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# QUASI CLIFFORD ALGEBRAS AND SYSTEMS OF ORTHOGONAL DESIGNS

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**Abstract**

The representation theory of Clifford algebras has been used to obtain information on the possible orders of amicable pairs of orthogonal designs on given numbers of variables. If, however, the same approach is tried on more complex systems of orthogonal designs, such as product designs and amicable triples, algebras which properly generalize the Clifford algebras are encountered. In this paper a theory of such generalizations is developed and applied to the theory of systems of orthogonal designs, and in particular to the theory of product designs.

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## 1. Introduction

(1.1) DEFINITION. An *orthogonal design* of order $n$, *type* $(u_1, \dots, u_p)$ on the distinct commuting variables $x_1, \dots, x_p$, is an $n \times n$ matrix $X$ with entries from $\{0, \pm x_1, \dots, \pm x_p\}$ satisfying

$$XX^T = \sum_i u_i x_i^2 I$$

($u_i$ positive integers, $I$ the identity matrix).

(1.2) DEFINITION. A pair of orthogonal designs $X$, $Y$ of order $n$ on disjoint sets of variables is called *amicable* when $XY^T = YX^T$ and *anti-amicable* when $XY^T = -YX^T$.

Thus a *product design* as defined in [6], [8] is a triple of orthogonal designs one pair of which is amicable and the other two pairs anti-amicable. A further requirement in [6], [8] is that each anti-amicable pair have zero Hadamard product (the *Hadamard product* $*$ being defined as component wise multiplication $(a_{ij}) * (b_{ij}) = (a_{ij}b_{ij})$).

The amicable triples studied in [9] are triples of orthogonal designs which are pairwise amicable.

Both product designs and amicable triples may be embraced in a wider concept.

(1.3) DEFINITION. A *K-system* of orthogonal designs of order $n$, *genus* $(\delta_{ij})$, $1 \leqslant i < j \leqslant K$ (where each $\delta_{ij} = 0$ or 1), is a $K$-tuple of order $n$ orthogonal designs $(X_1, \ldots, X_K)$ on disjoint sets of variables, satisfying $X_j X_i^T = (-1)^{\delta_{ij}} X_i X_j^T$ for all $i < j$, (thus each pair is either amicable or anti-amicable). The system is called *regular* if in addition the Hadamard product $X_i * X_j$ is zero whenever $\delta_{ij} = 1$ (thus the non-zero entries of anti-amicable pairs nowhere "overlap").

It may be noted that if $X_i, X_j, \ldots, X_k$ are pairwise anti-amicable members of a *regular* system, they may be added to make a design $X_i + \cdots + X_k$. This design is orthogonal for

$$(X_i + \cdots + X_k)(X_i + \cdots + X_k)^T = \Big(\sum_a u_{ia} x_{ia}^2 + \cdots + \sum_c u_{kc} x_{kc}^2\Big) I,$$

where $(x_{ia})$, $(u_{ia}), \ldots, (x_{kc})$, $(u_{kc})$ are respectively the variables, types of $X_i, \ldots, X_k$ (since for $\lambda \neq \mu$, $X_\lambda X_\mu^T + X_\mu X_\lambda^T = 0$).

So a *product design* $(X, Y, Z)$ defined by

$$\text{(1.4)} \qquad \begin{gathered} YX^T = -XY^T, \qquad ZX^T = -XZ^T, \qquad ZY^T = YZ^T, \\ X * Y = X * Z = 0, \quad \text{each } X, Y, Z \text{ orthogonal,} \end{gathered}$$

is a regular 3-system of genus ($\delta_{ij}$: $\delta_{12} = \delta_{13} = 1$, $\delta_{23} = 0$), and $X + Y$, $X + Z$ are orthogonal designs.

If, for each $i$, $X_i$ is of type $(u_{i1}, \ldots, u_{ip_i})$ then we say the $K$-system (1.3) has type $(u_{ij}) = (u_{11}, \ldots, u_{1p_1}; u_{21}, \ldots, u_{2p_2}; \cdots; u_{K1}, \ldots, u_{Kp_K})$.

(1.5) LEMMA. *A system of type* $(1, 1, \ldots, 1; \cdots; 1, 1, \ldots, 1)$ *is regular.*

PROOF. With this type each $X_i$ has each of its variables occurring just once in each row. So if $X_i$, $X_j$ are anti-amicable then since the diagonal elements of $X_j X_i^T = -X_i X_j^T$ are zero the non-zero elements of $X_i$, $X_j$ can nowhere overlap. The result follows.

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Let $(X_1, \dots, X_K)$ be a $K$-system of order $n$, genus $(\delta_{ij})$, type $(u_{ij})$, on the $p_1 + 1$, $p_2, \dots, p_K$ variables $x_{10}$, $x_{11}, \dots, x_{1p_1}$; $x_{21}, \dots, x_{2p_2}$; $\cdots$; $x_{K1}, \dots, x_{Kp_K}$. For each $i = 1, \dots, K$ write $X_i = \sum_j A_{ij}x_{ij}$ where each $A_{ij}$ is a constant $\{0, \pm 1\}$-matrix. Substitution into the defining equations $X_iX_i^T = \sum_j u_{ij}x_{ij}^2 I$, $X_jX_i^T = (-1)^{\delta_{ij}}X_iX_j^T$ $(i < j)$ and comparison of coefficients of each $x_{ij}$ yields the Hurwitz–Radon like relations:

$$A_{ij}A_{ij}^T = u_{ij}I, \qquad A_{ik}A_{ij}^T = -A_{ij}A_{ik}^T, \qquad (j \neq k)$$
$$A_{jl}A_{ik}^T = (-1)^{\delta_{ij}}A_{ik}A_{jl}^T, \qquad (i < j).$$

