag variety, respectively.

Let $\Lambda = \mathbb{Z}[c_1, c_2, \ldots]$ be the polynomial ring in countably many generators, with c_i in degree $2i$. Let $p_k = p_k(c)$ be the polynomial

$$p_k(c) = (-1)^{k-1} \det \begin{pmatrix} c_1 & 1 & 0 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 & 0 \\ 3c_3 & c_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ kc_k & c_{k-1} & \cdots & c_2 & c_1 \end{pmatrix}. \tag{1.1}$$

One can identify Λ with the ring of symmetric functions in some other set of variables, making c_k the complete homogeneous symmetric function, so that p_k becomes the power sum symmetric function via the Newton relations. But until Section 3 we remain agnostic about the choice of such an identification.

Fixing n, consider the polynomials

$$p_k(c|y) = p_k(c) + p_{k-1}(c)\, e_1(y_1, \ldots, y_n) + \cdots \tag{1.2}$$
$$+ p_2(c)\, e_{k-2}(y_1, \ldots, y_n) + p_1(c)\, e_{k-1}(y_1, \ldots, y_n)$$

$$= \sum_{i=1}^{k} p_i(c)\, e_{k-i}(y_1, \ldots, y_n) \tag{1.3}$$

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in $\Lambda[y_1,\ldots,y_n]$, where $e_i(y_1,\ldots,y_n)$ is the elementary symmetric polynomial in the indicated variables.

Let V be a complex vector space with basis e_i, for $i \in \mathbb{Z}$, and let $V_{\le 0}$ be the subspace spanned by e_i for $i \le 0$. The torus $T = (\mathbb{C}^*)^n$ acts by scaling the basis vector e_i by the character $y_{i (\mathrm{mod}\ n)}$, using representatives $1,\ldots,n$ for residues mod n. Let $\mathrm{Gr}^d = \mathrm{Gr}^d(V)$ be the corresponding Sato Grassmannian parameterizing subspaces of index d, with the induced action of T. The dth component of the affine Grassmannian embeds T-equivariantly in Gr^d. (Definitions of these spaces are reviewed in Section 2 below.) We write $\mathbb{S}_d \subset V$ for the tautological bundle on Gr^d, and recycle the same notation for the tautological bundle on subvarieties, when the context is clear.

The equivariant cohomology of the Sato Grassmannian is $H_T^*\mathrm{Gr}^d = \Lambda[y_1,\ldots,y_n]$, identifying c_k with the Chern class $c_k^T(V_{\le 0} - \mathbb{S}_d)$.

Theorem *The inclusion $\widetilde{\mathrm{Gr}}{}_n^d \hookrightarrow \mathrm{Gr}^d$ induces a surjection $H_T^*\mathrm{Gr}^d \twoheadrightarrow H_T^*\widetilde{\mathrm{Gr}}{}_n^d$, whose kernel is generated by $p_k(c|y)$ for $k > n$, together with $p_n(c|y) + d e_n(y)$.*

In particular, the map $c_k \mapsto c_k^T(V_{\le 0} - \mathbb{S}_d)$ defines an isomorphism of $H_T^*(\mathrm{pt}) = \mathbb{Z}[y_1,\ldots,y_n]$-algebras

$$\Lambda[y_1,\ldots,y_n]/I_n^d \xrightarrow{\sim} H_T^*(\widetilde{\mathrm{Gr}}{}_n^d),$$

where I_n^d is the ideal generated by $p_k(c|y)$ for $k > n$ and $p_n(c|y) + d e_n(y)$.

All the generators of I_n^d are symmetric in the y variables. It follows that the GL_n-equivariant cohomology has essentially the same presentation. Write $H_{GL_n}^*(\mathrm{pt}) = \mathbb{Z}[e_1,\ldots,e_n]$, with e_k in degree $2k$, regarded as a subring of $H_T^*(\mathrm{pt})$ by sending e_k to the elementary symmetric polynomial $e_k(y)$. Define elements $p_k(c|e) \in \Lambda[e_1,\ldots,e_n]$ by the same formula (1.3), with $e_k = 0$ for $k > n$.

Corollary A *Let $J_n^d \subset \Lambda[e_1,\ldots,e_n]$ be the ideal generated by $p_k(c|e)$ for $k > n$ and $p_n(c|e) + d e_n$. Then the map $c_k \mapsto c_k^{GL_n}(V_{\le 0} - \mathbb{S}_d)$ defines an isomorphism of $H_{GL_n}^*(\mathrm{pt})$-algebras*

$$\Lambda[e_1,\ldots,e_n]/J_n^d \xrightarrow{\sim} H_{GL_n}^*(\widetilde{\mathrm{Gr}}{}_n^d).$$

This follows from the theorem by an application of the general fact that $H_{GL_n}^*X \subset H_T^*X$ is the invariant ring for the natural $\mathfrak{S}_n$ action on y variables (see, e.g., [AF, Section 15.6]).

A presentation for the equivariant cohomology of $\widetilde{\mathrm{Fl}}_n$ also follows from the theorem. Let $\mathbb{S}_\bullet : \cdots \subset \mathbb{S}_{-1} \subset \mathbb{S}_0 \subset \mathbb{S}_1 \subset \cdots$ be the tautological flag on $\widetilde{\mathrm{Fl}}_n$.

Corollary B *Evaluating $c_k \mapsto c_k^T(V_{\le 0} - \mathbb{S}_0)$ and $x_i \mapsto c_1^T(\mathbb{S}_i/\mathbb{S}_{i-1})$, we have*

$$H_T^*\widetilde{\mathrm{Fl}}_n = \Lambda[x_1,\ldots,x_n,y_1,\ldots,y_n]/I_n^{\mathrm{Fl}},$$

where I_n^{Fl} is generated by $p_k(c|y)$ for $k \ge n$ along with $e_i(x) - e_i(y)$ for $i = 1,\ldots,n$.

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For GL_n-equivariant cohomology, the presentation is similar:

$$H^*_{GL_n}\widetilde{\mathrm{Fl}}_n = \Lambda[x_1,\ldots,x_n,e_1,\ldots,e_n]/J_n^{\mathrm{Fl}},$$

where J_n^{Fl} is generated by $p_k(c|e)$ for $k \geq n$ along with $e_i(x) - e_i$ for $i = 1,\ldots,n$.

This can be deduced from the theorem by examining the action of the shift morphism on cohomology (see Section 2).

A presentation for the non-equivariant cohomology ring $H^*\widetilde{\mathrm{Gr}}_n^0$ was given by Bott [Bo], who used a natural coproduct structure to identify this ring with the infinite symmetric power of the cohomology of projective space. Since $H^*\mathbb{P}^{n-1} \cong \mathbb{Z}[\xi]/(\xi^n)$, this is easily seen to be equivalent to the result of setting the y variables to 0 in the statement of the main theorem above. (One makes the indicated identifications with symmetric functions in variables $\xi_1, \xi_2, \ldots$, and Bott's relations become $p_k(\xi) = 0$ for $k \geq n$.)

