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NON-COCOMMUTATIVE C*-BIALGEBRA DEFINED AS THE DIRECT SUM OF FREE GROUP C*-ALGEBRAS

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Abstract. Let $\mathbb{F}_n$ be the free group of rank n and let $\bigoplus C^*(\mathbb{F}_n)$ denote the direct sum of full group C*-algebras $C^*(\mathbb{F}_n)$ of $\mathbb{F}_n$ $(1 \le n < \infty)$. We introduce a new comultiplication Δ_φ on $\bigoplus C^*(\mathbb{F}_n)$ such that $(\bigoplus C^*(\mathbb{F}_n), \Delta_\varphi)$ is a non-cocommutative C*-bialgebra. With respect to Δ_φ, the tensor product $\pi \otimes_\varphi \pi'$ of any two representations π and π' of free groups is defined. The operation $\otimes_\varphi$ is associative and non-commutative. We compute its tensor product formulas of several representations.

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1. Introduction. A C*-bialgebra is a generalisation of bialgebra in the theory of C*-algebras, which was introduced in C*-algebraic framework for quantum groups [**15**, **18**]. For example, if G is a locally compact group, then the full group C*-algebra $C^*(G)$ of G is a cocommutative C*-bialgebra with respect to the standard (diagonal) comultiplication.

In this paper, a C*-bialgebra arising from certain group homomorphisms among free groups is given as follows: Let $\mathbb{F}_n$ denote the free group of rank n with free generators $g_1^{(n)}, \dots, g_n^{(n)}$. For $n, m \ge 1$, define the group homomorphism $\phi_{n,m}$ from $\mathbb{F}_{nm}$ to $\mathbb{F}_n \times \mathbb{F}_m$ by

$$\phi_{n,m}\big(g^{(nm)}_{m(i-1)+j}\big) := \big(g_i^{(n)}, g_j^{(m)}\big) \quad (i = 1, \dots, n,\ j = 1, \dots, m). \tag{1.1}$$

The map $\phi_{n,m}$ is well defined on the whole of $\mathbb{F}_{nm}$ by the universality of $\mathbb{F}_{nm}$. Then, the following diagram is commutative for each $n, m, l \ge 1$:

Figure 1.1.

$$\begin{tikzcd} & \mathbb{F}_n \times \mathbb{F}_{ml} \arrow[dr, "id_n \times \phi_{m,l}"] & \\ \mathbb{F}_{nml} \arrow[ur, "\phi_{n,ml}"] \arrow[dr, "\phi_{nm,l}"'] & & \mathbb{F}_n \times \mathbb{F}_m \times \mathbb{F}_l. \\ & \mathbb{F}_{nm} \times \mathbb{F}_l \arrow[ur, "\phi_{n,m} \times id_l"'] & \end{tikzcd}$$

Group homomorphisms in (1.1) can be lifted as $*$-homomorphisms $\varphi_{n,m}$ among full group C*-algebras and their minimal tensors. For $\{\varphi_{n,m}\}$, the following diagram is also commutative for each $n, m, l \ge 1$:

Figure 1.2.

$$\begin{tikzcd}[column sep=small]
 & C^*(\mathbb{F}_n)\otimes C^*(\mathbb{F}_{ml}) \arrow[dr, "id_n\otimes\varphi_{m,l}"] & \\
C^*(\mathbb{F}_{nml}) \arrow[ur, "\varphi_{n,ml}"] \arrow[dr, "\varphi_{nm,l}"'] & & C^*(\mathbb{F}_n)\otimes C^*(\mathbb{F}_m)\otimes C^*(\mathbb{F}_l). \\
 & C^*(\mathbb{F}_{nm})\otimes C^*(\mathbb{F}_l) \arrow[ur, "\varphi_{n,m}\otimes id_l"'] &
\end{tikzcd}$$

By using $\{\varphi_{n,m}\}$, we can construct a new comultiplication Δ_φ on the direct sum

$$\bigoplus C^*(\mathbb{F}_n) = C^*(\mathbb{F}_1)\oplus C^*(\mathbb{F}_2)\oplus C^*(\mathbb{F}_3)\oplus\cdots \tag{1.2}$$

for all finite-rank free groups $\{\mathbb{F}_n : 1\le n<\infty\}$ such that $(\bigoplus C^*(\mathbb{F}_n),\ \Delta_\varphi)$ is a non-cocommutative C*-bialgebra without antipode (Theorem 1.6).

For any two unitary representations of free groups, we can define the tensor product $\otimes_\varphi$ by using the comultiplication Δ_φ which is *not* commutative (Fact 3.1). Especially, the $\otimes_\varphi$-tensor product of any two quasi-regular representations is a direct sum of quasi-regular representations (Theorem 3.2):

$$\lambda_{\mathbb{F}_n/H'}\otimes_\varphi \lambda_{\mathbb{F}_m/H''}\cong\bigoplus_\mu \lambda_{\mathbb{F}_{nm}/H_\mu}. \tag{1.3}$$

In this section, we show our motivation, definitions and the main theorem.

1.1. Motivation. According to [**9**], given two representations of a group G, their tensor product is a new representation of G, which decomposes into a direct sum of indecomposable representations. The problem of finding this decomposition is called the *Clebsch–Gordan problem* and the resulting formula for the decomposition is called the *tensor product formula* (or *Clebsch–Gordan formula* [**9**]). A generalisation of the Clebsch–Gordan problem for groups is to consider modules over associative algebras instead of group algebras. However, there lies an obvious obstruction in that there is no known way to define the tensor product of two left modules over an arbitrary associative algebra. For group algebras, the extra structure coming from the group yields the tensor product. For a bialgebra A, the associative tensor product of representations (=special modules) of A can be defined by using the comultiplicatoin. Hence, one of most important motivations of the study of bialgebras is the tensor product of their representations.

We have studied a new kind of C*-bialgebras which are defined as direct sums of well-known C*-algebras, for example, Cuntz algebras, UHF algebras, matrix algebras [**12**] and Cuntz–Krieger algebras [**13**]. They are non-commutative and non-cocommutative, and there never exist antipodes on them. Such bialgebra structures do not appear before one takes direct sums. With respect to their comultiplications, new tensor products among representations of these C*-algebras and their tensor product formulas were obtained [**11**, **14**]. In [**12**], we gave a general method to construct a C*-bialgebra from a given system of C*-algebras and special $*$-homomorphisms among them. The essential part of this construction is how to construct such $*$-homomorphisms for each concrete example. One of our interests is to construct new examples of C*-bialgebra from various C*-algebras.

On the other hand, group C*-algebras are important examples of C*-algebras [**4**, **6**, **20**]. Furthermore, quantum groups in the C*-algebra approach are founded on the study of group C*-algebras [**15**, **18**].

Hence, we consider to construct a new C*-bialgebra associated with group C*-algebras by using a new comultiplication instead of their standard comultiplications. In this paper, we choose free group C*-algebras for this purpose, and try to construct a new comultiplication on them according to our method [**12**].

1.2. C*-bialgebra. In this subsection, we review terminology about C*-bialgebra according to [**7**, **15**, **18**]. For two C*-algebras A and B, we write $\mathrm{Hom}(A, B)$ as the set of all $*$-homomorphisms from A to B. We assume that every tensor product $\otimes$ as below means the minimal C*-tensor product.

Definition 1.3. A pair (A, Δ) is a *C*-bialgebra* if A is a C*-algebra and $\Delta \in \mathrm{Hom}(A, M(A \otimes A))$, where $M(A \otimes A)$ denotes the multiplier algebra of $A \otimes A$, such that the linear span of $\{\Delta(a)(b \otimes c) : a, b, c \in A\}$ is norm dense in $A \otimes A$ and the following holds:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \tag{1.4}$$

We call Δ the *comultiplication* of A.

We say that a C*-bialgebra (A, Δ) is *strictly proper* if $\Delta(a) \in A \otimes A$ for any $a \in A$; (A, Δ) is *unital* if A is unital and Δ is unital; (A, Δ) is *counital* if there exists $\varepsilon \in \mathrm{Hom}(A, \mathbb{C})$ such that

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta. \tag{1.5}$$

We call ε the *counit* of A and write (A, Δ, ε) as the counital C*-bialgebra (A, Δ) with the counit ε. Remark that Definition 1.3 does not mean $\Delta(A) \subset A \otimes A$. If A is unital, then (A, Δ) is strictly proper. A *bialgebra* in the purely algebraic theory [**1**, **10**] means a unital counital strictly proper bialgebra with the unital counit with respect to the algebraic tensor product, which does not need to have an involution. Hence, a C*-bialgebra is not a bialgebra in general. In Definition 1.3, if A is unital and Δ is unital, then the condition of the dense subspace in $A \otimes A$ can be omitted.

According to [**12**], we recall several notions of C*-bialgebra.

Definition 1.4.

(i) For two C*-bialgebras (A_1, Δ_1) and (A_2, Δ_2), f is a *C*-bialgebra morphism* from (A_1, Δ_1) to (A_2, Δ_2) if f is a non-degenerate $*$-homomorphism from A_1 to $M(A_2)$ such that $(f \otimes f) \circ \Delta_1 = \Delta_2 \circ f$. In addition, if $f(A_1) \subset A_2$, then f is called *strictly proper*.

(ii) A map f is a *C*-bialgebra endomorphism* of a C*-bialgebra (A, Δ) if f is a C*-bialgebra morphism from A to A. In addition, if $f(A) \subset A$ and f is bijective, then f is called a *C*-bialgebra automorphism* of (A, Δ).