These imply

$$A_{ij}^TA_{ij} = u_{ij}I, \qquad A_{ik}^TA_{ij} = -A_{ij}^TA_{ik}, \qquad (j \neq k)$$
$$A_{jl}^TA_{ik} = (-1)^{\delta_{ij}}A_{ik}^TA_{jl}, \qquad (i < j).$$

Put

$$E_{ij} = \frac{1}{\sqrt{u_{ij}u_{10}}}A_{ij}A_{10}^T. \tag{1.6}$$

Then the $(p_1 + 1) + p_2 + \cdots + p_K$ real $n \times n$ matrices $E_{10}$, $E_{11}$, $\dots, E_{ij}, \dots, E_{Kp_K}$ are found to satisfy the following:

$$\begin{aligned} &E_{10} = I, \qquad E_{ij}^2 = (-1)^{\delta_{1i}}I, && (j \neq 0) \\ &E_{ik}E_{ij} = -E_{ij}E_{ik}, && (0 < j < k) \\ &E_{jl}E_{ik} = (-1)^{\delta_{1i}+\delta_{1j}+\delta_{ij}}E_{ik}E_{jl}, && (i < j;\ k \neq 0), \end{aligned} \tag{1.7}$$

where it is assumed that if $i = 1$, $\delta_{1i} = 1$.

Thus we have an order $n$ representation of an algebra which is reminiscent of the Clifford algebras studied in [7]. But while in [7] the generators were assumed pairwise anti-commutative, here some pairs of generators commute (unless all $\delta_{1i} + \delta_{1j} + \delta_{ij} = 1$).

In the following as structure and representation theory of such algebras is developed. Using this theory it is then shown how to find for given $p_1, \dots, p_K$ and $(\delta_{ij})$ the possible orders $n$ of $K$-systems of genus $(\delta_{ij})$ on $(p_1, \dots, p_K)$ variables. Further it is to be shown that, for each such $n$, such a $K$-system may be constructed which has type $(1, 1, \dots, 1; \cdots; 1, \dots, 1)$ and hence, by (1.5), is regular.

The particular class of 3-systems defined by (1.4) are then considered in more detail (the product designs).

## 2. Quasi Clifford algebras

(2.1) DEFINITION. Let $F$ be a commutative field of characteristic not 2, $m$ a positive integer, $(k_i)_{1 \leqslant i \leqslant m}$ a family of non-zero elements of $F$, and $(\delta_{ij})_{1 \leqslant i < j \leqslant m}$ a family of elements from $\{0, 1\}$. The *Quasi Clifford*, or QC, algebra $\mathcal{C} = \mathcal{C}_F[m, (k_i), (\delta_{ij})]$ is the algebra (associative, with a 1) over $F$ on $m$ generators $\alpha_1, \ldots, \alpha_m$, with defining relations

$$\text{(i)} \qquad \alpha_i^2 = k_i, \quad \alpha_j\alpha_i = (-1)^{\delta_{ij}}\alpha_i\alpha_j \qquad (i < j)$$

(where $k_i$ of $F$ is identified with $k_i$ times the 1 of $\mathcal{C}$).

If all $\delta_{ij} = 1$ we have a Clifford algebra corresponding to some non-singular quadratic form on $F^m$ (see [2]). If in addition each $k_i = \pm 1$ we have those special Clifford algebras studied in [7]. In this paper we are particularly interested in QC algebras for which each $k_i = \pm 1$, and we call such algebras *Special quasi Clifford*, or SQC, algebras. First however we develop some theory of QC algebras in general.

The QC algebra $\mathcal{C}$ of (2.1) is defined to within isomorphism by the following properties.

(a) It has $m$ elements, $\alpha_1, \ldots, \alpha_m$, which generate $\mathcal{C}$ (that is, each element of $\mathcal{C}$ is expressible as a polynomial in the $\alpha_i$ over $F$) and which satisfy (2.1)(i).

(2.2) (b) If $\mathcal{D}$ is any algebra over $F$ containing elements $\beta_1, \ldots, \beta_m$ which satisfy $\beta_i^2 = k_i$, $\beta_j\beta_i = (-1)^{\delta_{ij}}\beta_i\beta_j$ $(i < j)$, there is an $F$-algebra homomorphism $\mathcal{C} \to \mathcal{D}$ which maps $\alpha_i$ to $\beta_i$ for each $i$.

Because of (2.1)(i), any product of the $\alpha_i$ reduces to an $F$-multiple of one of the $2^m$ elements $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ (each $\varepsilon_i = 0$ or 1). So as a vector space over $F$, $\mathcal{C}$ is spanned by these $2^m$ elements and $\dim \mathcal{C} \leqslant 2^m$. Let us define $\mathcal{D}$ to be the vector space of all formal $F$-linear combinations of the $2^m$ formal expressions $\beta_1^{\varepsilon_1} \cdots \beta_m^{\varepsilon_m}$ $(\varepsilon_i = 0, 1)$. Make $\mathcal{D}$ an algebra by defining products

$$(\beta_1^{\varepsilon_1} \cdots \beta_m^{\varepsilon_m})(\beta_1^{\eta_1} \cdots \beta_m^{\eta_m}) = h\beta_1^{\xi_1} \cdots \beta_m^{\xi_m},$$

where $\xi_i$ is $\varepsilon_i + \eta_i$ reduced mod 2, and

$$h = \prod_{i=1}^{m} \left( k_i^{\varepsilon_i} \prod_{j=i+1}^{m} (-1)^{\varepsilon_j \delta_{ij}} \right)^{\eta_i}.$$