Several authors have given different presentations of the equivariant cohomology ring, sometimes with field coefficients, using localization or representation theory [LS, Y, YZ]. In the context of the moduli stack of vector bundles on $\mathbb{P}^1$, Larson described the integral cohomology ring as a subring of a polynomial ring with rational coefficients [La]. In fact, Larson's description is equivalent to the quotient ring appearing in Corollary A; the precise translation is given in Section 6 below.

In this note, the main contributions are to provide a concise presentation of $H^*_T\widetilde{\mathrm{Gr}}_n^d$ as a quotient of a polynomial ring, and to show how Bott's method extends naturally to the equivariant setting. We also describe a new basis of double monomial symmetric functions which are well-adapted to the presentation of $H^*_T\widetilde{\mathrm{Gr}}_n$. Apart from some elementary calculations with symmetric functions, the only additional input required is a well-known presentation of the equivariant cohomology of projective space.

2 Infinite and affine flag varieties

We follow [A], which in turn is based on [LLS, PS] (see also [SS]). As in the introduction, V is a complex vector space with basis e_i for $i \in \mathbb{Z}$. For any interval $[a,b]$ in $\mathbb{Z}$, we write $V_{[a,b]}$ for the subspace spanned by e_i for i in $[a,b]$. We will especially use subspaces $V_{\leq p}$ (or $V_{>q}$), spanned by e_i for $i \leq p$ (or $i > q$, respectively).

2.1 Definitions

The *Sato Grassmannian* Gr^d is the set of subspaces $E \subset V$ of index d. This means (1) $V_{\leq -m} \subset E \subset V_{\leq m}$ for some (and hence all) $m \gg 0$, and (2) $\dim E/(V_{\leq 0} \cap E) - \dim V_{\leq 0}/(V_{\leq 0} \cap E) = d$. The Sato Grassmannian is naturally topologized as an ind-variety.

The *Sato flag variety* is the subvariety $\mathrm{Fl} \subset \prod_{d\in\mathbb{Z}} \mathrm{Gr}^d$ consisting of chains of subspaces $E_\bullet : \cdots \subset E_{-1} \subset E_0 \subset E_1 \subset \cdots$, with $E_d \subset V$ belonging to Gr^d. It is naturally a pro-ind-variety, and comes with projection morphisms $\pi_d \colon \mathrm{Fl} \to \mathrm{Gr}^d$.

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The *shift automorphism* $\mathrm{sh}\colon V \to V$, defined by $\mathrm{e}_i \mapsto \mathrm{e}_{i-1}$, induces an automorphism of Fl, by $\mathrm{sh}(E_\bullet)_k = \mathrm{sh}(E_{k+1})$. For a fixed positive n, the *affine flag variety* is the fixed locus of sh^n:

$$\widetilde{\mathrm{Fl}}_n = \{E_\bullet \in \mathrm{Fl} \mid \mathrm{sh}^n(E_\bullet) = E_\bullet\}.$$

The *affine Grassmannian* is the image of $\widetilde{\mathrm{Fl}}_n$ under the projection map:

$$\widetilde{\mathrm{Gr}}_n^d = \pi_d(\widetilde{\mathrm{Fl}}_n) = \{E \in \mathrm{Gr}^d \mid \mathrm{sh}^n(E) \subset E\}.$$

A torus $T_{\mathbb{Z}} = \prod_{i\in\mathbb{Z}} \mathbb{C}^*$ acts on V by scaling the coordinate e_i by the character y_i. This induces actions on Fl and Gr^d. We cyclically embed $T = (\mathbb{C}^*)^n$ in $T_{\mathbb{Z}}$, by specializing characters $y_i \mapsto y_{i (\mathrm{mod}\ n)}$, using representatives $1, \dots, n$ for residues mod n. So $T \subset T_{\mathbb{Z}}$ is the fixed subgroup for the automorphism induced by sh^n, and T therefore acts on $\widetilde{\mathrm{Fl}}_n$ and $\widetilde{\mathrm{Gr}}_n^d$.

The T-fixed points of Fl (which are the same as the $T_{\mathbb{Z}}$-fixed points) are indexed by the set Inj^0 consisting of all injections $w\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\#\{i \le 0 \mid w(i) > 0\} = \#\{j > 0 \mid w(j) \le 0\},$$

and both these cardinalities are finite.[1] The flag $E_\bullet^w$ corresponding to $w \in \mathrm{Inj}^0$ consists of subspaces E_k spanned by $\mathrm{e}_{w(i)}$ for $i \le k$, together with all e_j for $j \le 0$ not in the image of w. The condition defining Inj^0 guarantees $E_\bullet^w$ lies in Fl. (See [A, Section 6].)

The T-fixed points of $\widetilde{\mathrm{Fl}}_n$ are indexed by the group of affine permutations. This is the group $\widetilde{\mathbb{S}}_n$ consisting of bijections w from $\mathbb{Z}$ to itself, such that $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$, and such that $\sum_{i=1}^n w(i) = \binom{n}{2}$. Among many other equivalent descriptions, this is the subset of n-shift-invariant elements in Inj^0:

$$\widetilde{\mathbb{S}}_n = \{w \in \mathrm{Inj}^0 \mid w(i+n) = w(i) + n \text{ for all } i\}.$$

Similarly, GL_n acts on V, extending the standard action on $V_{[1,n]} \cong \mathbb{C}^n$ by blocks, so $V = \cdots \oplus V_{[-n+1,0]} \oplus V_{[1,n]} \oplus V_{[n+1,2n]} \oplus \cdots$. This induces actions on the Sato and affine flag varieties and Grassmannians.

Often we will omit the superscript when focusing on the degree $d = 0$ component, writing $\mathrm{Gr} = \mathrm{Gr}^0$ and $\widetilde{\mathrm{Gr}}_n = \widetilde{\mathrm{Gr}}_n^0$.

2.2 Chern classes and cohomology

We write $c_k^{(d)} = c_k^T(V_{\le 0} - \mathbb{S}_d)$ in $H_T^*\mathrm{Gr}^d$, and we use the same notation for the pullbacks to other varieties. For $d = 0$, or when the index is understood, we omit the superscript. We have canonical isomorphisms

$$H_T^*\mathrm{Gr}^d = \Lambda[y_1, \dots, y_n] \quad \text{and} \quad H_T^*\mathrm{Fl} = \Lambda[\dots, x_{-1}, x_0, x_1, \dots; y_1, \dots, y_n],$$

where $\Lambda = \mathbb{Z}[c_1, c_2, \dots]$ and $x_i = c_1^T(\mathbb{S}_i/\mathbb{S}_{i-1})$ as before. (See [A, Section 3], but note that our sign convention on x_i is opposite the one used there.)