(iii) A pair (B, Γ) is a *right comodule-C*-algebra* of a C*-bialgebra (A, Δ) if B is a C*-algebra and Γ is a non-degenerate $*$-homomorphism from B to $M(B \otimes A)$

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such that the following holds:

$$(\Gamma \otimes id) \circ \Gamma = (id \otimes \Delta) \circ \Gamma, \tag{1.6}$$

where both $\Gamma \otimes id$ and $id \otimes \Delta$ are extended to unital $*$-homomorphisms from $M(B \otimes A)$ to $M(B \otimes A \otimes A)$. The map Γ is called the *right coaction* of A on B.

(iv) A proper C*-bialgebra (A, Δ) satisfies the *cancellation law* if $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ are dense in $A \otimes A$ where $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ denote the linear spans of sets $\{\Delta(a)(I \otimes b) : a, b \in A\}$ and $\{\Delta(a)(b \otimes I) : a, b \in A\}$, respectively.

Let $(B, m, \eta, \Delta, \varepsilon)$ be a bialgebra in the purely algebraic theory, where m is a multiplication and η is a unit of the algebra B. An endomorphism S of B is called an *antipode* for $(B, m, \eta, \Delta, \varepsilon)$ if S satisfies $m \circ (id \otimes S) \circ \Delta = \eta \circ \varepsilon = m \circ (S \otimes id) \circ \Delta$ [**1**, **10**].

1.3. Free group algebras and homomorphisms among them. In this subsection, we briefly review free group C*-algebras [**4**, **6**], and introduce new homomorphisms among them in order to define a comultiplication.

For $n = \infty, 1, 2, 3, \ldots$, let $\mathbb{F}_n$ denote the free group of rank n where we use the symbol '∞' as the countable infinity for convenience in this paper. Let $(\mathcal{K}_n, \eta_n)$ denote a direct sum of all irreducible representations (up to unitary equivalence) of the Banach algebra $\ell^1(\mathbb{F}_n)$. Let $C^*(\mathbb{F}_n)$ denote the *full group C*-algebra* of $\mathbb{F}_n$, which is defined as the C*-algebra generated by the image of $\ell^1(\mathbb{F}_n)$ by η_n. Remark that $C^*(\mathbb{F}_1)$ is $*$-isomorphic to the C*-algebra $C(\mathbb{T})$ of all complex-valued continuous functions on the torus $\mathbb{T}$. With respect to the natural identification of the group algebra $\mathbb{C}\mathbb{F}_n$ over the coefficient field $\mathbb{C}$ with a subalgebra of $C^*(\mathbb{F}_n)$, $\mathbb{C}\mathbb{F}_n$ is dense in $C^*(\mathbb{F}_n)$. For $n = \infty, 1, 2, 3, \ldots$, let $\{g_i^{(n)}\}$ be the free generators of $\mathbb{F}_n$. We also identify $g_i^{(n)}$ with the unitary $\eta_n(g_i^{(n)})$ in $C^*(\mathbb{F}_n)$.

We introduce $*$-homomorphisms among $C^*(\mathbb{F}_n)$'s as follows.

LEMMA 1.5.

(i) *For $1 \le n, m < \infty$, define the map $\varphi_{n,m}$ from $C^*(\mathbb{F}_{nm})$ to the minimal tensor product $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$ by*

$$\varphi_{n,m}\big(g^{(nm)}_{m(i-1)+j}\big) := g_i^{(n)} \otimes g_j^{(m)} \quad (i = 1, \ldots, n,\ j = 1, \ldots, m). \tag{1.7}$$

Then, it is well defined on the whole of $C^*(\mathbb{F}_{nm})$ as a unital $*$-homomorphism.

(ii) *For $1 \le n < \infty$, define the map $\varphi_{\infty,n}$ from $C^*(\mathbb{F}_\infty)$ to the minimal tensor product $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_n)$ by*

$$\varphi_{\infty,n}(g^{(\infty)}_{n(i-1)+j}) := g_i^{(\infty)} \otimes g_j^{(n)} \quad (i \ge 1,\ j = 1, \ldots, n). \tag{1.8}$$

Then, it is well defined on the whole of $C^*(\mathbb{F}_\infty)$ as a unital $*$-homomorphism.

(iii) *If $n, m \ge 2$, then $\varphi_{n,m}$ is not injective.*

(iv) *Assume $n, m \ge 2$. Let $C_r^*(\mathbb{F}_n)$ denote the reduced group C*-algebra of $\mathbb{F}_n$, which is defined as the C*-algebra generated by the image of the left regular representation of $\mathbb{F}_n$. Then, the map $\varphi_{n,m}$ in (1.7) cannot be extended as a $*$-homomorphism from $C_r^*(\mathbb{F}_{nm})$ to $C_r^*(\mathbb{F}_n) \otimes C_r^*(\mathbb{F}_m)$.*

Especially, $\varphi_{1,1}$ equals the standard comultiplication of $C^*(\mathbb{F}_1)$. The proof of Lemma 1.5 will be given in Section 2.2.

1.4. Main theorem. In this subsection, we show our main theorem. Let $C^*(\mathbb{F}_n)$, $\{g_i^{(n)}\}_{i=1}^n$, $\mathbb{C}\mathbb{F}_n$, $\{\varphi_{n,m}\}_{n,m\geq 1}$ and $\{\varphi_{\infty,n}\}_{n\geq 1}$ be as in Section 1.3.

THEOREM 1.6. *Define the C*-algebra $\mathcal{A}$ as the direct sum*

$$\mathcal{A} := \bigoplus_{1\leq n<\infty} C^*(\mathbb{F}_n) \tag{1.9}$$

and define $\Delta_\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}\otimes\mathcal{A})$ and $\varepsilon \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ by

$$\Delta_\varphi(x) := \sum_{m,l;\, ml=n} \varphi_{m,l}(x) \quad \text{when } x \in C^*(\mathbb{F}_n), \tag{1.10}$$

$$\varepsilon := \varepsilon_1 \circ E_1 \tag{1.11}$$

where $\varepsilon_1 \in \mathrm{Hom}(C^*(\mathbb{F}_1), \mathbb{C})$ is defined as $\varepsilon_1|_{\mathbb{F}_1} = 1$, and E_1 is the projection from $\mathcal{A}$ onto $C^*(\mathbb{F}_1)$. Then the following holds:

(i) *The C*-algebra $\mathcal{A}$ is a strictly proper counital C*-bialgebra with the comultiplication Δ_φ and the counit ε.*
(ii) *The C*-bialgebra $(\mathcal{A}, \Delta_\varphi)$ satisfies the cancellation law.*
(iii) *By the smallest unitisation, $(\mathcal{A}, \Delta_\varphi, \varepsilon)$ can be extended to the unital counital C*-bialgebra $(\tilde{\mathcal{A}}, \hat{\Delta}_\varphi, \tilde{\varepsilon})$.*
(iv) *There never exists any antipode for any dense unital counital subbialgebra of $(\tilde{\mathcal{A}}, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ in (iii).*
(v) *Define the algebraic direct sum $\mathbb{C}\mathbb{F}_* := \oplus_{alg}\{\mathbb{C}\mathbb{F}_n : 1 \leq n < \infty\}$. Then, $\Delta_\varphi(\mathbb{C}\mathbb{F}_*) \subset \mathbb{C}\mathbb{F}_* \odot \mathbb{C}\mathbb{F}_*$ where $\odot$ means the algebraic tensor product, and $\mathbb{C}\mathbb{F}_*$ is identified with a $*$-subalgebra of $\mathcal{A}$ with respect to the canonical embedding.*
(vi) *Define $\Gamma_\varphi \in \mathrm{Hom}(C^*(\mathbb{F}_\infty), M(C^*(\mathbb{F}_\infty)\otimes\mathcal{A}))$ by*

$$\Gamma_\varphi(x) := \prod_{1\leq n<\infty} \varphi_{\infty,n}(x) \quad (x \in C^*(\mathbb{F}_\infty)), \tag{1.12}$$

where we identify the multiplier $M(C^*(\mathbb{F}_\infty)\otimes\mathcal{A})$ with the direct product $\prod_{n\geq 1} C^*(\mathbb{F}_\infty)\otimes C^*(\mathbb{F}_n)$. Then, $C^*(\mathbb{F}_\infty)$ is a right comodule-C-algebra of $(\mathcal{A}, \Delta_\varphi)$ with respect to the coaction Γ_φ.*

REMARK 1.7.
(i) The R.H.S. in (1.10) is always a finite sum when $x \in C^*(\mathbb{F}_n)$.
(ii) The C*-bialgebra $(\mathcal{A}, \Delta_\varphi)$ is non-cocommutative. In fact, the following holds:

$$\Delta_\varphi(g_2^{(6)}) = g_1^{(1)}\otimes g_2^{(6)} + g_1^{(2)}\otimes g_2^{(3)} + g_1^{(3)}\otimes g_2^{(2)} + g_2^{(6)}\otimes g_1^{(1)}. \tag{1.13}$$

(iii) In (1.9), every free group C*-algebras $C^*(\mathbb{F}_n)$ $(1 \leq n < \infty)$ appear at once. This is an essentially new structure of the class of free group C*-algebras. On the other hand, $C^*(\mathbb{F}_\infty)$ appears as a comodule-C*-algebra of $(\mathcal{A}, \Delta_\varphi)$. This shows a certain naturality of this bialgebra structure.

(iv) From Theorem 1.6(iv), the C*-bialgebra $(\mathcal{A}, \Delta_\varphi)$ is not a locally compact quantum group in the sense of Kustermans–Vaes [**15**] and Masuda–Nakagami–Woronowicz [**18**] because any locally compact quantum group has an antipode ([**15**], p550).

In Section 2, we prove Theorem 1.6. In Section 3, we show tensor product formulas of representations of $\mathbb{F}_n$'s with respect to Δ_φ, and show some C*-bialgebra automorphisms.