It may be verified that this product is associative with a 1 (which is in fact $\beta_1^0 \cdots \beta_m^0$), and that if we write $\beta_i$ for $\beta_1^0 \cdots \beta_{i-1}^0 \beta_i^1 \beta_{i+1}^0 \cdots \beta_m^0$, the $\beta_i$ generate $\mathfrak{D}$ and satisfy (2.2)(b). It follows that $\mathfrak{D}$ is an epimorphic image of $\mathcal{C}$, so dim $\mathcal{C} \geqslant$ dim $\mathfrak{D} = 2^m$. We have proved:

(2.3) THEOREM. *The QC algebra* $\mathcal{C}$ *of* (2.1) *has dimension* $2^m$ *as a vector space over* $F$, *and a basis is* $\{\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}: \varepsilon_i = 0$ *or* $1\}$.

We use $[\cdots]$ for $F$-algebra generators, and $\langle \cdots \rangle$ for $F$ vector space generators. Thus

$$\mathcal{C} = [\alpha_1, \ldots, \alpha_m] = \langle 1, \alpha_1, \ldots, \alpha_m, \alpha_1\alpha_2, \ldots, \alpha_1\alpha_2 \cdots \alpha_m \rangle.$$

In discussing the structure and representation theory of QC algebras the concepts and elementary theory of semi-simple (associative) algebras, direct sums and tensor products are assumed (see [3], [4]).

Certain QC algebras of low dimension are of special importance. For $b$ non-zero, in $F$, let $\mathbb{C}_b$ denote the QC algebra on one generator, $\beta$ say, satisfying $\beta^2 = b$. For $c, d$ non-zero, in $F$, let $\mathbb{Q}_{c,d}$ denote the QC algebra on two generators $\gamma, \delta$ say, where $\gamma^2 = c$, $\delta^2 = d$, $\delta\gamma = -\gamma\delta$. These algebras are in fact Clifford algebras and their structures are known (see [1]). An outline of the facts follows.

(2.4) (i) *If $b$ is a square in $F$, say $b = p^2$, there is an isomorphism $\mathbb{C}_b \simeq 2F$ $(= F \oplus F)$ a direct sum of two copies of $F$.* For

$$\mathbb{C}_b = [\beta] = \langle 1, \beta \rangle = \left\langle \frac{p+\beta}{2p}, \frac{p-\beta}{2p} \right\rangle$$
$$= \left\langle \frac{p+\beta}{2p} \right\rangle \oplus \left\langle \frac{p-\beta}{2p} \right\rangle$$

(since $(p+\beta)/2p \cdot (p-\beta)/2p = 0) = F \oplus F$ (since $((p+\beta)/2p)^2 = (p+\beta)/2p$). *There are two irreducible matrix representations, of order* 1, *given by* $\beta \to (p)$ *and* $\beta \to (-p)$.

(ii) *If $b$ is not a square in $F$, then $\mathbb{C}_b \simeq F[\sqrt{b}\,]$, a field extension of rank* 2, since any $x + y\beta \neq 0$ $(x, y$ in $F)$ has an inverse $(x^2 - by^2)^{-1}(x - y\beta)$; $(x^2 - by^2$ cannot be 0). *To within equivalence the regular representation is the only irreducible representation. It is of order* 2, *and is given by* $\beta \to \left(\begin{smallmatrix} 0 & b \\ 1 & 0 \end{smallmatrix}\right)$.

(iii) *If $d = p^2 - cq^2$ for some $p, q$ in $F$, there is an isomorphism $\mathbb{Q}_{c,d} \simeq F_2$, the full $2 \times 2$ matrix algebra over $F$. The correspondence*

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma \leftrightarrow \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix},$$
$$\delta \leftrightarrow \begin{pmatrix} p & cq \\ -q & -p \end{pmatrix}, \qquad \gamma\delta \leftrightarrow \begin{pmatrix} -cq & -cp \\ p & cq \end{pmatrix}$$

*gives an isomorphism, and to within equivalence gives the only irreducible representation which is of order* 2. (These four matrices are linearly independent since the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & c & 1 & 0 \\ p & cq & -q & -p \\ -cq & -cp & p & cq \end{vmatrix} = 4c(p^2 - cq^2) = 4cd$$

is non-zero.)

(iv) *If there is no $p, q$ in $F$ with $d = p^2 - cq^2$, then $\mathbb{Q}_{c,d}$ is a division algebra,* since any $t1 + x\gamma + y\delta + z\gamma\delta \neq 0$ ($t, x, y, z$ in $F$) has an inverse

$$(t^2 - cx^2 - dy^2 + cdz^2)^{-1}(t1 - x\gamma - y\delta - z\gamma\delta).$$

($t^2 - cx^2 - dy^2 + cdz^2$ cannot be zero. For suppose it is zero with not all of $t, x, y, z = 0$. Now if $t = z = 0$ then $cx^2 + dy^2 = 0$, $y \neq 0$, and $d = 0^2 - c(xy^{-1})^2$; while if $t^2 + cdz^2 = 0 \neq z$ then $d = 0^2 - c(tz^{-1}c^{-1})^2$. If $t^2 + cdz^2 \neq 0$ then $cu^2 + dv^2 = 1$ where