[1] This implies $\#\{i \le d \mid w(i) > 0\} - \#\{j > d \mid w(j) \le 0\} = d$ for any integer d.

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For each fixed point $w \in \mathrm{Inj}^0$, there is a localization homomorphism $H_T^*\mathrm{Fl} \to \mathbb{Z}[y_1, \ldots, y_n]$, given by

$$x_i \mapsto y_{w(i)} \qquad \text{and} \qquad c_k \mapsto [t^k]\left(\prod_{\substack{i \le 0, w(i) > 0 \\ j > 0, w(j) \le 0}} \frac{1 + y_{w(j)}t}{1 + y_{w(i)}t} \right).$$

Here, the operator $[t^k]$ extracts the coefficient of t^k, and we always understand y_a as $y_{a (\mathrm{mod}\ n)}$. Since $w \in \mathrm{Inj}^0$, the RHS is a finite product. The same formulas define localization homomorphisms for Gr, $\widetilde{\mathrm{Fl}}_n$, and $\widetilde{\mathrm{Gr}}_n$. For Gr^d and $\widetilde{\mathrm{Gr}}_n^d$ with $d \neq 0$, we use

$$c_k^{(d)} \mapsto [t^k]\left(\prod_{\substack{i \le d, w(i) > 0 \\ j > d, w(j) \le 0}} \frac{1 + y_{w(j)}t}{1 + y_{w(i)}t} \right).$$

We do not logically require these localization homomorphisms, but they are useful for checking that relations hold, and comparing them against other sources.

The shift morphism determines an automorphism $\gamma = \mathrm{sh}^*$ of $\Lambda[x, y]$, by

$$\gamma(x_i) = x_{i+1}, \quad \gamma(y_i) = y_{i+1}, \quad \text{and} \quad \gamma(C(t)) = C(t) \cdot \frac{1 + y_1 t}{1 + x_1 t},$$

where $C(t) = \sum_{k \ge 0} c_k t^k$ is the generating series for c.

The inclusions $\widetilde{\mathrm{Fl}}_n \hookrightarrow \mathrm{Fl}$ and $\widetilde{\mathrm{Gr}}_n^d \hookrightarrow \mathrm{Gr}^d$ determine pullback homomorphisms on cohomology: we have maps

$$\Lambda[x; y] = H_T^*\mathrm{Fl} \to H_T^*\widetilde{\mathrm{Fl}}_n \quad \text{and} \quad \Lambda[y] = H_T^*\mathrm{Gr}^d \to H_T^*\widetilde{\mathrm{Gr}}_n^d.$$

The main theorems assert that these homomorphisms are surjective, and specify the kernels. One relation is immediately evident: since sh^n fixes $\widetilde{\mathrm{Fl}}_n \subset \mathrm{Fl}$, we have $\gamma^n(c) = c$, so

$$\prod_{i=1}^{n} \frac{1 + y_i t}{1 + x_i t} = 1$$

in $H_T^*\widetilde{\mathrm{Fl}}_n$. As promised in the introduction, this shows that Corollary B follows from the Theorem.

(An alternative argument uses the fact, not needed here, that the projection $\widetilde{\mathrm{Fl}}_n \to \widetilde{\mathrm{Gr}}_n$ is topologically identified with the trivial fiber bundle $\widetilde{\mathrm{Gr}}_n \times Fl(\mathbb{C}^n) \to \widetilde{\mathrm{Gr}}_n$.)

2.3 Coproduct

There is a co-commutative coproduct structure on Λ, where the map $\Lambda \to \Lambda \otimes \Lambda$ is given by $c_k \mapsto c_k \otimes 1 + c_{k-1} \otimes c_1 + \cdots + 1 \otimes c_k$. This extends $\mathbb{Z}[y]$-linearly to a coproduct on $\Lambda[y] = H_T^*\mathrm{Gr}$. As explained in [A, Section 8], this can be interpreted as an (equivariant) cohomology pullback via the direct sum morphism $\mathrm{Gr} \times \mathrm{Gr} \to \mathrm{Gr}$.

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Likewise, there is a co-commutative coproduct structure on $H_T^*\widetilde{\mathrm{Gr}}_n$, coming from a homotopy equivalence with the based loop group, $\widetilde{\mathrm{Gr}}_n \sim \Omega SU(n)$ (see [PS, Section 8.6]). The homotopy equivalence is equivariant with respect to the compact torus $(S^1)^n \subset T$. So the group structure on $\Omega SU(n)$ determines a coproduct on $H^*_{(S^1)^n}\Omega SU(n) = H_T^*\widetilde{\mathrm{Gr}}_n$. (This coproduct can also be realized algebraically, but the construction is somewhat more involved than the direct sum map for Gr (see, e.g., [YZ]).)

The coproducts on $H_T^*\mathrm{Gr}$ and $H_T^*\widetilde{\mathrm{Gr}}_n$ are compatible, in the sense that the inclusion $\widetilde{\mathrm{Gr}}_n \hookrightarrow \mathrm{Gr}$ induces a pullback homomorphism of co-algebras (and in fact, of Hopf algebras): the diagram

$$\begin{tikzcd}
H_T^*\mathrm{Gr} \arrow[r] \arrow[d] & H_T^*\mathrm{Gr} \otimes_{\mathbb{Z}[y]} H_T^*\mathrm{Gr} \arrow[d] \\
H_T^*\widetilde{\mathrm{Gr}}_n \arrow[r] & H_T^*\widetilde{\mathrm{Gr}}_n \otimes_{\mathbb{Z}[y]} H_T^*\widetilde{\mathrm{Gr}}_n
\end{tikzcd}$$

commutes.

3 Some algebra of symmetric functions

In this section, we introduce some polynomials which appear in the presentations of equivariant cohomology rings, and establish some identities which imply isomorphisms among different such presentations. Most of this comes from basic facts about symmetric functions, and can be found in standard sources (e.g., [Mac, Chapter I]). We indicate proofs for facts not easily found there.

Recall $\Lambda = \mathbb{Z}[c_1, c_2, \ldots]$ and $\Lambda[y] = \Lambda[y_1, \ldots, y_n]$.

3.1 Some identities in $\Lambda[y]$

We define elements $h_k \in \Lambda[y_1, \ldots, y_n]$ by

$$h_k = \sum_{i=0}^{k-1} \binom{k-1}{i} y_0^i \, c_{k-i}, \tag{3.1}$$

writing $y_0 = y_n$ to emphasize stability with respect to n.

Let $H(t) = \sum_{k\geq 0} h_k t^k$ and $C(t) = \sum_{k\geq 0} c_k t^k$ be the generating series, with $h_0 = c_0 = 1$. Then (3.1) is equivalent to $H(t) = C\left(t/(1-y_0 t)\right)$. Both the h's and the c's are algebraically independent generators of $\Lambda[y]$ as a $\mathbb{Z}[y]$-algebra.