2. Proofs of theorems. In this section, we prove Lemma 1.5 and Theorem 1.6.

2.1. C*-weakly coassociative system. According to Section 3 in [**12**], we recall a general method to construct a C*-bialgebra from a set of C*-algebras and ∗-homomorphisms among them. A *monoid* is a set M equipped with a binary associative operation $\mathsf{M} \times \mathsf{M} \ni (a, b) \mapsto ab \in \mathsf{M}$, and a unit with respect to the operation. For example, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is an abelian monoid with respect to the multiplication. In order to show Theorem 1.6, we give a new definition of C*-weakly coassociative system which is a generalisation of Definition 3.1 of [**12**].

DEFINITION 2.1. Let M be a monoid with the unit e. A data $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$ is a *C*-weakly coassociative system (= C*-WCS)* over M if A_a is a unital C*-algebra for $a \in \mathsf{M}$ and $\varphi_{a,b}$ is a unital ∗-homomorphism from A_{ab} to $A_a \otimes A_b$ for $a, b \in \mathsf{M}$ such that

(i) for all $a, b, c \in \mathsf{M}$, the following holds:

$$(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,bc} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c}, \tag{2.1}$$

where id_x denotes the identity map on A_x for $x = a, c$,

(ii) there exists a counit ε_e of A_e such that $(A_e, \varphi_{e,e}, \varepsilon_e)$ is a counital C*-bialgebra,

(iii) for each $a \in \mathsf{M}$, the following holds:

$$(\varepsilon_e \otimes id_a) \circ \varphi_{e,a} = id_a = (id_a \otimes \varepsilon_e) \circ \varphi_{a,e}. \tag{2.2}$$

The condition (2.2) is weaker than the older, '$\varphi_{e,a}(x) = I_e \otimes x$ and $\varphi_{a,e}(x) = x \otimes I_e$ for $x \in A_a$ and $a \in \mathsf{M}$' ([**12**], Definition 3.1). In fact, the older definition satisfies (2.2). From the new definition, the same result holds as follows.

THEOREM 2.2 ([**12**], Theorem 3.1). *Let* $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$ *be a C*-WCS over a monoid* M. *Assume that* M *satisfies that*

$$\#\mathcal{N}_a < \infty \text{ for each } a \in \mathsf{M} \tag{2.3}$$

where $\mathcal{N}_a := \{(b, c) \in \mathsf{M} \times \mathsf{M} : bc = a\}$. *Define C*-algebras*

$$A_* := \oplus\{A_a : a \in \mathsf{M}\}, \quad C_a := \oplus\{A_b \otimes A_c : (b, c) \in \mathcal{N}_a\} \quad (a \in \mathsf{M}).$$

Define $\Delta_\varphi^{(a)} \in \mathrm{Hom}(A_a, C_a)$*,* $\Delta_\varphi \in \mathrm{Hom}(A_*, A_* \otimes A_*)$ *and* $\varepsilon \in \mathrm{Hom}(A_*, \mathbb{C})$ *by*

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$$\Delta_\varphi^{(a)}(x) := \sum_{(b,c)\in\mathcal{N}_a} \varphi_{b,c}(x) \quad (x \in A_a), \qquad \Delta_\varphi := \oplus\{\Delta_\varphi^{(a)} : a \in \mathsf{M}\},$$
$$\varepsilon := \varepsilon_e \circ E_e \tag{2.4}$$

where E_e denotes the projection from A_* onto A_e. Then $(A_*, \Delta_\varphi, \varepsilon)$ is a strictly proper counital C^*-bialgebra.

Proof. By (2.3), $\Delta_\varphi^{(a)}$ is well defined. Furthermore, C_a is unital and $\Delta_\varphi^{(a)}$ is unital for each a. Since $\mathsf{M} \times \mathsf{M} = \coprod_{a\in\mathsf{M}} \mathcal{N}_a$, $A_* \otimes A_* = \oplus\{A_f \otimes A_g : f, g \in \mathsf{M}\} = \oplus\{C_a : a \in \mathsf{M}\}$. Since $\Delta_\varphi^{(a)}$ is unital for each a, Δ_φ is non-degenerate. From (2.1), the following holds for $x \in A_a$:

$$\begin{aligned}\{(\Delta_\varphi \otimes id) \circ \Delta_\varphi\}(x) &= \textstyle\sum_{b,c,d\in\mathsf{M},\, bcd=a}(\varphi_{b,c} \otimes id_d)(\varphi_{bc,d}(x))\\ &= \textstyle\sum_{b,c,d\in\mathsf{M},\, bcd=a}(id_b \otimes \varphi_{c,d})(\varphi_{b,cd}(x))\\ &= \{(id \otimes \Delta_\varphi) \circ \Delta_\varphi\}(x).\end{aligned} \tag{2.5}$$

Hence, $(\Delta_\varphi \otimes id) \circ \Delta_\varphi = (id \otimes \Delta_\varphi) \circ \Delta_\varphi$ on A_*. Therefore, Δ_φ is a comultiplication of A_*. On the other hand, for $x \in A_a$, we see that

$$\begin{aligned}\{(\varepsilon \otimes id) \circ \Delta_\varphi\}(x) &= (\varepsilon \otimes id)(\Delta_\varphi^{(a)}(x))\\ &= \textstyle\sum_{(b,c)\in\mathcal{N}_a}(\varepsilon \otimes id)(\varphi_{b,c}(x))\\ &= (\varepsilon_e \otimes id_a)(\varphi_{e,a}(x))\\ &= x \qquad \text{(from (2.2))}.\end{aligned} \tag{2.6}$$

Hence, $(\varepsilon \otimes id) \circ \Delta_\varphi = id$. In like wise, we see that $(id \otimes \varepsilon) \circ \Delta_\varphi = id$. Therefore, ε is a counit of (A_*, Δ_φ). In consequence, we see that $(A_*, \Delta_\varphi, \varepsilon)$ is a counital C*-bialgebra. By definition, (A_*, Δ_φ) is strictly proper. □

We call $(A_*, \Delta_\varphi, \varepsilon)$ in Theorem 2.2 by a (counital) *C*-bialgebra* associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$.

The following lemma holds independently of the generalisation in Definition 2.1(iii).

LEMMA 2.3. *For the following C^*-WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$, we assume the condition (2.3).*

(i) *([**12**], Lemma 2.2). For a given strictly proper non-unital counital C^*-bialgebra (A, Δ, ε), let $\tilde{A} := A \oplus \mathbb{C}$ denote the smallest unitisation of A. Then there exist a unique extension $(\hat{\Delta}, \tilde{\varepsilon})$ of (Δ, ε) on $\tilde{A}$ such that $(\tilde{A}, \hat{\Delta}, \tilde{\varepsilon})$ is a strictly proper unital counital C^*-bialgebra.*

(ii) *([**12**], Lemma 3.2). For a C^*-WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$ over M, let $(A_*, \Delta_\varphi, \varepsilon)$ be as in Theorem 2.2 and let $(\tilde{A}_*, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ be the smallest unitisation of $(A_*, \Delta_\varphi, \varepsilon)$ in (i). Assume that any element in M has no left inverse except the unit e. Then the antipode for any dense unital counital subbialgebra of $(\tilde{A}_*, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ never exists.*

(iii) *([**12**], Lemma 3.1). Let $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$ be a C^*-WCS over a monoid M and let (A_*, Δ_φ) be as in Theorem 2.2 associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$. Define*

$$X_{a,b} := \varphi_{a,b}(A_{ab})(A_a \otimes I_b), \qquad Y_{a,b} := \varphi_{a,b}(A_{ab})(I_a \otimes A_b) \quad (a, b \in \mathsf{M}) \tag{2.7}$$

where $\varphi_{a,b}(A_{ab})(A_a \otimes I_b)$ and $\varphi_{a,b}(A_{ab})(I_a \otimes A_b)$ mean the linear spans of $\{\varphi_{a,b}(x)(y \otimes I_b) : x \in A_{ab},\ y \in A_a\}$ and $\{\varphi_{a,b}(x)(I_a \otimes y) : x \in A_{ab},\ y \in A_b\}$,

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respectively. If both $X_{a,b}$ and $Y_{a,b}$ are dense in $A_a \otimes A_b$ for each $a, b \in \mathsf{M}$, then (A_*, Δ_φ) satisfies the cancellation law.

(iv) *([12], Theorem 3.2). For a C^*-WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathsf{M}\}$ over a monoid M, assume that B is a unital C^*-algebra and a set $\{\varphi_{B,a} : a \in \mathsf{M}\}$ of unital $*$-homomorphisms such that $\varphi_{B,a} \in \mathrm{Hom}(B, B \otimes A_a)$ for each $a \in \mathsf{M}$ and the following holds:*

$$(\varphi_{B,a} \otimes id_b) \circ \varphi_{B,b} = (id_B \otimes \varphi_{a,b}) \circ \varphi_{B,ab} \quad (a, b \in \mathsf{M}). \tag{2.8}$$

Then, B is a right comodule-C^*-algebra of the C^*-bialgebra (A_*, Δ_φ) with the unital coaction $\Gamma_\varphi := \prod_{a \in \mathsf{M}} \varphi_{B,a}$.

2.2. Homomorphisms among free groups. In this subsection, we show properties of $\phi_{n,m}$ in (1.1) and prove Lemma 1.5.

LEMMA 2.4. *For each $n \geq 1$, we write 1 as the unit of $\mathbb{F}_n$.*

(i) *For any $x \in \mathbb{F}_n$, there exists $(y, z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x, y)$.*
(ii) *For any $y \in \mathbb{F}_m$, there exists $(x, z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x, y)$.*
(iii) *For any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(x', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(x', 1) = (x, y)$.*
(iv) *For any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(y', z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(1, y') = (x, y)$.*
(v) *When $n, m \geq 2$, $\phi_{n,m}$ is not injective.*

Proof.