$$u = (tx + dyz)/(t^2 + cdz^2), \quad v = (ty - cxz)/(t^2 + cdz^2).$$

If $v \neq 0$ then $d = (v^{-1})^2 - c(uv^{-1})^2$, and if $v = 0$ then $d = ((d+1)/2)^2 - c((d-1)u/2)^2$. In all cases there are $p, q$ in $F$ with $d = p^2 - cq^2$.) *To within equivalence the regular representation is the only irreducible representation. It is of order* 4 *and is given by*

$$\gamma \to \begin{bmatrix} 0 & c & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \delta \to \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & -d \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The importance of the algebras $\mathbb{C}_b$, $\mathbb{Q}_{c,d}$ lies in the fact that any QC algebra decomposes into a tensor product of such algebras. To show this the following well-known property of tensor products will be used (see [1], Ch. I, §5).

(2.5) LEMMA. *Let $\mathcal{A}$ be a finite dimensional algebra over $F$, and $\mathcal{H}$, $\mathcal{K}$ subalgebras each containing the* 1 *of $\mathcal{A}$. Suppose* (i) *each element of $\mathcal{H}$ commutes with every element of $\mathcal{K}$,* (ii) *$\mathcal{H}\mathcal{K} = \mathcal{A}$ (where $\mathcal{H}\mathcal{K}$ is the set of finite sums $\sum h_i k_i$: $h_i \in \mathcal{H}$, $k_i \in \mathcal{K}$), and* (iii) $\dim \mathcal{A} = \dim \mathcal{H} \dim \mathcal{K}$. *Then there is an isomorphism $\mathcal{A} \cong \mathcal{H} \otimes_F \mathcal{K}$ (the tensor product of algebras over $F$), given by $hk \leftrightarrow h \otimes k$. (In such cases we shall identify $h$ with $h \otimes 1$, $k$ with $1 \otimes k$).*

The decomposition of $\mathcal{C} = \mathcal{C}[m, (k_i), (\delta_{ij})] = [\alpha_1, \ldots, \alpha_m]$ proceeds as follows.

Suppose that all $\delta_{ij} = 0$, so that $\mathcal{C}$ is commutative. If $m > 1$, $[\alpha_1, \ldots, \alpha_m] = [\alpha_1][\alpha_2, \ldots, \alpha_m]$ since each basis element $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2} \cdots \alpha_m^{\varepsilon_m}$ of $\mathcal{C}$ is the product $(\alpha_1^{\varepsilon_1})(\alpha_2^{\varepsilon_2} \cdots \alpha_m^{\varepsilon_m})$ of elements of $[\alpha_1]$, $[\alpha_2, \ldots, \alpha_m]$ respectively. Also $\dim \mathcal{C} = 2^m = 2 \times 2^{m-1} = \dim[\alpha_1]\dim[\alpha_2, \ldots, \alpha_m]$. So by (2.5)

$$\mathcal{C} \cong [\alpha_1] \otimes [\alpha_2, \ldots, \alpha_m] = \mathbb{C}_{k_1} \otimes \mathcal{C}[m-1, (k_i'), (\delta_{ij}')],$$

where $k_{i-1}' = k_i$ $(2 \leqslant i \leqslant m)$, $\delta_{i-1,j-1}' = \delta_{ij}$ $(2 \leqslant i < j \leqslant m)$.

Suppose on the other hand that some $\delta_{ij} = 1$. We may suppose (possibly after reordering the $\alpha_i$ and correspondingly the $k_i$, $\delta_{ij}$) that $\delta_{12} = 1$, so that $\alpha_1$, $\alpha_2$ anticommute. If $m > 2$, $[\alpha_1, \ldots, \alpha_m] = [\alpha_1, \alpha_2][\alpha_1^{\delta_{23}}\alpha_2^{\delta_{13}}\alpha_3, \ldots, \alpha_1^{\delta_{2m}}\alpha_2^{\delta_{1m}}\alpha_m]$ since each basis element $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ of $\mathcal{C}$ is the product: $(\alpha_1^{\eta_1}\alpha_2^{\eta_2})((\alpha_1^{\delta_{23}}\alpha_2^{\delta_{13}}\alpha_3)^{\varepsilon_3} \cdots (\alpha_1^{\delta_{2m}}\alpha_2^{\delta_{1m}}\alpha_m)^{\varepsilon_m})$ (perhaps divided by the plus or minus a product of some of the $k_i$), where $\eta_1 = \varepsilon_1 + \sum_{j=3}^m \delta_{2j}\varepsilon_j$ reduced mod 2, and $\eta_2 = \varepsilon_2 + \sum_{j=3}^m \delta_{1j}\varepsilon_j$ reduced mod 2. Now products of the $\alpha_1^{\delta_{2i}}\alpha_2^{\delta_{1i}}\alpha_i$ $(3 \leqslant i \leqslant m)$ yield non-zero multiples of precisely $2^{m-2}$ of the basis elements $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ of $\mathcal{C}$, so $\dim \mathcal{C} = 2^2 \times 2^{m-2} = \dim[\alpha_1, \alpha_2]\dim[\alpha_1^{\delta_{23}}\alpha_2^{\delta_{13}}\alpha_3, \ldots, \alpha_1^{\delta_{2m}}\alpha_2^{\delta_{1m}}\alpha_m]$. Also $\alpha_1$ and $\alpha_2$ commute with each $\alpha_1^{\delta_{2i}}\alpha_2^{\delta_{1i}}\alpha_i$. So by (2.5)