We have the elements $p_k \in \Lambda = \mathbb{Z}[c_1, c_2, \ldots]$ given by

$$P(t) := \sum_{k\geq 1} p_k t^{k-1} = \frac{d}{dt} \log C(t). \tag{3.2}$$

We define new elements $\widetilde{p}_k \in \mathbb{Z}[h_1, h_2, \ldots]$ by the analogous identity of generating series:

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$$\widetilde{P}(t) := \sum_{k\geq 1} \widetilde{p}_k t^{k-1} = \frac{d}{dt} \log H(t). \tag{3.3}$$

(These formulas are equivalent to the Newton relations (1.1) (see, e.g., [Mac, Section 2]).)

Let $E(t) = \prod_{i=1}^{n}(1 + y_i t)$ be the generating series for the elementary symmetric polynomials in $y_1, \ldots, y_n$, and let $\widetilde{E}(t) = \prod_{i=1}^{n}(1 + (y_i - y_0)t)$ be the corresponding series in variables $y_i - y_0$. So $\widetilde{E}(t) = E(t/(1 - y_0 t)) \cdot (1 - y_0 t)^n$.

Finally, let

$$p_k(c|y) = p_k + p_{k-1}e_1(y) + \cdots + p_1 e_{k-1}(y) \tag{3.4}$$

and

$$\begin{aligned}\widetilde{p}_k(h|y) = \widetilde{p}_k + \widetilde{p}_{k-1}e_1(y_1 - y_0, \ldots, y_n - y_0) + \cdots \\ + \widetilde{p}_1 e_{k-1}(y_1 - y_0, \ldots, y_n - y_0).\end{aligned} \tag{3.5}$$

Equivalently, the generating series for $p_k(c|y)$ and $\widetilde{p}_k(h|y)$ are given by

$$\boldsymbol{P}(t) = P(t) \cdot E(t) \qquad \text{and} \qquad \widetilde{\boldsymbol{P}}(t) = \widetilde{P}(t) \cdot \widetilde{E}(t),$$

respectively.

The polynomials $p_k(c|y)$ are those appearing in the main theorem from the introduction. The variations $\widetilde{p}_k(h|y)$ will be easier to interpret as relations in the equivariant cohomology ring of the affine Grassmannian (in Section 4). We wish to compare the ideals generated by $p_k(c|y)$ and $\widetilde{p}_k(h|y)$.

Lemma 3.1 *For $k \geq n$, we have*

$$\widetilde{p}_k(h|y) = \sum_{i=0}^{k-1} \binom{k-n}{i} y_0^i p_{k-i}(c|y) \tag{3.6}$$

and

$$p_k(c|y) = \sum_{i=0}^{k-1} \binom{k-n}{i} (-y_0)^i \widetilde{p}_{k-i}(h|y). \tag{3.7}$$

In particular, we have an equality

$$\Big(p_k(c|y)\Big)_{k\geq n} = \Big(\widetilde{p}_k(h|y)\Big)_{k\geq n}$$

of ideals in $\Lambda[y_1, \ldots, y_n]$.

Proof The second statement follows from the first, the RHS of (3.6) involves only $p_i(c|y)$ for $i \geq n$, and likewise the RHS of (3.7) involves only $\widetilde{p}_i(h|y)$ for $i \geq n$.

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To prove (3.6), we expand the definitions and compute

$$\begin{aligned}\widetilde{\boldsymbol{P}}(t) &= \widetilde{P}(t)\cdot\widetilde{E}(t)\\ &= \Big(\frac{d}{dt}\log H(t)\Big)\cdot E\Big(t/(1-y_0t)\Big)\cdot(1-y_0t)^n\\ &= \Big(\frac{d}{dt}\log C\Big(t/(1-y_0t)\Big)\Big)\cdot E\Big(t/(1-y_0t)\Big)\cdot(1-y_0t)^n\\ &= \frac{1}{(1-y_0t)^2}P\Big(t/(1-y_0t)\Big)\cdot E\Big(t/(1-y_0t)\Big)\cdot(1-y_0t)^n\\ &= (1-y_0t)^{n-2}\boldsymbol{P}\Big(t/(1-y_0t)\Big).\end{aligned}$$

Expanding the RHS, we obtain

$$\sum_{m\geq 1} p_m(c|y)t^{m-1}(1-y_0t)^{n-m-1} = \sum_{\substack{m\geq 1\\ i\geq 0}} p_m(c|y)\binom{n-m-1}{i}(-y_0)^i t^{m-1+i}.$$

Setting $k = m+i$, for $k \geq n$, the coefficient of t^{k-1} is

$$\sum_{i=0}^{k-1}\binom{n-k+i-1}{i}(-y_0)^i p_{k-i}(c|y) = \sum_{i=0}^{k-1}\binom{k-n}{i}y_0^i p_{k-i}(c|y),$$

as desired. (The last equality uses the extended binomial coefficient identity $\binom{-m}{i} = (-1)^i\binom{m+i-1}{i}$.) The proof of (3.7) is analogous. ∎

3.2 Notation for symmetric functions in ξ

Let $\Lambda^{(\xi)} = \mathbb{Z}[\xi_1, \xi_2, \ldots]^{\mathfrak{S}_\infty}$ be the ring of symmetric functions in countably many variables $\xi_1, \xi_2, \ldots$, each of degree 2. This is the inverse limit of $\Lambda_r^{(\xi)} = \mathbb{Z}[\xi_1, \ldots, \xi_r]^{\mathfrak{S}_r}$ as $r \to \infty$ (in the category of graded rings). It may be identified with the polynomial ring $\mathbb{Z}[h_1(\xi), h_2(\xi), \ldots]$, where $h_k(\xi)$ is the *complete homogeneous symmetric function* in ξ.

There is also a $\mathbb{Z}$-linear basis for $\Lambda^{(\xi)}$ consisting of the *monomial symmetric functions* $m_\lambda(\xi)$. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$, the function $m_\lambda(\xi)$ is the symmetrization of the monomial $\xi_1^{\lambda_1}\xi_2^{\lambda_2}\cdots\xi_r^{\lambda_r}$ – that is, the sum of all distinct permutations of this monomial.

The *power sum functions* $p_k(\xi) = \xi_1^k + \xi_2^k + \cdots$ also play an important role. They generate $\Lambda^{(\xi)}$ as a $\mathbb{Q}$-algebra, but not as a $\mathbb{Z}$-algebra. The function $p_k(\xi)$ is expressed in terms of the functions $h_k(\xi)$ via the Newton relations, which can be written as the determinant (1.1), substituting $h_k(\xi)$ for c_k in the matrix.