(i) Let $a_1, \ldots, a_n$, $b_1, \ldots, b_m$, $c_1, \ldots, c_{nm}$ be the free generators of $\mathbb{F}_n, \mathbb{F}_m, \mathbb{F}_{nm}$, respectively. Assume that $x \in \mathbb{F}_n$ is written as a reduced word $x = a_{i_1}^{\varepsilon_1} \cdots a_{i_l}^{\varepsilon_l}$ where $\varepsilon_i = 1$ or -1 for $i = 1, \ldots, l$. For example, define $(y, z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ by $y := b_1^{\varepsilon_1} \cdots b_1^{\varepsilon_l}$ and $z := c_{m(i_1-1)+1}^{\varepsilon_1} \cdots c_{m(i_l-1)+1}^{\varepsilon_l}$. Then y belongs to the abelian subgroup generated by the single element b_1, and it is not always a reduced word in $\mathbb{F}_m$. Then the statement holds for (y, z).
(ii) As the proof of (i), this is proved.
(iii) From (ii), we can find $(x'', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x'', y)$. Define $x' := (x'')^{-1}x$, then the statement holds.
(iv) As the proof of (iii), this is proved from (i).
(v) Let $c_1, \ldots, c_{nm}$ be as in the proof of (i). For $i, l \in \{1, \ldots, n\}$, $k, j \in \{1, \ldots, m\}$, define $x(i, l; j, k) \in \mathbb{F}_{nm}$ by

$$x(i, l; j, k) := c_{m(i-1)+j}\, c_{m(i-1)+k}^{-1}\, c_{m(l-1)+k}\, c_{m(l-1)+j}^{-1}. \tag{2.9}$$

Then, $x(i, l; j, k) \neq 1$ when $k \neq j$, $i \neq l$, but $x(i, l; j, k) \in \ker \phi_{n,m}$ for any i, l, j, k. □

In the proof of Lemma 2.4(v), if $n = m = 2$, then the reduced word $c_1 c_2^{-1} c_4 c_3^{-1}$ in $\mathbb{F}_4$ satisfies $\phi_{2,2}(c_1 c_2^{-1} c_4 c_3^{-1}) = (1, 1)$.

Proof of Lemma 1.5

(i) Let $\phi_{n,m}$ be as in (1.1) and let $(\mathcal{K}_n, \eta_n)$ be as in Section 1.3. Define the unitary representation $\varphi_{n,m}^0$ of $\mathbb{F}_{nm}$ on $\mathcal{K}_n \otimes \mathcal{K}_m$ by $\varphi_{n,m}^0 := (\eta_n \otimes \eta_m) \circ \phi_{n,m}$. The representation $\varphi_{n,m}^0$ is well defined by the universality of $\mathbb{F}_{nm}$. Since the

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image of $\varphi^0_{n,m}$ is included in $C^*(\mathbb{F}_n)\otimes C^*(\mathbb{F}_m)$, $\varphi^0_{n,m}$ is uniquely extended to $\varphi_{n,m}$ in (1.7) such that $\varphi_{n,m}(\eta_{nm}(x))=\varphi^0_{n,m}(x)$ for each $x\in\mathbb{F}_{nm}$ ([**4**], Proposition 2.5.2). Hence the statement holds.

(ii) In analogy with (i), the statement holds.

(iii) For $x(i,l;j,k)$ in (2.9), we see that $\varphi_{n,m}(x(i,l;j,k)-1)=0$ for each i,l,j,k. Hence the statement holds.

(iv) If such an extension $\tilde{\varphi}_{n,m}$ of $\varphi_{n,m}$ exists, then $\tilde{\varphi}_{n,m}$ must be injective because $C^*_r(\mathbb{F}_{nm})$ is simple when $nm\geq 2$ [**21**]. On the other hand, $\tilde{\varphi}_{n,m}$ never be injective for $m,n\geq 2$ by (iii).

□

2.3. Proof of Theorem 1.6. We prove Theorem 1.6 in this subsection. Let $\mathbb{N}:=\{1,2,3,\ldots\}$. Remark that (2.3) holds for any element in the multiplicative monoid $(\mathbb{N},\cdot)$.

(i) From Theorem 2.2, it is sufficient to show that $\{(C^*(\mathbb{F}_n),\varphi_{n,m}):n,m\in\mathbb{N}\}$ is a C*-WCS over the monoid $\mathbb{N}$. By the definition of $\varphi_{n,m}$ in (1.7), we can verify that $(\varphi_{n,m}\otimes id_l)\circ\varphi_{nm,l}=(id_n\otimes\varphi_{m,l})\circ\varphi_{n,ml}$ for $n,m,l\in\mathbb{N}$ where id_a denotes the identity map on $C^*(\mathbb{F}_a)$ for $a=n,l$. Hence (2.1) is satisfied. On the other hand, since $\varepsilon_1|_{\mathbb{F}_1}=1$, $\{(\varepsilon_1\otimes id_n)\circ\varphi_{1,n}\}(g^{(n)}_j)=(\varepsilon_1\otimes id_n)(g^{(1)}_1\otimes g^{(n)}_j)=\varepsilon_1(g^{(1)}_1)\,g^{(n)}_j=g^{(n)}_j$ for each $j=1,\ldots,n$ and $n\in\mathbb{N}$. By the same token, we obtain $(id_n\otimes\varepsilon_1)\circ\varphi_{n,1}=id_n$. Hence (2.2) is verified. Therefore $\{(C^*(\mathbb{F}_n),\varphi_{n,m}):n,m\in\mathbb{N}\}$ is a C*-WCS over the monoid $\mathbb{N}$.

(ii) For $n,m\in\mathbb{N}$, define three subsets $\mathcal{P}_{n,m}$, $\mathcal{Q}_{n,m}$, $\mathcal{R}_{n,m}$ of $C^*(\mathbb{F}_n)\otimes C^*(\mathbb{F}_m)$ by

$$\mathcal{P}_{n,m}:=\{\varphi_{n,m}(z)(x\otimes I_m):x\in\mathbb{F}_n,\ z\in\mathbb{F}_{nm}\},\tag{2.10}$$

$$\mathcal{Q}_{n,m}:=\{\varphi_{n,m}(z)(I_n\otimes y):y\in\mathbb{F}_m,\ z\in\mathbb{F}_{nm}\},\tag{2.11}$$

$$\mathcal{R}_{n,m}:=\qquad\{x\otimes y:x\in\mathbb{F}_n,\ y\in\mathbb{F}_m\}.\tag{2.12}$$

Then their linear spans are dense subspaces of $\varphi_{n,m}(C^*(\mathbb{F}_{nm}))(C^*(\mathbb{F}_n)\otimes I_m)$, $\varphi_{n,m}(C^*(\mathbb{F}_{nm}))(I_n\otimes C^*(\mathbb{F}_m))$ and $C^*(\mathbb{F}_n)\otimes C^*(\mathbb{F}_m)$, respectively. From Lemma 2.4(iii), it is sufficient to show that $\mathcal{R}_{n,m}\subset\mathcal{P}_{n,m}$ and $\mathcal{R}_{n,m}\subset\mathcal{Q}_{n,m}$.

We prove $\mathcal{R}_{n,m}\subset\mathcal{P}_{n,m}$ as follows: For $(x,y)\in\mathbb{F}_n\times\mathbb{F}_m$, there exists $(x',z)\in\mathbb{F}_n\times\mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(x',1)=(x,y)$ from Lemma 2.4(iii). By definitions of $\phi_{n,m}$ and $\varphi_{n,m}$, this implies $\varphi_{n,m}(z)(x'\otimes I_m)=x\otimes y$. Therefore $\mathcal{R}_{n,m}\subset\mathcal{P}_{n,m}$. In a similar fashion, we obtain $\mathcal{R}_{n,m}\subset\mathcal{Q}_{n,m}$ from Lemma 2.4(iv). Hence the statement holds.

(iii) From (i) and Lemma 2.3(i), the statement holds.

(iv) Remark that the monoid $\mathbb{N}$ has no left invertible element except the unit 1. From the proof of (i) and Lemma 2.3(ii), the statement holds.

(v) From the proof of (i) and the definition of $\varphi_{n,m}$ in (1.7), we see that $\varphi_{n,m}(\mathbb{C}\mathbb{F}_{nm})\subset\mathbb{C}\mathbb{F}_n\odot\mathbb{C}\mathbb{F}_m$. This implies the statement.

(vi) By definition, we see that $(\varphi_{\infty,n}\otimes id_m)\circ\varphi_{\infty,m}=(id_\infty\otimes\varphi_{n,m})\circ\varphi_{\infty,nm}$ for $n,m\in\mathbb{N}$. From Lemma 2.3(iv) for $\varphi_{C^*(\mathbb{F}_\infty),n}:=\varphi_{\infty,n}$, the statement holds.

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3. Tensor product formulas of representations, and automorphisms. In this section, we show tensor product formulas of unitary representations of $\mathbb{F}_n$'s with respect to the comultiplication Δ_φ in Theorem 1.6, and C*-bialgebra automorphisms.

3.1. General facts about representations. We introduce a new tensor product of representations of $\mathbb{F}_n$'s and show tensor product formulas of quasi-regular representations.