$$\mathcal{C} \cong [\alpha_1, \alpha_2] \otimes [\alpha_1^{\delta_{23}}\alpha_2^{\delta_{13}}\alpha_3, \ldots, \alpha_1^{\delta_{2m}}\alpha_2^{\delta_{1m}}\alpha_m] = \mathbb{Q}_{k_1,k_2} \otimes \mathcal{C}[m-2, (k_i'), (\delta_{ij}')],$$

where $k_{i-2}' = (-1)^{\delta_{1i}\delta_{2i}}k_1^{\delta_{2i}}k_2^{\delta_{1i}}k_i$ $(3 \leqslant i \leqslant m)$–found by evaluating $(\alpha_1^{\delta_{2i}}\alpha_2^{\delta_{1i}}\alpha_i)^2$, and $\delta_{i-2,j-2}' = \delta_{1i}\delta_{2j} + \delta_{1j}\delta_{2i} + \delta_{ij}$ reduced mod 2 $(3 \leqslant i < j \leqslant m)$–found by comparing $(\alpha_1^{\delta_{2j}}\alpha_2^{\delta_{1j}}\alpha_j)(\alpha_1^{\delta_{2i}}\alpha_2^{\delta_{1i}}\alpha_i)$ with $(\alpha_1^{\delta_{2i}}\alpha_2^{\delta_{1i}}\alpha_i)(\alpha_1^{\delta_{2j}}\alpha_2^{\alpha_{1j}}\alpha_j)$.

(2.6) Remark. If $\mathcal{C}$ is a Clifford algebra, with all $\delta_{ij} = 1$, then clearly all $\delta_{ij}' = 1$ and so the second factor in this tensor product is again a Clifford algebra.

In general however $\mathcal{C}$ decomposes into an algebra $\mathbb{C}_{b_1}$ or $\mathbb{Q}_{c_1,d_1}$ tensored by a QC algebra $\mathcal{C}[m-1$ or $m-2, (k_i'), (\delta_{ij}')]$ of dimension less than $2^m$. Note that the new parameters $k_i'$, as well as the $b_1$ or $c_1$, $d_1$ are each one of the original $k_i$, or plus or minus a product of some of them.

Induction on $m$ gives the following decomposition theorem:

(2.7) Theorem. *Any QC algebra* $\mathcal{C}[m, (k_i), (\delta_{ij})] = [\alpha_1, \ldots, \alpha_m]$ *is expressible as a tensor product over* $F$:

$$\text{(i)} \quad \mathcal{C} \cong \mathbb{C}_{b_1} \otimes \cdots \otimes \mathbb{C}_{b_r} \otimes \mathbb{Q}_{c_1,d_1} \otimes \cdots \otimes \mathbb{Q}_{c_s,d_s}$$
$$= [\beta_1] \otimes \cdots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \cdots \otimes [\gamma_s, \delta_s]$$

*say where* $r, s \geqslant 0$, $r + 2s = m$, *and each* $b_i$, $c_j$, $d_k$ *is plus or minus a product of some of the* $k_i$. *Each* $\beta_i$, $\gamma_j$, $\delta_k$ *(where* $\beta_i^2 = b_i$, $\gamma_j^2 = c_j$, $\delta_k^2 = d_k$, *and all pairs commute except* $\delta_i\gamma_i = -\gamma_i\delta_i$, $1 \leqslant i \leqslant s$) *is, to within multiplication by plus or* *minus a product of some of the* $k_i$, *one of the basis elements* $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ *of* $\mathcal{C}$. *Conversely each* $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ *is, to within division by plus or minus a product of* $k_i$*'s, one of* $\beta_1^{\theta_1} \cdots \beta_r^{\theta_r}\gamma_1^{\varphi_1}\delta_1^{\psi_1} \cdots \gamma_s^{\varphi_s}\delta_s^{\psi_s}$ *(each* $\theta_i, \varphi_j, \psi_k = 0$ *or* 1*). Thus the latter* $2^{r+2s} = 2^m$ *elements form a new basis of* $\mathcal{C}$, *and* $\{\beta_i, \gamma_j, \delta_k\}$ *is a new set of generators.*

The numbers $r, s$ are invariants of $\mathcal{C}$, as can be deduced from the following:

(2.8) LEMMA. *The centre of* $\mathcal{C} = [\beta_1] \otimes \cdots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \cdots \otimes [\gamma_s, \delta_s]$ $(\beta_i, \gamma_j, \delta_k$ *as in* 2.7) *is the* $2^r$*-dimensional subalgebra* $[\beta_1] \otimes \cdots \otimes [\beta_r]$.

PROOF. Clearly this subalgebra is contained in the centre. Conversely, let $\xi$ be in the centre. Expressing $\xi$ as a linear combination of the basis $\{\beta_1^{\theta_1} \cdots \beta_r^{\theta_r}\gamma_1^{\varphi_1}\delta_1^{\psi_1} \cdots \gamma_s^{\varphi_s}\delta_s^{\psi_s}\}$ we have $\xi = \sum h\beta_1^{\theta_1} \cdots \beta_r^{\theta_r}\gamma_1^{\varphi_1}\delta_1^{\psi_1} \cdots \gamma_s^{\varphi_s}\delta_s^{\psi_s}$ where $h = h(\theta_1, \ldots, \theta_r, \varphi_1, \psi_1, \ldots, \varphi_s, \psi_s)$ is in $F$. For each $i = 1, \ldots, s$, $\xi = \gamma_i^{-1}\xi\gamma_i = \sum \pm h\beta_1^{\theta_1} \cdots \beta_r^{\theta_r}\gamma_1^{\varphi_1}\delta_1^{\psi_1} \cdots \gamma_s^{\varphi_s}\delta_s^{\psi_s}$ where the sign is negative when $\psi_i = 1$ (since $\gamma_i$ commutes with all $\beta_j, \gamma_j, \delta_j$ except $\delta_i$).