There is an isomorphism $\Lambda[y] = \mathbb{Z}[c, y] \xrightarrow{\sim} \Lambda^{(\xi)}[y]$ determined by evaluating the generating series $C(t) = \sum c_k t^k$ as

$$C(t) = \prod_{i\geq 1}\frac{1+y_0t}{1-\xi_i t + y_0 t}. \tag{3.8}$$

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Under this identification, we have $H(t) = \prod_{i\geq 1} \frac{1}{1-\xi_i t}$, so h_k maps to $h_k(\xi)$, and it follows that $\widetilde{p}_k$ maps to the power sum function $p_k(\xi)$. In what follows, we will sometimes use this identification without further comment.

3.3 Another equality of ideals

In Section 4, we require another algebraic lemma. First, we consider finitely many variables $\xi_1, \ldots, \xi_r$, and the symmetric polynomial ring $\Lambda_r^{(\xi)} \subset \mathbb{Z}[\xi_1, \ldots, \xi_r]$.

Lemma 3.2 *Fix $n > 0$, and consider the ideal $(\xi_1^n, \ldots, \xi_r^n) \subset \mathbb{Z}[\xi_1, \ldots, \xi_r]$. As ideals in $\Lambda^{(\xi)}$, we have*

$$(\xi_1^n, \ldots, \xi_r^n) \cap \Lambda_r^{(\xi)} = \big(m_\lambda(\xi)\big)_{\lambda_1 \geq n} = \big(p_k(\xi)\big)_{k\geq n}.$$

Proof The first equality holds because monomials $\xi_1^{a_1} \cdots \xi_r^{a_r}$ with some $a_i \geq n$ form a $\mathbb{Z}$-linear basis for $(\xi_1^n, \ldots, \xi_r^n) \subset \mathbb{Z}[\xi_1, \ldots, \xi_r]$. For the second equality, the inclusion "$\supseteq$" is evident, because $p_k = m_{(k)}$. It remains to see that m_λ lies in the ideal $(p_k)_{k\geq n}$ whenever $\lambda_1 \geq n$, and this is proved by induction on the number of parts of λ. ■

Taking the inverse limit over r (in the category of graded rings), we obtain the following:

Corollary 3.3 *We have isomorphisms of graded rings*

$$\begin{aligned}\varprojlim_r \big(\mathbb{Z}[\xi_1, \ldots, \xi_r]/(\xi_1^n, \ldots, \xi_r^n)\big)^{\mathcal{S}_r} &= \Lambda^{(\xi)}/\big(m_\lambda(\xi)\big)_{\lambda_1\geq n} \\ &= \Lambda^{(\xi)}/\big(p_k(\xi)\big)_{k\geq n}.\end{aligned}$$

4 Proof of the main theorem

Given any variety X with basepoint p_0, Bott [Bo] considers a system of embeddings

$$X^{\times r} = X^{\times r} \times \{p_0\} \hookrightarrow X^{\times r+1}.$$

Assume T acts on X, fixing p_0, so these embeddings are T-equivariant. The symmetric group $\mathcal{S}_r$ acts on these products by permuting factors, and therefore on their (equivariant) cohomology rings. The inverse limit is written

$$\mathcal{S}H_T^* X := \varprojlim_r \big(H_T^* X^{\times r}\big)^{\mathcal{S}_r}. \tag{4.1}$$

We further assume $H_T^* X$ is free over $\mathbb{Z}[y] = H_T^*(\mathrm{pt})$, and has no odd cohomology. Then $H_T^* X^{\times r} = H_T^* X \otimes_{\mathbb{Z}[y]} \cdots \otimes_{\mathbb{Z}[y]} H_T^* X$ (r factors). In this case, given any T-equivariant morphism $f\colon X \to \widetilde{\mathrm{Gr}}_n$, there is a pullback homomorphism

$$H_T^* \widetilde{\mathrm{Gr}}_n \to H_T^* X \otimes_{\mathbb{Z}[y]} \cdots \otimes_{\mathbb{Z}[y]} H_T^* X,$$

obtained by factoring through the r-fold coproduct on $H_T^* \widetilde{\mathrm{Gr}}_n$. Since the coproduct is commutative, the image lies in the $\mathcal{S}_r$-invariant part of the tensor product. Taking

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the limit over r produces a homomorphism

$$f^*: H_T^*\widetilde{\mathrm{Gr}}_n \to \mathcal{S}H_T^* X.$$

For X, we take projective space $\mathbb{P}(V_{[0,n-1]}) \cong \mathbb{P}^{n-1}$, with basepoint p_0 corresponding to the line $\mathbb{C}\cdot \mathrm{e}_0 \subset V_{[0,n-1]}$, which is scaled by the character $y_0 = y_n$. (Recall that we treat indices of y_i modulo n.)

Let $H = \mathbb{P}(V_{[1,n-1]}) \subset \mathbb{P}(V_{[0,n-1]}) = \mathbb{P}^{n-1}$ be the hyperplane defined by $e_0^* = 0$, and let $\xi = [H]$ be its class in $H_T^2\mathbb{P}^{n-1}$. So $\xi = c_1^T(\mathcal{O}(1)) + y_0$, where $\mathcal{O}(1)$ is the dual of the tautological bundle on $\mathbb{P}^{n-1}$. The equivariant cohomology ring of $\mathbb{P}^{n-1}$ has a well-known presentation, which in our notation takes the form

$$H_T^*\mathbb{P}^{n-1} = \mathbb{Z}[y][\xi]/\big(\xi(\xi + y_1 - y_0)\cdots(\xi + y_{n-1} - y_0)\big). \tag{4.2}$$

Written slightly differently, the defining relation is

$$\begin{aligned} \xi^n + \xi^{n-1} e_1(y_1 - y_0, \dots, y_n - y_0) + \cdots \\ + \xi\, e_{n-1}(y_1 - y_0, \dots, y_n - y_0) = 0, \end{aligned} \tag{4.3}$$

which one should compare with (3.5). Similarly, let $H_i \subset (\mathbb{P}^{n-1})^{\times r}$ be the hyperplane defined by $e_0^* = 0$ on the ith factor, and let $\xi_i = [H_i]$ be its class in $H_T^*(\mathbb{P}^{n-1})^{\times r}$, which has a presentation with one relation of the form (4.3) for each ξ_i. Taking symmetric invariants leads to the following calculation:

Lemma 4.1 *The ring $\mathcal{S}H_T^*\mathbb{P}^{n-1}$ is a free $\mathbb{Z}[y]$-algebra. Letting $\widetilde{p}_k(\xi|y)$ be the polynomials defined by (3.5), where $\widetilde{p}_k = p_k(\xi) = \xi_1^k + \xi_2^k + \cdots$, it has the presentation*

$$\mathcal{S}H_T^*\mathbb{P}^{n-1} = \Lambda^{(\xi)}[y]/\big(\widetilde{p}_k(\xi|y)\big)_{k\geq n}.$$