3.1.1. Representations of $\mathbb{F}_n$. We identify $\mathbb{F}_n$ with the unitary subgroup of $C^*(\mathbb{F}_n)$ with respect to the canonical embedding. Let $\mathrm{Rep}_u\mathbb{F}_n$ denote the class of all unitary representations of $\mathbb{F}_n$. For $(\pi, \pi') \in \mathrm{Rep}_u\mathbb{F}_n \times \mathrm{Rep}_u\mathbb{F}_m$, define the new representation $\pi \otimes_\varphi \pi' \in \mathrm{Rep}_u\mathbb{F}_{nm}$ by

$$\pi \otimes_\varphi \pi' := (\pi \otimes \pi') \circ \varphi_{n,m}, \tag{3.1}$$

where $\varphi_{n,m}$ is as in (1.7). Then we see that the new operation $\otimes_\varphi$ is associative, and it is distributive with respect to the direct sum. Furthermore, $\otimes_\varphi$ is well defined on the unitary equivalence classes of representations. It will be shown that $\otimes_\varphi$ is non-commutative in Fact 3.1.

3.1.2. Tensor product formulas of one-dimensional unitary representations of free groups. In this subsection, we show tensor product formulas of one-dimensional unitary representations of free groups with respect to $\otimes_\varphi$ in (3.1) as a basic example of tensor product formula. For groups G and H, let $\mathrm{Hom}(G, H)$ denote the set of all homomorphisms from G to H. Define

$$\mathrm{Ch}(\mathbb{F}_n) := \mathrm{Hom}(\mathbb{F}_n, U(1)). \tag{3.2}$$

Since an element in $\mathrm{Ch}(\mathbb{F}_n)$ can be regarded as a one-dimensional unitary representation of $\mathbb{F}_n$, we can regard $\pi_1 \otimes_\varphi \pi_2$ as an element in $\mathrm{Ch}(\mathbb{F}_{nm})$ for $\pi_1 \in \mathrm{Ch}(\mathbb{F}_n)$ and $\pi_2 \in \mathrm{Ch}(\mathbb{F}_m)$.

Let $g_1, \ldots, g_n$ be free generators of $\mathbb{F}_n$. For $z = (z_1, \ldots, z_n) \in \mathbb{T}^n := U(1)^n$, define $\chi_z^{(n)} \in \mathrm{Ch}(\mathbb{F}_n)$ by

$$\chi_z^{(n)}(g_i) := z_i \quad (i = 1, \ldots, n). \tag{3.3}$$

Clearly, $\chi_z^{(n)}$ is irreducible for any $z \in \mathbb{T}^n$. For $z, w \in \mathbb{T}^n$, $\chi_z^{(n)}$ and $\chi_w^{(n)}$ are unitarily equivalent if and only if $z = w$. By the correspondence

$$\mathbb{T}^n \ni z \mapsto \chi_z^{(n)} \in \mathrm{Ch}(\mathbb{F}_n), \tag{3.4}$$

we see that $\mathrm{Ch}(\mathbb{F}_n)$ is equivalent to $\mathbb{T}^n$ as a set.

For $z \in \mathbb{T}^n$ and $w \in \mathbb{T}^m$, define the Kronecker product $z \boxtimes w \in \mathbb{T}^{nm}$ as $(z \boxtimes w)_{m(i-1)+j} := z_i w_j$ for $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. By definition, the following holds:

$$\chi_z^{(n)} \otimes_\varphi \chi_w^{(m)} = \chi_{z \boxtimes w}^{(nm)} \quad (z \in \mathbb{T}^n,\ w \in \mathbb{T}^m). \tag{3.5}$$

This implies that the correspondence in (3.4) gives a semigroup isomorphism from $(\bigcup_{n \geq 1} \mathrm{Ch}(\mathbb{F}_n),\ \otimes_\varphi)$ to $(\bigcup_{n \geq 1} \mathbb{T}^n,\ \boxtimes)$. This shows a naturality of the operation $\otimes_\varphi$.

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FACT 3.1. The operation $\otimes_\varphi$ in (3.1) is non-commutative as the following sense: There exist $n, m \geq 2$ and representations π_1, π_2 of $\mathbb{F}_n$ and $\mathbb{F}_m$, respectively such that $\pi_1 \otimes_\varphi \pi_2$ and $\pi_2 \otimes_\varphi \pi_1$ are not unitarily equivalent.

Proof. In (3.5), let $(n, m) = (2, 3)$ and $z = (1, -1) \in \mathbb{T}^2$ and $w = (1, 1, 1) \in \mathbb{T}^3$. Then $\chi_z^{(2)} \otimes_\varphi \chi_w^{(3)} = \chi_{z\boxtimes w}^{(6)} \not\cong \chi_{w\boxtimes z}^{(6)} = \chi_w^{(3)} \otimes_\varphi \chi_z^{(2)}$ because $z \boxtimes w = (1, 1, 1, -1, -1, -1)$ and $w \boxtimes z = (1, -1, 1, -1, 1, -1)$. □

3.1.3. Quasi-regular representations. In this subsection, we review quasi-regular representations of discrete groups, and show the general formula of the $\otimes_\varphi$-tensor product of quasi-regular representations of free groups.

For a discrete group Γ and a subgroup Γ_0, let Γ/Γ_0 denote the left coset space, that is, $\Gamma/\Gamma_0 := \{x\Gamma_0 : x \in \Gamma\}$. Define the *quasi-regular representation* **[8]** (or *permutation representation* **[16]**) $(\ell^2(\Gamma/\Gamma_0), \lambda_{\Gamma/\Gamma_0})$ of Γ associated with Γ_0 as the natural (unitary) left action of Γ on the standard basis of $\ell^2(\Gamma/\Gamma_0)$:

$$\lambda_{\Gamma/\Gamma_0} : \Gamma \curvearrowright \ell^2(\Gamma/\Gamma_0). \tag{3.6}$$

Especially, the regular representation λ_Γ and the trivial representation $\mathbf{1}$ of Γ are quasi-regular representations associated with subgroups $\{e\}$ and Γ of Γ, respectively. For the trivial representation $\mathbf{1}_{\Gamma_0}$ of Γ_0, $\lambda_{\Gamma/\Gamma_0}$ coincides with the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{1}_{\Gamma_0})$.

Since Γ/Γ_0 is a Γ-homogeneous space, $\lambda_{\Gamma/\Gamma_0}$ is a cyclic representation. When Γ acts on a set X, the permutation representation of Γ on $\ell^2(X)$ are decomposed into the direct sum of quasi-regular representations as follows:

$$\ell^2(X) \cong \bigoplus_\mu \ell^2(\Gamma/H_\mu), \tag{3.7}$$

where H_μ is a subgroup of Γ such that $X_\mu \cong \Gamma/H_\mu$ for the orbit decomposition $X = \coprod_\mu X_\mu$ with respect to the Γ-action. About the irreducibility and unitary equivalence of quasi-regular representations, see Appendix A.

Next, we consider the tensor product $\otimes_\varphi$ in (3.1) for quasi-regular representations of free groups.

THEOREM 3.2. *Let H', H'' be subgroups of $\mathbb{F}_n$ and $\mathbb{F}_m$, respectively. For $\phi_{n,m}$ in (1.1), define the left action $\tilde{\phi}_{n,m}$ of $\mathbb{F}_{nm}$ on the direct product set $X := \mathbb{F}_n/H' \times \mathbb{F}_m/H''$ by*

$$\tilde{\phi}_{n,m}(g)(xH', yH'') := (g'xH', g''yH'') \quad ((xH', yH'') \in X,\ g \in \mathbb{F}_{nm}), \tag{3.8}$$

where $(g', g'') := \phi_{n,m}(g)$. With respect to the action $\tilde{\phi}_{n,m}$, let $X = \coprod_\mu X_\mu$ be the orbit decomposition and choose H_μ as a stabiliser subgroup of $\mathbb{F}_{nm}$ associated with X_μ. Then the following holds:

$$\lambda_{\mathbb{F}_n/H'} \otimes_\varphi \lambda_{\mathbb{F}_m/H''} \cong \bigoplus_\mu \lambda_{\mathbb{F}_{nm}/H_\mu}. \tag{3.9}$$

Proof. Let $\{\xi'_a : a \in \mathbb{F}_n/H'\}$ and $\{\xi''_b : b \in \mathbb{F}_{nm}/H''\}$ denote standard bases of $\ell^2(\mathbb{F}_n/H')$ and $\ell^2(\mathbb{F}_{nm}/H'')$, respectively. By definition, $(\lambda_{\mathbb{F}_n/H'} \otimes_\varphi \lambda_{\mathbb{F}_m/H''})(g)(\xi'_a \otimes \xi''_b) = \xi'_{g'a} \otimes \xi''_{g''b}$ for $g \in \mathbb{F}_{nm}$ where $(g', g'') := \phi_{n,m}(g)$. On the other hand, the permutation representation $(\ell^2(X), L)$ of $\mathbb{F}_{nm}$ by $\tilde{\phi}_{n,m}$ satisfies $L_g\xi_{(a,b)} = \xi_{\tilde{\phi}_{n,m}(g)(a,b)} = \xi_{(g'a,\, g''b)}$ where $\{\xi_{(a,b)} : (a, b) \in X\}$ denotes the standard basis of $\ell^2(X)$. By the natural identification of $\ell^2(X)$ with $\ell^2(\mathbb{F}_n/H') \otimes \ell^2(\mathbb{F}_m/H'')$, we see that L and $\lambda_{\mathbb{F}_n/H'} \otimes_\varphi \lambda_{\mathbb{F}_m/H''}$ are unitarily equivalent.

By definition, $X_\mu \cong \mathbb{F}_{nm}/H_\mu$ as $\mathbb{F}_{nm}$-homogeneous spaces, and the statement holds from the decomposition in (3.7). □

Theorem 3.2 states that the $\otimes_\varphi$-tensor product of any two quasi-regular representations is decomposed into the direct sum of quasi-regular representations. That is, the category of direct sums of quasi-regular representations of free groups is closed with respect to the $\otimes_\varphi$-tensor product. This shows a naturality of the $\otimes_\varphi$-tensor product. In Sections 3.2 and 3.3, we will show concrete examples of the formula (3.9).