Comparison of the two expressions for $\xi$ shows that $h = 0$ for any basis element with $\psi_i = 1$. A similar argument using $\xi = \delta_i^{-1}\xi\delta_i$ shows that for each $i$, $h = 0$ for any basis element with $\varphi_i = 1$. So $\xi \in [\beta_1, \ldots, \beta_r] = [\beta_1] \otimes \cdots \otimes [\beta_r]$ as required.

(2.9) REMARK. The converse of (2.7) is obviously also true–that is, *any algebra of the form* (2.7)(i) *is a QC algebra*. Indeed, regarded as an algebra on the generators $\{\beta_i, \gamma_j, \delta_k\}$, $\mathcal{C}$ of the form (2.7)(i) is the QC algebra $\mathcal{C}[r + 2s, (k_i), (\delta_{ij})]$ where $k_1, \ldots, k_{r+2s} = b_1, \ldots, b_r, c_1, d_1, \ldots, c_s, d_s$, respectively, and all $\delta_{ij} = 0$ except $\delta_{r+2i-1,r+2i} = 1$ for $1 \leqslant i \leqslant s$.

From these facts it is clear that *a tensor product of QC algebras is itself a QC algebra*.

If $\mathcal{C}$ is a Clifford algebra it follows from (2.6) that the decomposition process splits off only $\mathbf{Q}_{c_i,d_i}$ type algebras, except for a possible final commutative algebra on one generator. In other words: *r of* (2.7) *must be* 0 *or* 1 *if* $\mathcal{C}$ *is a Clifford algebra*. Conversely, *any algebra of the form* (2.7)(i) *with* $r = 0$ *or* 1 *is a Clifford algebra*. This can be proved by induction on $s$. For suppose $\mathcal{C}$ is a Clifford algebra on $2s$ generators $[\alpha_1, \ldots, \alpha_{2s}]$. Then

$$\begin{aligned}\mathcal{C} \otimes \mathbf{Q}_{c,d} &= [\alpha_1, \ldots, \alpha_{2s}] \otimes [\gamma, \delta] \text{ say,}\\ &\cong [\alpha_1, \ldots, \alpha_{2s}, \alpha_1 \cdots \alpha_{2s}\gamma, \alpha_1 \cdots \alpha_{2s}\delta]\end{aligned}$$

using (2.5). This is a Clifford algebra on $2s + 2$ generators since, as is easily verified, $\alpha_1, \ldots, \alpha_{2s}, \alpha_1 \cdots \alpha_{2s}\gamma, \alpha_1 \cdots \alpha_{2s}\delta$ all anti-commute. Similarly $\mathcal{C} \otimes \mathbb{C}_b = [\alpha_1, \ldots, \alpha_{2s}] \otimes [\beta]$ say, $\cong [\alpha_1, \ldots, \alpha_{2s}, \alpha_1 \cdots \alpha_{2s}\beta]$ is a Clifford algebra.

Collecting these facts we have the following relationships between the classes of QC algebras and Clifford algebras over a field $F$ of characteristic not 2.

(2.10) THEOREM. *The class of QC algebras over $F$ is the smallest class which is closed under tensor products over $F$ and which contains the Clifford algebras corresponding to non-degenerate quadratic forms over $F$. It is the smallest class which is closed under tensor products over $F$ and contains the algebras $\mathbb{C}_b$, $\mathbb{Q}_{c,d}$ ($b, c, d$ non-zero, in $F$). The Clifford algebras are the QC algebras with 1- or 2-dimensional centres (general QC algebras can have $2^r$-dimensional centres, $r$ any non-negative integer).*

(2.11) THEOREM. *Every QC algebra $\mathcal{C}_F[m, (k_i), (\delta_{ij})]$ ($k_i \neq 0, \in F$; characteristic of $F$ not 2) is semi-simple.*

PROOF. This can be shown by a suitable adaptation of the familiar proof (see [4]) of Maschke's Theorem for group algebras (the $2^m$ basis elements $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$ of a QC algebra behave somewhat like a group with product closed to within non-zero $F$ multiples). Alternatively the decomposition theorem (2.7) may be used:

Let $K$ be the algebraic closure of $F$. Over $K$ all algebras $\mathbb{C}_b = \mathbb{C}_b(K)$ are $2K$ (2.4(i)), and all algebras $\mathbb{Q}_{c,d} = \mathbb{Q}_{c,d}(K)$ are $K_2$ (2.4(iii)). So

$$\begin{aligned} K \otimes_F \mathcal{C} = \mathcal{C}(K) &\cong \mathbb{C}_{b_1}(K) \otimes_K \cdots \otimes_K \mathbb{C}_{b_r}(K) \\ &\quad \otimes_K \mathbb{Q}_{c_1,d_1}(K) \otimes_K \cdots \otimes_K \mathbb{Q}_{c_s,d_s}(K) \\ &\cong 2K \otimes_K \cdots \otimes_K 2K \otimes_K K_2 \otimes_K \cdots \otimes_K K_2 \\ &\cong 2^r K_{2^s} \end{aligned}$$

since $\otimes$ distributes over $\oplus$ and $K_m \otimes_K K_n = K_{mn}$.