Proof The homomorphism $\Lambda^{(\xi)}[y] \to \mathcal{S}H_T^*\mathbb{P}^{n-1}$ is the limit of homomorphisms $\mathbb{Z}[\xi_1,\dots,\xi_r]^{\mathcal{S}_r} \to \big(H_T(\mathbb{P}^{n-1})^{\times r}\big)^{\mathcal{S}_r}$ defined by $\xi_i \mapsto [H_i]$. The relations $\widetilde{p}_k(\xi|y) = 0$ hold in $\mathcal{S}H_T^*\mathbb{P}^{n-1}$, because they symmetrize relations of the form (4.3), so there is a well-defined homomorphism modulo the ideal $\big(\widetilde{p}_k(\xi|y)\big)_{k\geq n}$. Modulo the y-variables, this reduces to the isomorphism described in Corollary 3.3. The assertion follows by graded Nakayama. ■

One embeds $\mathbb{P}(V_{[0,n-1]})$ in Gr by sending $L \subset V_{[0,n-1]}$ to $V_{<0} \oplus L \subset V$, and this embedding factors through $\widetilde{\mathrm{Gr}}_n$, all T-equivariantly. So we have homomorphisms

$$\Lambda[y] = H_T^*\mathrm{Gr} \to H_T^*\widetilde{\mathrm{Gr}}_n \xrightarrow{f^*} \mathcal{S}H_T^*\mathbb{P}^{n-1}. \tag{4.4}$$

The map $\Lambda[y] = H_T^*\mathrm{Gr} \to H_T^*\mathbb{P}^{n-1}$ sends the generating series

$$C(t) = c^T(V_{\leq 0} - \mathbb{S}) \quad \text{to} \quad c^T(\mathbb{C}\cdot \mathrm{e}_0 - \mathcal{O}(-1)) = \frac{1 + y_0 t}{1 - \xi t + y_0 t}.$$

The map $\Lambda[y] \to \mathcal{S}H_T^*\mathbb{P}^{n-1}$ is determined by the evaluation (3.8).

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Proposition 4.2 *The homomorphism $f^*\colon H_T^*\widetilde{\mathrm{Gr}}_n \to \mathcal{S}H_T^*\mathbb{P}^{n-1}$ is an isomorphism of $\mathbb{Z}[y]$-algebras. In particular, we have*

$$H_T^*\widetilde{\mathrm{Gr}}_n = \Lambda[y]/\big(\widetilde{p}_k(h|y)\big)_{k\geq n}.$$

Proof The affine Grassmannian has a T-invariant Schubert cell decomposition, with finitely many cells in each dimension, so $H_T^*\widetilde{\mathrm{Gr}}_n$ is a free $\mathbb{Z}[y]$-module. It follows that the non-equivariant cohomology is recovered by setting y-variables to 0: we have an isomorphism $(H_T^*\widetilde{\mathrm{Gr}}_n)/(y) \cong H^*\widetilde{\mathrm{Gr}}_n$, and likewise $(\mathcal{S}H_T^*\mathbb{P}^{n-1})/(y) \cong \mathcal{S}H^*\mathbb{P}^{n-1}$. The induced map $H^*\widetilde{\mathrm{Gr}}_n \to \mathcal{S}H^*\mathbb{P}^{n-1}$ was shown to be an isomorphism by Bott [Bo, Proposition 8.1]. So the first statement of the proposition follows by another application of graded Nakayama. The second statement is a combination of the first with the presentation of $\mathcal{S}H_T^*\mathbb{P}^{n-1}$ from Lemma 4.1. ■

The $d = 0$ case of the main theorem follows from Proposition 4.2 together with the equality of ideals $\big(\widetilde{p}_k(h|y)\big)_{k\geq n} = \big(p_k(c|y)\big)_{k\geq n}$ established in Lemma 3.1.

For the general d case, we use the shift morphism sh^d, which defines isomorphisms

$$\begin{array}{ccc} \mathrm{Gr}^d & \xrightarrow{\sim} & \mathrm{Gr} \\ \uparrow & & \uparrow \\ \widetilde{\mathrm{Gr}}_n^d & \xrightarrow{\sim} & \widetilde{\mathrm{Gr}}_n. \end{array}$$

These are equivariant with respect to the corresponding automorphism of T which cyclically permutes coordinates. The action on cohomology rings is given by the homomorphism γ^d, as described in Section 2.2. The presentation of $H_T^*\widetilde{\mathrm{Gr}}_n$ is mapped to

$$H_T^*\widetilde{\mathrm{Gr}}_n^d = \Lambda[y]/\big(\gamma^d p_k(c|y)\big)_{k\geq n},$$

where now $\Lambda = \mathbb{Z}[c_1^{(d)}, c_2^{(d)}, \ldots]$, and the variables map by $c_k^{(d)} = c_k^T(V_{\leq 0} - \mathbb{S}_d)$. It remains to express $\gamma^d p_k(c|y)$ in terms of the polynomials $p_k(c^{(d)}|y)$.

Since $(\mathrm{sh}^d)^* c_k^T(V_{\leq 0} - \mathbb{S}_0) = c_k^T(V_{\leq d} - \mathbb{S}_d)$, we have

$$\gamma^d(C(t)) = C^{(d)}(t)\cdot(1+y_1t)\cdots(1+y_dt),$$

where $C^{(d)}(t) = \sum_{k\geq 0} c_k^{(d)} t^k$ is the generating series. So, using notation from Section 3, we have

$$\begin{aligned} \gamma^d \boldsymbol{P}(t) &= \big(\gamma^d P(t)\big)\cdot\big(\gamma^d E(t)\big) \\ &= \left(\frac{d}{dt}\log \gamma^d C(t)\right)\cdot E(t) \\ &= \frac{d}{dt}\log\left(C^{(d)}(t)\prod_{i=1}^d(1+y_it)\right)\cdot E(t) \\ &= P^{(d)}(t)\cdot E(t) + \sum_{i=1}^d y_i(1+y_1t)\cdots\widehat{(1+y_it)}\cdots(1+y_nt), \end{aligned}$$

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where $P^{(d)}(t) = \sum_{k\geq 1} p_k(c^{(d)}|y)t^{k-1}$. Extracting the coefficients of t^{k-1}, we find

$$y^d p_n(c|y) = p_n(c^{(d)}|y) + d \cdot e_n(y_1, \ldots, y_n)$$

and

$$y^d p_k(c|y) = p_k(c^{(d)}|y)$$

for $k > n$, as claimed.

Remark 4.3 Consider the $\mathbb{Z}[y]$-algebra automorphism of $\Lambda[y]$ defined by sending $p_k(c)$ to $p_k(c) - (-1)^k p_k(y)$, where $p_k(y) = y_1^k + \cdots + y_n^k$. Using [Mac, (2.11')], this sends

$$p_k(c|y) \mapsto p_k(c|y) + k\, e_k(y).$$

So we have an isomorphism of $\mathbb{Z}[y]$-algebras $\Lambda[y]/I_n^d \xrightarrow{\sim} \Lambda[y]/I_n^{d+n}$.