3.2. Tensor product of some irreducible representations. In this subsection, we show tensor product formulas of some irreducible quasi-regular representations of $\mathbb{F}_n$'s as examples of Theorem 3.2.

We review some irreducible quasi-regular representations of $\mathbb{F}_n$ [**2**, **19**, **22**]. Let $g_1, \ldots, g_n$ be the free generators of $\mathbb{F}_n$. Fix $i \in \{1, \ldots, n\}$ and let $H_i^{(n)}$ denote the abelian subgroup of $\mathbb{F}_n$ generated by the single element g_i:

$$H_i^{(n)} := \{g_i^l : l \in \mathbb{Z}\} \subset \mathbb{F}_n. \tag{3.10}$$

PROPOSITION 3.3.

(i) *For any* $i = 1, \ldots, n$, $\lambda_{\mathbb{F}_n/H_i^{(n)}}$ *is irreducible.*

(ii) $\lambda_{\mathbb{F}_n/H_i^{(n)}} \cong \lambda_{\mathbb{F}_n/H_j^{(n)}}$ *if and only if* $i = j$.

Proof. See Appendix A. □

We show the tensor product formula of $\lambda_{\mathbb{F}_n/H_i^{(n)}}$'s in Proposition 3.3. For the map $\phi_{n,m}$ in (1.1), define the subgroup $G_{n,m}$ of $\mathbb{F}_{nm}$ by

$$G_{n,m} := \ker \phi_{n,m}. \tag{3.11}$$

From Lemma 2.4(v), $G_{n,m} \neq \{1\}$ when $n, m \geq 2$ and $G_{n,1} = G_{1,n} = \{1\}$ for any $n \geq 1$. Since, $G_{n,m}$ is a normal subgroup of $\mathbb{F}_{nm}$, $G_{n,m}H := \{gh : (g, h) \in G_{n,m} \times H\}$ is also a subgroup of $\mathbb{F}_{nm}$ and $G_{n,m}H = HG_{n,m}$ for any subgroup H of $\mathbb{F}_{nm}$.

THEOREM 3.4. *Let* $H_i^{(n)}$ *and* $G_{n,m}$ *be as in (3.10) and (3.11), respectively. Define* $K_{n,m,l} := G_{n,m}H_l^{(nm)}$ *for* $l = 1, \ldots, nm$.

(i) *For* $n, m \geq 1$ *and* $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$,

$$\lambda_{\mathbb{F}_n/H_i^{(n)}} \otimes_\varphi \lambda_{\mathbb{F}_m/H_j^{(m)}} \cong \lambda_{\mathbb{F}_{nm}/K_{n,m,m(i-1)+j}}. \tag{3.12}$$

(ii) *For any* $l = 1, \ldots, nm$, $\lambda_{\mathbb{F}_{nm}/K_{n,m,l}}$ *is irreducible.*

(iii) $\lambda_{\mathbb{F}_{nm}/K_{n,m,l}}$ *and* $\lambda_{\mathbb{F}_{nm}/K_{n,m,l'}}$ *are unitarily equivalent if and only if* $l = l'$.

Proof. See Appendix B. □

3.3. Tensor product of regular representations. In this subsection, we consider regular representations of $\mathbb{F}_n$'s as examples of Theorem 3.2.

We recall characterisation of representations by using von Neumann algebras. A nondegenerate representation π of a C*-algebra $\mathcal{A}$ is said to be *(pure) type* X if $\pi(\mathcal{A})''$ is of type X for X=I, II, III, II_1, II_∞ [3]. For a group G and a unitary representation U of G, U is said to be *type* X if $U(G)''$ is of type X for X=I, II, III, II_1, II_∞. U is *factor* if the centre of $U(G)''$ is trivial.

In the proof of Theorem 3.2, $\lambda_{\mathbb{F}_n} = \lambda_{\mathbb{F}_n/H'}$ and $\lambda_{\mathbb{F}_m} = \lambda_{\mathbb{F}_m/H''}$ when $H' = \{1\} \subset \mathbb{F}_n$ and $H'' = \{1\} \subset \mathbb{F}_m$.

PROPOSITION 3.5. *Let $n, m \geq 1$.*

(i) $\lambda_{\mathbb{F}_n} \otimes_\varphi \lambda_{\mathbb{F}_m} \cong (\lambda_{\mathbb{F}_{nm}/G_{n,m}})^{\oplus\infty}$ *where $G_{n,m}$ is as in (3.11).*
(ii) *If $n, m \geq 2$, then $\lambda_{\mathbb{F}_{nm}/G_{n,m}}$ is a type* II_1 *factor representation.*

Proof.

(i) Let $X := \mathbb{F}_n \times \mathbb{F}_m$ and let $\tilde{\phi}_{n,m}$ be the action of $\mathbb{F}_{nm}$ on X in (3.8). From Lemma 2.4(iii), we see that the orbit decomposition is given as follows:

$$X = \coprod_{x \in \mathbb{F}_n} \mathcal{O}_x, \quad \mathcal{O}_x := \{\tilde{\phi}_{n,m}(y)(x, 1) : y \in \mathbb{F}_{nm}\}. \tag{3.13}$$

Furthermore, the equivalence $(\mathcal{O}_x, \tilde{\phi}_{n,m}|_{\mathcal{O}_x}) \cong (\mathbb{F}_{nm}/G_{n,m}, L)$ holds as $\mathbb{F}_{nm}$-homogeneous spaces for all $x \in \mathbb{F}_n$ where L denotes the natural left action of $\mathbb{F}_{nm}$ on $\mathbb{F}_{nm}/G_{n,m}$. Therefore, the equivalence $(X, \tilde{\phi}_{n,m}) \cong (\mathbb{F}_{nm}/G_{n,m}, L)^{\#\mathbb{F}_n}$ of $\mathbb{F}_{nm}$-homogeneous spaces holds. From this and the proof of Theorem 3.2, the statement holds.

(ii) See Appendix C.

□

3.4. Automorphisms. In this subsection, we show examples of some C*-bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$. For $t \in \mathbb{R}$, define $\alpha_t^{(n)} \in \mathrm{Aut}C^*(\mathbb{F}_n)$ by

$$\alpha_t^{(n)}(g_i^{(n)}) := e^{\sqrt{-1}t\log n}\, g_i^{(n)} \quad (i = 1, \ldots, n). \tag{3.14}$$

Then, $\alpha_t^{(*)} := \oplus_{n\geq 1}\alpha_t^{(n)}$ is a C*-bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$ such that $\alpha_t^{(*)} \circ \alpha_s^{(*)} = \alpha_{t+s}^{(*)}$ for $s, t \in \mathbb{R}$.

Define $\beta^{(n)} \in \mathrm{Aut}C^*(\mathbb{F}_n)$ by

$$\beta^{(n)}(g_i^{(n)}) := g_{n-i+1}^{(n)} \quad (i = 1, \ldots, n). \tag{3.15}$$

Then, $\beta^{(*)} := \oplus_{n\geq 1}\beta^{(n)}$ is a C*-bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$ such that $\beta^{(*)} \circ \beta^{(*)} = id$.

The automorphism $\beta^{(*)}$ commutes $\alpha_t^{(*)}$ for each t. Hence, these give the action of the group $\mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$ on the C*-bialgebra $(\mathcal{A}, \Delta_\varphi)$.

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Appendix

A. Applications of commensurator to quasi-regular representations of discrete groups. Recall that two subgroups Γ_0 and Γ_1 of a group Γ are *commensurable* if $\Gamma_0 \cap \Gamma_1$ is of finite index in both Γ_0 and Γ_1, that is, $[\Gamma_0, \Gamma_0 \cap \Gamma_1] \cdot [\Gamma_1, \Gamma_0 \cap \Gamma_1] < \infty$, in such case, we write $\Gamma_0 \approx \Gamma_1$. Define the *commensurator* $\mathrm{Com}_\Gamma(\Gamma_0)$ of Γ_0 in Γ as

$$\mathrm{Com}_\Gamma(\Gamma_0) := \{g \in \Gamma : \Gamma_0 \approx g\Gamma_0 g^{-1}\}. \tag{A.1}$$

Then the inclusions $\Gamma_0 \subset \mathrm{Com}_\Gamma(\Gamma_0) \subset \Gamma$ of subgroups hold. According to [**5**], we review applications of commensurator to quasi-regular representations of discrete groups by Mackey [**16**].

THEOREM A.1. ([**5**], Theorem 2.1). *Let Γ be a discrete group and let Γ_0, Γ_1 be subgroups of Γ.*

(i) *The representation $(\ell^2(\Gamma/\Gamma_0), \lambda_{\Gamma/\Gamma_0})$ of Γ is irreducible if and only if* $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$.

(ii) *Assume* $\mathrm{Com}_\Gamma(\Gamma_i) = \Gamma_i$ *for* $i = 0, 1$. *Then $\lambda_{\Gamma/\Gamma_0}$ and $\lambda_{\Gamma/\Gamma_1}$ are unitarily equivalent if and only if Γ_0 and Γ_1 are quasiconjugate in Γ, that is, there exists $g \in \Gamma$ such that $\Gamma_0 \approx g\Gamma_1 g^{-1}$.*

LEMMA A.2. *Let $H_i^{(n)}$ be as in (3.10).*

(i) *If $a, b \in \mathbb{F}_n$ satisfy $ab = ba$, then there exists $w \in \mathbb{F}_n$ such that $a = w^l$ and $b = w^{l'}$ for some $l, l' \in \mathbb{Z}$.*

(ii) *If $a, b \in \mathbb{F}_n$ satisfy $a^l b^k = b^k a^l$ for some $l, k \in \mathbb{Z}$, $l, k \neq 0$, then $ab = ba$.*

(iii) *If $a \in \mathbb{F}_n$ satisfies $ab = ba$ for some $b \in H_i^{(n)}$ and $b \neq 1$, then $a \in H_i^{(n)}$.*

(iv) *If $i, j \in \{1, \ldots, n\}$ and $g \in \mathbb{F}_n$ satisfy $H_i^{(n)} \approx gH_j^{(n)}g^{-1}$, then $i = j$ and $g \in H_i^{(n)}$.*

Proof. Let $c_1, \ldots, c_n$ be free generators of $\mathbb{F}_n$ and let $H_i := H_i^{(n)}$.