Hence $K \otimes_F \mathcal{C}$ being a direct sum of full matrix algebras over $K$ is semi-simple. Hence $\mathcal{C} = \mathcal{C}(F)$ is semi-simple (since if $\mathcal{J}$ was a null ideal of $\mathcal{C}$, $K \otimes_F \mathcal{J}$ would be a null ideal of $K \otimes_F \mathcal{C}$).

## 3. Special quasi Clifford algebras

Being semi-simple any QC algebra has a Wedderburn structure as a direct sum of full matrix algebras over division algebras. We now discuss the possible

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structures of SQC algebras (algebras $\mathcal{C}[m, (k_i), (\delta_{ij})]$ with each $k_i = \pm 1$). (See [5] for a study of general QC algebras.)

As in [7] it is convenient to classify the possible fields $F$ into three types:

I. *$F$ contains $p$ such that $-1 = p^2$;*

II. *$F$ is not of type* I, *but contains $p$, $q$ such that $-1 = p^2 + q^2$;*

III. *$F$ is not of type* I *or* II.

From (2.4) the following summary of structures and representations of $\mathbb{C}_b$, $\mathbb{Q}_{c,d}$ for $b, c, d = \pm 1$ is readily deduced:

(3.1)

| *Algebra* | *Field Type* | *Structure* | *Irreducible Representation(s)* | *Order* |
|---|---|---|---|---|
| $\mathbb{C}_1 = [\beta]$ | any | $2F$ | $\beta \to (1)$ or $\beta \to (-1)$ | 1 |
| $\mathbb{C}_{-1} = [\beta]$ | I | $2F$ | $\beta \to (p)$ or $\beta \to (-p)$ | 1 |
| | II or III | $\mathbb{C}$ | $\beta \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 2 |
| $\mathbb{Q}_{\pm 1,1} = [\gamma, \delta]$ | any | $F_2$ | $\gamma \to \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}$, $\delta \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | 2 |
| $\mathbb{Q}_{1,-1} = [\gamma, \delta]$ | any | $F_2$ | $\gamma \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\delta \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 2 |
| $\mathbb{Q}_{-1,-1} = [\gamma, \delta]$ | I | $F_2$ | $\gamma \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\delta \to \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$ | 2 |
| | II | $F_2$ | $\gamma \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\delta \to \begin{pmatrix} p & -q \\ -q & -p \end{pmatrix}$ | 2 |
| | III | $\mathbb{Q}$ | $\gamma \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\delta \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ | 4 |

*where $\mathbb{C}$ denotes the field $F[\sqrt{-1}\,]$ ($F$ type* II *or* III*), and $\mathbb{Q}$ denotes the quaternion division algebra over $F$ ($F$ type* III*).*

(3.2) REMARK. For fields of type III there are irreducible representations of $\mathbb{C}_b$, $\mathbb{Q}_{c,d}$ ($b, c, d = \pm 1$) in which $\beta, \gamma, \delta$ are each represented by $\{0, \pm 1\}$ matrices with just one non-zero entry in each row and column.

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Now in the decomposition of an SQC algebra, each of the $b_k$, $c_j$, $d_k$ of (2.7(i)) is $\pm 1$. From (3.1) it follows that the decomposition of an SQC algebra takes (possibly after reordering the factors) the form:

(3.3)

$$\begin{aligned}
\mathcal{C} &= [\alpha_1, \ldots, \alpha_m] \\
&\cong 2F \otimes \cdots \otimes 2F \otimes F_2 \otimes \cdots \otimes F_2 \quad (F \text{ type I}) \text{ or} \\
&\quad\; 2F \otimes \cdots \otimes 2F \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes F_2 \otimes \cdots \otimes F_2 \quad (\text{II}) \text{ or} \\
&\quad\; 2F \otimes \cdots \otimes 2F \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes \mathbb{Q} \otimes \cdots \otimes \mathbb{Q} \otimes F_2 \otimes \cdots \otimes F_2 \quad (\text{III}) \\
&\cong [\beta_1] \otimes \cdots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \cdots \otimes [\gamma_s, \delta_s] \quad \text{say}
\end{aligned}$$

*where each $\beta_i$, $\gamma_j$, $\delta_k$ is plus or minus a product of the $\alpha_i$, and conversely each $\alpha_i$ is plus or minus a product of the $\beta_i$, $\gamma_j$, $\delta_k$.*

(3.4) LEMMA. (i) $\mathbb{C} \otimes \mathbb{C} \cong 2F \otimes \mathbb{C}$ (*F type* II *or* III).
(ii) $\mathbb{C} \otimes \mathbb{Q} \cong \mathbb{C} \otimes F_2$ (*F type* III).
(iii) $\mathbb{Q} \otimes \mathbb{Q} \cong F_2 \otimes F_2$ (*F type* III).

PROOF. (i) $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}_{-1} \otimes \mathbb{C}_{-1} = [\beta_1] \otimes [\beta_2]$ (where $\beta_i^2 = -1$) $\cong [\beta_1, \beta_2] \cong [\beta_1\beta_2] \otimes [\beta_1]$ (by 2.5 since $\beta_1\beta_2$, $\beta_1$ commute and generate $[\beta_1, \beta_2]$) $\cong 2F \otimes \mathbb{Q}$ (since $(\beta_1\beta_2)^2 = 1$).