5 Double monomial symmetric functions

The monomial symmetric functions $m_\lambda(\xi)$, with $\lambda_1 < n$, form a basis for $\mathcal{S}H_T^*\mathbb{P}^{n-1}$ over $\mathbb{Z}[y]$ – so they also form a basis for $H_T^*\widetilde{\mathrm{Gr}}_n$. (This follows from the arguments above, and it is also easy to see directly from the fact that $1, \xi, \ldots, \xi^{n-1}$ forms a basis for $H_T^*\mathbb{P}^{n-1}$ over $\mathbb{Z}[y]$.) It is useful to work with a deformation of this basis of $\Lambda[y]$, which extends a basis for the defining ideal of $H_T^*\widetilde{\mathrm{Gr}}_n$.

For the general definition, we use variables $a_1, a_2, \ldots$ in degree 2. Given a sequence $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive integers, let $n_i(\alpha)$ be the number of occurrences of i in α, and set $n(\alpha) := n_1(\alpha)!n_2(\alpha)!\cdots$. (So $n(\alpha)$ is the number of permutations fixing α.) For a partition λ with r parts, so $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, we write $\alpha \subset \lambda$ to mean $\alpha_i \leq \lambda_i$ for all i. Let

$$e_{\lambda-\alpha}(a) = e_{\lambda_1-\alpha_1}(a_1, \ldots, a_{\lambda_1 - 1})\cdots e_{\lambda_r-\alpha_r}(a_1, \ldots, a_{\lambda_r - 1}),$$

where e_k is the elementary symmetric polynomial.

Definition 5.1 The *double monomial symmetric function* is

$$m_\lambda(\xi|a) = \sum_{(1^r)\subset\alpha\subset\lambda} \frac{n(\alpha)}{n(\lambda)} e_{\lambda-\alpha}(a)\, m_\alpha(\xi),$$

an element of $\Lambda^{(\xi)}[a_1, a_2, \ldots]$.

For a given α, the coefficient $n(\alpha)/n(\lambda)$ need not be an integer, but in the sum over all α, the coefficients are integers. In fact, $m_\lambda(\xi|a)$ is the symmetrization of the "monomial"

$$(\xi|a)^\lambda = \prod_{i=1}^{r} \xi_i(\xi_i + a_1)\cdots(\xi_i + a_{\lambda_i - 1}) = \sum_{(1^r)\subset\alpha\subset\lambda} e_{\lambda-\alpha}(a)\, \xi^\alpha, \tag{5.1}$$

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i.e., it is the sum of $\sigma\big((\xi|a)^\lambda\big)$ over all distinct permutations σ of λ, where σ acts in the usual way by permuting the ξ variables.

For instance, the functions corresponding to λ with a single row are

$$m_k(\xi|a) = m_k(\xi) + e_1(a_1,\dots,a_{k-1})\, m_{k-1}(\xi) + \cdots + e_{k-1}(a_1,\dots,a_{k-1})\, m_1(\xi).$$

Other examples are

$$\begin{aligned} m_{21}(\xi|a) &= m_{21}(\xi) + 2a_1\, m_{11}(\xi), \\ m_{22}(\xi|a) &= m_{22}(\xi) + a_1\, m_{21}(\xi) + a_1^2\, m_{11}(\xi), \\ m_{31}(\xi|a) &= m_{31}(\xi) + (a_1+a_2)\, m_{21}(\xi) + 2a_1a_2\, m_{11}(\xi), \\ m_{32}(\xi|a) &= m_{32}(\xi) + 2(a_1+a_2)\, m_{22}(\xi) + a_1\, m_{31}(\xi) \\ &\qquad + a_1(a_1+2a_2)\, m_{21} + 2a_1^2a_2\, m_{11}(\xi). \end{aligned}$$

From now on, we evaluate the a variables as $a_i = y_i - y_0$, with the indices taken mod n as usual. In the single-row case, this recovers the double power sum function defined by (3.5) in Section 3 above: $m_k(\xi|a) = \widetilde{p}_k(\xi|y)$.

We use the isomorphism $\Lambda^{(\xi)}[y] \cong \Lambda[y]$ from (3.8) to identify the functions $m_\lambda(\xi|y)$ in $\Lambda^{(\xi)}[y]$ with elements $m_\lambda(c|y)$ in $\Lambda[y]$, also called double monomial functions.

Proposition 5.2 *The double monomial functions $m_\lambda(c|y)$ form a $\mathbb{Z}[y]$-linear basis for $\Lambda[y]$. The $m_\lambda(c|y)$ with $\lambda_1 \geq n$ form a $\mathbb{Z}[y]$-linear basis for the ideal $I_n \subset \Lambda[y]$, the kernel of the surjective homomorphism $\Lambda[y] = H_T^*\mathrm{Gr} \to H_T^*\widetilde{\mathrm{Gr}}_n$.*

The ideal here is $I_n = I_n^0$, in the notation of the main Theorem from the introduction. In particular, Proposition 5.2 implies that every class in $H_T^*\widetilde{\mathrm{Gr}}_n$ has a canonical lift to a polynomial in $\Lambda[y]$, by taking an expansion in the monomial basis as a normal form, using only those $m_\lambda(c|y)$ with $\lambda_1 < n$.

Proof The first statement is proved by setting $y = 0$, since the monomial functions m_λ form a basis for Λ. For the second statement, it suffices to check that each $m_\lambda(c|y)$ lies in the ideal. This follows from the characterization of $m_\lambda(\xi|a)$ as the symmetrization of the monomial $(\xi|a)^\lambda$ defined in (5.1). Indeed, after setting $\xi_i = [H_i]$ and $a_i = y_i - y_0$, as in Section 4, each $(\xi|a)^\lambda$ with $\lambda_1 \geq n$ lies in the ideal defining $H_T^*(\mathbb{P}^{n-1})^{\times r}$, so the symmetrization lies in the defining ideal of $\mathcal{S}H_T^*\mathbb{P}^{n-1}$. ■

Remark 5.3 Up to sign and reindexing variables, the single-row functions $m_k(\xi|a)$ nearly agree with the functions $\widetilde{m}_k(x\|a)$ in [LS, Section 4.5]. To make the identification, use an isomorphism of our $\Lambda^{(\xi)}[a]$ with their $\Lambda(x\|a)$ which sends $m_k(\xi) \mapsto m_k[x - a_{>0}]$ and $a_i \mapsto -a_{1-i}$. Then the image of our $m_k(\xi|a)$ is the result of setting $a_1 = 0$ in $\widetilde{m}_k(x\|a)$. In general, however, the double monomial functions defined here differ from those of [LS], which are more analogous to power-sum functions. For instance, the latter are a basis only over $\mathbb{Q}[a]$.