(i) See [**17**], p42, 6.

(ii) See [**17**], p41, 4.

(iii) By (i), both a and b can be written as $a = w^l$ and $b = w^{l'}$ for some $w \in \mathbb{F}_n$ and $l, l' \in \mathbb{Z}$. By the choice of b, $b = c_i^k$ for some $k \in \mathbb{Z}$ and $k \neq 0$. Hence $c_i^k = w^{l'}$. Therefore $c_i^k w^{l'} = w^{l'} c_i^k$. From (ii), $c_i w = w c_i$. From (i), we see $w \in H_i$. Therefore $a = w^l \in H_i$.

(iv) If i, j and g satisfy the assumption, then $[H_i, H_i \cap gH_j g^{-1}] < \infty$ by definition. Since $\#H_i = \infty$, there exists $x \in H_i \cap gH_j g^{-1}$ such that $x \neq 1$. Then, $c_i^l = x = gc_j^{l'} g^{-1}$ for some $l, l' \in \mathbb{Z}$. For $\chi_z^{(n)}$ in (3.3), $z_i^l = \chi_z^{(n)}(c_i^l) = \chi_z^{(n)}(gc_j^{l'} g^{-1}) = \chi_z^{(n)}(c_j^{l'}) = z_j^{l'}$ for any $z \in \mathbb{T}^n$. This implies $i = j$ and $l = l'$. In consequence, $c_i^l = gc_i^l g^{-1}$ for some $l \in \mathbb{Z}$. By the choice of x and l, $l \neq 0$. From (iii), $g \in H_i$. □

Proof of Proposition 3.3

(i) From Lemma A.2(iv), $\mathrm{Com}_{\mathbb{F}_n}(H_i^{(n)}) = H_i^{(n)}$. (This has been shown by Lemma 6 in p15, [**19**].) By Theorem A.1(i), the statement holds.

(ii) From Lemma A.2(iv), $H_i^{(n)}$ and $H_j^{(n)}$ are quasiconjugate if and only if $i = j$. From this and Theorem A.1(ii), the statement holds.

□

B. Proof of Theorem 3.4. In this section, we prove Theorem 3.4. Let $\phi_{n,m}$, $H_i^{(n)}$, $G_{n,m}$ be as in (1.1), (3.10) and (3.11), respectively. Since, $G_{n,m}$ is a normal subgroup of $\mathbb{F}_{nm}$, we can define the quotient group

$$Q_{n,m} := \mathbb{F}_{nm}/G_{n,m}. \tag{B.1}$$

Then, $Q_{1,n} \cong Q_{n,1} \cong \mathbb{F}_n$ for any $n \geq 1$.

Let $g_1, \ldots, g_n$ be free generators of $\mathbb{F}_n$. Define $p^{(n)} \in \mathrm{Hom}(\mathbb{F}_n, \mathbb{Z})$ by

$$p^{(n)}(g) := \varepsilon_1 + \cdots + \varepsilon_k \quad \text{when } g = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k} \in \mathbb{F}_n \tag{B.2}$$

where $\varepsilon_i \in \{1, -1\}$ for $i = 1, \ldots, k$. By definition, the following holds.

FACT B.1.

(i) For $g \in \mathbb{F}_{nm}$, if $\phi_{n,m}(g) = (g', g'')$, then $p^{(nm)}(g) = p^{(n)}(g') = p^{(m)}(g'')$.

(ii) If $g \in G_{n,m}$, then $p^{(nm)}(g) = 0$.

(iii) For any $i = 1, \ldots, n$, the restriction $p^{(n)}|_{H_i^{(n)}} : H_i^{(n)} \to \mathbb{Z}$ is an isomorphism.

(iv) Let $\hat{p}^{(nm)} : Q_{n,m} \to \mathbb{Z}$ by $\hat{p}^{(nm)}(\hat{g}) := p^{(nm)}(g)$ for $\hat{g} = gG_{n,m} \in Q_{n,m}$ and define the subgroup $\hat{H}_l^{(nm)}$ of $Q_{n,m}$ by

$$\hat{H}_l^{(nm)} := \{hG_{n,m} \in Q_{n,m} : h \in H_l^{(nm)}\}. \tag{B.3}$$

Then, $\hat{p}^{(nm)}$ is well defined and is a group homomorphism. Furthermore, the restriction $\hat{p}^{(nm)}|_{\hat{H}_l^{(nm)}} : \hat{H}_l^{(nm)} \to \mathbb{Z}$ is an isomorphism for any $l = 1, \ldots, nm$.

Remark that $\hat{H}_l^{(1\cdot n)} \cong \hat{H}_l^{(n\cdot 1)} \cong H_l^{(n)}$ for any $n \geq 1$.

LEMMA B.2. *Let $n, m \geq 1$,$(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ and $g \in \mathbb{F}_{nm}$.*

(i) *If $\phi_{n,m}(g) \in H_i^{(n)} \times H_j^{(m)}$, then there exists $h \in H_{m(i-1)+j}^{(nm)}$ such that $\phi_{n,m}(h) = \phi_{n,m}(g)$.*

(ii) *If $\phi_{n,m}(g) \in H_i^{(n)} \times H_j^{(m)}$, then $g \in G_{n,m}H_{m(i-1)+j}^{(nm)}$.*

Proof.

(i) Let $(g', g'') := \phi_{n,m}(g)$. From Fact B.1(i), $p^{(n)}(g') = p^{(nm)}(g) = p^{(m)}(g'')$. When $l := p^{(nm)}(g)$, $g' = (g_i^{(n)})^l$ and $g'' = (g_j^{(m)})^l$ by Fact B.1(iii). Hence, $h := (g_{m(i-1)+j}^{(nm)})^l \in H_{m(i-1)+j}^{(nm)}$ satisfies the relation.

(ii) From (i), $\phi_{n,m}(h^{-1}g) = (1, 1)$ for some $h \in H_{m(i-1)+j}^{(nm)}$. Therefore, $h^{-1}g \in G_{n,m}$ and $g \in hG_{n,m} \subset H_{m(i-1)+j}^{(nm)}G_{n,m} = G_{n,m}H_{m(i-1)+j}^{(nm)}$.

□

LEMMA B.3. *For $n, m \geq 1$ and $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, two $\mathbb{F}_{nm}$-homogeneous spaces $(\mathbb{F}_{nm}/(G_{n,m}H_{m(i-1)+j}^{(nm)}), L)$ and $(\mathbb{F}_n/H_i^{(n)} \times \mathbb{F}_m/H_j^{(m)}, \tilde{\phi}_{n,m})$ are equivalent where L denotes the natural left action of $\mathbb{F}_{nm}$ on $\mathbb{F}_{nm}/K$ and $\tilde{\phi}_{n,m}$ is as in (3.8).*

Proof. Rewrite $K := G_{n,m}H_{m(i-1)+j}^{(nm)}$, $\phi := \phi_{n,m}$, $H' := H_i^{(n)}$ and $H'' := H_j^{(m)}$ here. Define the map

$$\theta : \mathbb{F}_{nm}/K \to \mathbb{F}_n/H' \times \mathbb{F}_m/H''; \quad \theta([x]) := ([x'], [x'']), \tag{B.4}$$

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where $(x', x'') := \phi(x)$ and $[a]$ denotes the coset with the representative a. By definition, we see that the map θ is well defined and satisfies $\theta \circ L_g = \tilde{\phi}_{n,m}(g) \circ \theta$ for all $g \in \mathbb{F}_{nm}$. It is sufficient to show that θ is injective and surjective.

(i) Injectivity: For $[x], [y] \in \mathbb{F}_{nm}/K$, assume $\theta([x]) = \theta([y])$. Then $[x'] = [y']$ and $[x''] = [y'']$. Hence $x' = y'h'$ and $x'' = y''h''$ for some $(h', h'') \in H' \times H''$. Therefore $\phi(x) = (x', x'') = (y'h', y''h'') = \phi(y)(h', h'')$. Hence $\phi(y^{-1}x) = (h', h'')$. From Lemma B.2(ii), $y^{-1}x \in K$. Therefore $[x] = [y]$. Hence θ is injective.

(ii) Surjectivity: For $([x'], [x'']) \in \mathbb{F}_n/H' \times \mathbb{F}_m/H''$, there exists $(w', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi(z)(w', 1) = (x', x'')$ from Lemma 2.4(iii). Let $g_1^{(n)}, \ldots, g_n^{(n)}$ be free generators of $\mathbb{F}_n$. Assume $w' = (g_{j_1}^{(n)})^{\varepsilon_1} \cdots (g_{j_l}^{(n)})^{\varepsilon_l}$ for $\varepsilon_i \in \{1, -1\}$. Let $h'' := (g_j^{(m)})^{\varepsilon_1} \cdots (g_j^{(m)})^{\varepsilon_l} = (g_j^{(m)})^{\varepsilon_1 + \cdots + \varepsilon_l}$. Then $(x', x''h'') = \phi(z)(w', h'') = \phi(z)\phi(w) = \phi(zw)$ for some $w \in \mathbb{F}_{nm}$. Therefore $([x'], [x'']) = ([x'], [x''h'']) = \theta([zw])$. Hence θ is surjective.

$\square$

Let $\chi_z^{(n)}$ be as in (3.3) and let $\mathbb{T}^n \boxtimes \mathbb{T}^m := \{z \boxtimes w : (z, w) \in \mathbb{T}^n \times \mathbb{T}^m\}$. For $u \in \mathbb{T}^n \boxtimes \mathbb{T}^m$, we see $G_{n,m} \subset \ker \chi_u^{(nm)}$ by (3.5). From this, $\chi_u^{(nm)}(gh) = \chi_u^{(nm)}(g)$ for any $g \in \mathbb{F}_{nm}$ and $h \in G_{n,m}$. Hence

$$\hat{\chi}_u^{(nm)} : Q_{n,m} \to U(1); \quad \hat{\chi}_u^{(nm)}(gG_{n,m}) := \chi_u^{(nm)}(g), \tag{B.5}$$

is well defined for any $u \in \mathbb{T}^n \boxtimes \mathbb{T}^m$ as a homomorphism.

LEMMA B.4. *Let $n, m \geq 1$ and let $\hat{H}_l^{(nm)}$ be as in (B.3). If $i, j \in \{1, \ldots, nm\}$ and $\hat{g} \in Q_{n,m}$ satisfy $\hat{H}_i^{(nm)} \approx \hat{g}\hat{H}_j^{(nm)}\hat{g}^{-1}$, then $i = j$ and $\hat{g} \in \hat{H}_i^{(nm)}$.*

Proof. Let $H_i := H_i^{(nm)}$, $G := G_{n,m}$ and let $c_1, \ldots, c_{nm}$ be free generators of $\mathbb{F}_{nm}$. For $g \in \mathbb{F}_{nm}$, let $\hat{g} := gG \in Q_{n,m}$. If i, j and $\hat{g}$ satisfy the assumption, then $[\hat{H}_i, \hat{H}_i \cap \hat{g}\hat{H}_j\hat{g}^{-1}] < \infty$ by definition. Since $\#\hat{H}_i = \infty$, there exists $\hat{x} \in \hat{H}_i \cap \hat{g}\hat{H}_j\hat{g}^{-1}$ such that $\hat{x} \neq 1$. Then, $c_i^l G = xG = gc_j^{l'}g^{-1}G$ for some $l, l' \in \mathbb{Z}$. For $\hat{\chi}_u^{(nm)}$ in (B.5), $u_i^l = \hat{\chi}_u^{(nm)}(c_i^l G) = \hat{\chi}_u^{(nm)}(gc_j^{l'}g^{-1}G) = u_j^{l'}$ for any $u = (u_1, \ldots, u_{nm}) \in \mathbb{T}^n \boxtimes \mathbb{T}^m$. This implies $i = j$ and $l = l'$. From this, the first statement is verified. In consequence, $c_i^l G = gc_i^l g^{-1}G$ for some $l \in \mathbb{Z}$. By the choice of x and l, $l \neq 0$. Since $c_i^l gG = gc_i^l G$, $c_i^l g = gc_i^l w$ for some $w \in G$. For $(g', g'') := \phi_{n,m}(g)$,

$$(a_{i'}^l g', b_{i''}^l g'') = \phi_{n,m}(c_i^l g) = \phi_{n,m}(gc_i^l w) = \phi_{n,m}(gc_i^l) = (g'a_{i'}^l, g''b_{i''}^l), \tag{B.6}$$

where $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are free generators of $\mathbb{F}_n$ and $\mathbb{F}_m$, respectively, and $(i', i'') \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ is defined as $i = m(i' - 1) + i''$. Hence $a_{i'}^l g' = g'a_{i'}^l$ and $b_{i''}^l g'' = g''b_{i''}^l$. From these and Lemma A.2(iii), $g' \in H_{i'}^{(n)}$ and $g'' \in H_{i''}^{(m)}$. Hence $\phi_{n,m}(g) = (g', g'') \in H_{i'}^{(n)} \times H_{i''}^{(m)}$. Therefore, $g \in H_i G$ by Lemma B.2(ii). Hence $\hat{g} \in \hat{H}_i$. $\square$

PROPOSITION B.5. *Let $Q_{n,m}$ and $\hat{H}_l^{(nm)}$ be as in (B.1) and (B.3), respectively.*

(i) *For any $l \in \{1, \ldots, nm\}$, $\lambda_{Q_{n,m}/\hat{H}_l^{(nm)}}$ is irreducible.*

(ii) *For $l, l' \in \{1, \ldots, nm\}$, $\lambda_{Q_{n,m}/\hat{H}_l^{(nm)}} \cong \lambda_{Q_{n,m}/\hat{H}_{l'}^{(nm)}}$ if and only if $l = l'$.*

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Proof.

(i) From Lemma B.4, $\mathrm{Com}_{Q_{n,m}}(\hat{H}_l^{(nm)}) = \hat{H}_l^{(nm)}$. From this and Theorem A.1(i), the statement holds.

(ii) From Lemma B.4, $\hat{H}_l^{(nm)}$ and $\hat{H}_{l'}^{(nm)}$ are quasiconjugate if and only if $l = l'$. From this and Theorem A.1(ii), the statement holds.

$\square$

Proof of Theorem 3.4

(i) From Lemma B.3, the statement holds.

(ii) Let $K_l := G_{n,m}H_l^{(nm)}$. By identifying $\mathbb{F}_{nm}/K_l$ with $Q_{n,m}/\hat{H}_l^{(nm)}$ as a $\mathbb{F}_{nm}$-homogeneous space, we see that

$$\lambda_{\mathbb{F}_{nm}/K_l}(g_l^{(nm)}) = \lambda_{Q_{n,m}/\hat{H}_l^{(nm)}}(\hat{g}_l^{(nm)}) \quad (l = 1, \ldots, nm). \tag{B.7}$$

Hence $\lambda_{\mathbb{F}_{nm}/K_l}(\mathbb{F}_{nm}) = \lambda_{Q_{n,m}/\hat{H}_l^{(nm)}}(Q_{n,m})$. From Proposition B.5(i), $\lambda_{\mathbb{F}_{nm}/K_l}$ is also irreducible.

(iii) From Proposition B.5(ii) and (B.7), the statement holds.

$\square$

C. Proof of Proposition 3.5(ii). In this section, we prove Proposition 3.5(ii). For this purpose, we prove the following proposition.

PROPOSITION C.1. *Let $Q_{n,m}$ be as in (B.1). If $n, m \geq 2$, then the group $Q_{n,m}$ is ICC, that is, every conjugacy class in $Q_{n,m}$, other than its unit is infinite [**3**].*

Proof. Rewrite $\phi := \phi_{n,m}$ and $G := G_{n,m}$, and let $\hat{g} := gG \in Q_{n,m}$ and $(g', g'') := \phi(g)$ for $g \in \mathbb{F}_{nm}$. Fix $\hat{g} \in Q_{n,m} \setminus \{1\}$. By the choice of g, $(g', g'') \neq (1, 1)$.

Assume $g' \neq 1$. Choose an infinite sequence $\{a_l : l \geq 1\} \subset \mathbb{F}_n$ such that $a_l g' a_l^{-1} \neq a_{l'} g' a_{l'}^{-1}$ when $l \neq l'$. Since $\mathbb{F}_n$ is ICC, such a sequence always exists. By the choice of $\{a_l\}$, $a_l \neq a_{l'}$ when $l \neq l'$. From Lemma 2.4(i), there exists $\{(b_l, c_l) : l \geq 1\} \subset \mathbb{F}_m \times \mathbb{F}_{nm}$ such that $\phi(c_l) = (a_l, b_l)$ for any $l \geq 1$. By the choice of $\{c_l\}$, $\hat{\phi}(c_l g c_l^{-1} G) = (a_l g' a_l^{-1}, b_l g'' b_l^{-1})$ where $\hat{\phi} : Q_{n,m} \to \mathbb{F}_n \times \mathbb{F}_m$ denotes the natural homomorphism induced by ϕ. From this and the choice of $\{a_l\}$, $\hat{\phi}(c_l g c_l^{-1} G) \neq \hat{\phi}(c_{l'} g c_{l'}^{-1} G)$ when $l \neq l'$. From this, $\hat{c}_l \hat{g} \hat{c}_l^{-1} = c_l g c_l^{-1} G \neq c_{l'} g c_{l'}^{-1} G = \hat{c}_{l'} \hat{g} \hat{c}_{l'}^{-1}$ when $l \neq l'$. Therefore, $\{\hat{c}_l \hat{g} \hat{c}_l^{-1} : l \geq 1\}$ is an infinite subset of the conjugacy class of $\hat{g}$.

If $g' = 1$, then $g'' \neq 1$. In a similar way, we can construct an infinite subset of the conjugacy class of $\hat{g}$ from g'' by using Lemma 2.4(ii). Hence, the statement is verified. $\square$

Proof of Proposition 3.5(ii) By definition, the group $Q_{n,m}$ in (B.1) acts on $\ell^2(\mathbb{F}_{nm}/G_{n,m}) = \ell^2(Q_{n,m})$ by its left regular representation $\lambda_{Q_{n,m}}$. For the natural left action L' of $Q_{n,m}$ on $Q_{n,m}$, $L(g)(hG_{n,m}) = ghG_{n,m} = L'(gG_{n,m})(hG_{n,m})$ for any $g, h \in \mathbb{F}_{nm}$ where L denotes the natural left action of $\mathbb{F}_{nm}$ on $\mathbb{F}_{nm}/G_{n,m}$. Hence $\lambda_{\mathbb{F}_{nm}/G_{n,m}}(\mathbb{F}_{nm}) = \lambda_{Q_{n,m}}(Q_{n,m})$. By Proposition C.1, $\lambda_{Q_{n,m}}$ is a type II_1 factor representation ([**3**], III.3.3.7 Proposition). Hence, the statement holds. $\square$

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