The $m_\lambda(\xi|a)$ are closer to the double monomial functions $m_\lambda(x\|a)$ introduced by Molev [M, Section 5], which are defined non-explicitly via Hopf algebra duality, but do form a basis over $\mathbb{Z}[a]$. They are not quite identical, as can be seen from the

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table in [LS, Section 8.1], but in small examples the image of our $m_\lambda(\xi|a)$ under the substitution $a_i \mapsto -a_{1-i}$ agrees with the result of setting $a_1 = 0$ in Molev's function $m_\lambda(x\|a)$. It would be interesting to know if this pattern persists.

6 Moduli of vector bundles

The affine Grassmannian $\widetilde{\mathrm{Gr}}_n^d$ is homotopy-equivalent to the moduli stack parameterizing rank-n, degree d vector bundles on $\mathbb{P}^1$ together with a trivialization at ∞. Forgetting the trivialization identifies the moduli stack of vector bundles on $\mathbb{P}^1$ with the quotient stack $[GL_n\backslash\widetilde{\mathrm{Gr}}_n^d]$. (See, e.g., [La] for constructions of the moduli stacks, as well as further references, and [Z, Section 4] for a careful exposition of the relation between moduli of bundles and affine Grassmannians.)

Larson gave an algebraic description of the Chow ring of the moduli stack $\mathcal{B}_{n,d}^\dagger$ of rank n, degree d vector bundles on $\mathbb{P}^1$, as a certain subring of a polynomial ring [La]. In our context, the Chow and singular cohomology rings are isomorphic, and it follows from the above considerations that this ring must be isomorphic to the equivariant cohomology ring $H^*_{GL_n}\widetilde{\mathrm{Gr}}_n^d$. Here we will show that Larson's description is equivalent to the presentation given above in Corollary A, using some basic identities of symmetric functions.

Consider the polynomial ring $\mathbb{Q}[e_1,\dots,e_n,q_1,\dots,q_{n-1}]$, with e_i and q_i in degree $2i$. Larson shows that $H^*\mathcal{B}_{n,d}^\dagger = H^*_{GL_n}\widetilde{\mathrm{Gr}}_n^d$ is isomorphic to the subring generated over $\mathbb{Z}[e_1,\dots,e_n]$ by the coefficients of a series $\overline{C}(t) = \sum_{k\geq 0}\overline{c}_k\,t^k$, defined by

$$\exp\left(\int \frac{-d(e_1+e_2t+\cdots+e_nt^{n-1})+(q_1+q_2\,t+\cdots+q_{n-1}\,t^{n-2})}{1+e_1t+\cdots+e_nt^n}\,dt\right). \tag{6.1}$$

(To compare with Larson's notation, our $\overline{c}_i$ is her e_i, our e_i is her a_i, and our q_i is her $-a'_{i+1}$.)

Proposition 6.1 *The ideal J_n^d is the kernel of the $\mathbb{Z}[e_1,\dots,e_n]$-algebra homomorphism $\Lambda[e_1,\dots,e_n]\to\mathbb{Q}[e_1,\dots,e_n,q_1,\dots,q_{n-1}]$ which sends c_k to $\overline{c}_k$. In particular, the $\mathbb{Z}[e_1,\dots,e_n]$-subalgebra of $\mathbb{Q}[e_1,\dots,e_n,q_1,\dots,q_{n-1}]$ generated by the $\overline{c}_k$ is isomorphic to $\Lambda[e_1,\dots,e_n]/J_n^d\cong H^*_{GL_n}\widetilde{\mathrm{Gr}}_n^d$.*

Proof Consider a generating series

$$Q(t)=\sum_{k>0}q_k\,t^{k-1},$$

along with

$$C(t)=\exp\left(\int\frac{-d(e_1+e_2t+\cdots+e_nt^{n-1})+Q(t)}{E(t)}\,dt\right), \tag{6.2}$$

where $E(t)=\sum_{k=0}^n e_kt^k$ as usual. The coefficients c_k are algebraically independent, so this formula defines an embedding $\Lambda[e_1,\dots,e_n]\hookrightarrow\mathbb{Q}[e_1,\dots,e_n,q_1,q_2,\dots]$. The

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elements $\overline{c}_k$ defined by (6.1) are the images of c_k under the projection

$$\mathbb{Q}[e_1, \dots, e_n, q_1, q_2, \dots] \to \mathbb{Q}[e_1, \dots, e_n, q_1, \dots, q_{n-1}]$$

which sets q_k to 0 for $k \geq n$. So it suffices to identify these q_k with the generators of J_n^d.

Rewriting the expression (6.2), we find

$$t\,Q(t) = t\,P(t)\,E(t) + d\big(E(t) - 1\big),$$

where the series $P(t) = \frac{d}{dt} \log C(t)$ is determined by the Newton relations, in the form given in (3.2). Extracting the coefficient of t^k, we see $q_k = p_k(c|e) + d\,e_k$ for all $k \geq 1$. In particular, $q_n = p_n(c|e) + d\,e_n$, and $q_k = p_k(c|e)$ for $k > n$. ∎

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References

[A] D. Anderson, *Infinite flags and Schubert polynomials*. Preprint, 2023. https://arxiv.org/abs/2105.11404

[AF] D. Anderson and W. Fulton, *Equivariant cohomology in algebraic geometry*, Cambridge University Press, Cambridge, 2024.

[Bo] R. Bott, *The space of loops on a Lie group*. Michigan Math. J. 5(1958), 35–61.

[LLS] T. Lam, S. Lee, and M. Shimozono, *Back stable Schubert calculus*. Compos. Math. **157**(2021), 883–962.

[LS] T. Lam and M. Shimozono, *k-double Schur functions and equivariant (co)homology of the affine Grassmannian*. Math. Ann. **356**(2013), no. 4, 1379–1404.

[La] H. K. Larson, *The intersection theory of the moduli stack of vector bundles on* $\mathbb{P}^1$. Canad. Math. Bull. **66**(2023), 359–379.

[Mac] I. G. Macdonald, *Symmetric functions and hall polynomials*, 2nd ed., Oxford University Press, Oxford, 1995.

[M] A. I. Molev, *Comultiplication rules for the double Schur functions and Cauchy identities*. Electron. J. Comb. **16**(2009), #R13.

[PS] A. Pressley and G. Segal, *Loop groups*, Oxford University Press, Oxford, 1986.

[SS] M. Sato and Y. Sato, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold*. In: *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, North-Holland, Amsterdam, 1983, pp. 259–271.

[Y] Z. Yun, *Goresky–MacPherson calculus for the affine flag varieties*. Canad. J. Math. **62**(2010), no. 2, 473–480.

[YZ] Z. Yun and X. Zhu, *Integral homology of loop groups via Langlands dual groups*. Represent. Theory **15**(2011), 347–369.

[Z] X. Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*. In: Geometry of moduli spaces and representation theory, IAS/Park City Mathematics Series, 24, American Mathematical Society, Providence, RI, 2017, pp. 59–154.

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