(ii) $\mathbb{C} \otimes \mathbb{Q} = \mathbb{C}_{-1} \otimes \mathbb{Q}_{-1,-1} = [\beta] \otimes [\gamma, \delta]$ (say) $\cong [\beta, \gamma, \delta] \cong [\beta] \otimes [\beta\gamma, \delta]$ (by 2.5) $\cong \mathbb{C} \otimes F_2$ (by 3.1 since $(\beta\gamma)^2 = 1$).

(iii) $\mathbb{Q} \otimes \mathbb{Q} = \mathbb{Q}_{-1,-1} \otimes \mathbb{Q}_{-1,-1} = [\gamma_1, \delta_1] \otimes [\gamma_2, \delta_2]$ (say) $\cong [\gamma_1, \delta_1, \gamma_2, \delta_2] \cong [\gamma_1\delta_2, \delta_1] \cong [\delta_1\gamma_2, \delta_2]$ (by 2.5) $\cong \mathbb{Q}_{1,-1} \otimes \mathbb{Q}_{1,-1} \cong F_2 \otimes F_2$.

Note that in the proof of each part of (3.4) tensor products of algebras in terms of certain generators are converted to tensor products of algebras in terms of new generators. In each case the new generators are certain products of the old, and each old generator is plus or minus a product of the new. Hence

(3.5) COROLLARY. *If* (3.4) *is applied to pairs of factors in* (3.3), *the new generators are still plus or minus products of the original $\alpha_i$, and each $\alpha_i$ is plus or minus a product of the new generators.*

Application of (3.4) sufficiently often to (3.3) yields:

$$(3.6)\qquad C \cong \begin{array}{ll} \text{(i) } 2F \otimes \cdots \otimes 2F \otimes F_2 \otimes \cdots \otimes F_2 & (\text{any type of field}) \text{ or} \\ \text{(ii) } 2F \otimes \cdots \otimes 2F \otimes \mathbb{C} \otimes F_2 \otimes \cdots \otimes F_2 & (\text{type II or III}) \text{ or} \\ \text{(iii) } 2F \otimes \cdots \otimes 2F \otimes \mathbb{Q} \otimes F_2 \otimes \cdots \otimes F_2 & (\text{type III}). \end{array}$$

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If we use the fact that $\otimes$ distributes over $\oplus$, and $F_m \otimes F_n = F_{mn}$ we immediately obtain the possible Wedderburn structures of SQC algebras:

(3.7) THEOREM. *The Wedderburn structure of an SQC algebra* $\mathcal{C} = \mathcal{C}[m, (k_i), (\delta_{ij})]$ $(k_i = \pm 1)$ *as a direct sum of full matrix algebras over division algebras is (depending on* $m$, $(k_i)$, $(\delta_{ij})$*) one of*

(i) $2^r F_{2^s}$ *(any type of field),*

(ii) $2^{r-1}\mathbb{C} \otimes F_{2^s}$ *(type* II *or* III*), or*

(iii) $2^r \mathbb{Q} \otimes F_{2^{s-1}}$ *(type* III*),*

*where in each case* $r + 2s = m$, *and* $2^r$ *is the dimension of the centre. Conversely (as in* 2.9*) any such algebra* (i), (ii) *or* (iii) *is an SQC algebra* $\mathcal{C}[r + 2s, (k_i), (\delta_{ij})]$ $(k_i = \pm 1)$ *with respect to certain generators. Also (as in* 2.10*) the subclass of algebras with structures* (i), (ii) *or* (iii) *for which* $r \leqslant 1$ *is precisely the class of algebras isomorphic to Special Clifford algebras on* $r + 2s$ *generators (the generators anticommuting and squaring to* $\pm 1$*).*

(3.8) COROLLARY. *In case* (i) *of* (3.7) *there are* $2^r$ *inequivalent irreducible representations, of order* $2^s$; *in case* (ii) $2^{r-1}$ *of order* $2^{s+1}$, *and in case* (iii) $2^r$ *of order* $2^{s+1}$. *Any representation must be of order a multiple of* (i) $2^s$, (ii) $2^{s+1}$, (iii) $2^{s+1}$ *respectively.*

Explicit matrix representations may be constructed using (3.1) and (3.6) as follows.

Suppose

$$\begin{aligned}\mathcal{C} &\cong 2F \otimes \cdots \otimes 2F \otimes F_2 \otimes \cdots \otimes F_2 \quad \text{as in 3.6(i)}\\ &= [\beta_1] \otimes \cdots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \cdots \otimes [\gamma_s, \delta_s] \quad \text{say.}\end{aligned}$$

By (3.1) we have irreducible matrix representations for each $[\beta_i]$, $[\gamma_j, \delta_j]$ of orders 1, 2 respectively. Suppose $\beta_i \to B_i$, $\gamma_j \to C_j$ and $\delta_k \to D_k$. Then a representation $\lambda$ say of $\mathcal{C}$ is defined by setting

$$\begin{aligned}&\lambda(\beta_1^{\theta_1} \cdots \beta_r^{\theta_r}\gamma_1^{\varphi_1}\delta_1^{\psi_1} \cdots \gamma_s^{\varphi_s}\delta_s^{\psi_s})\\ &\quad = B_1^{\theta_1} \otimes \cdots \otimes B_r^{\theta_r} \otimes C_1^{\varphi_1}D_1^{\psi_1} \otimes \cdots \otimes C_s^{\varphi_s}D_s^{\psi_s},\end{aligned} \tag{3.9}$$

(each $\theta_i$, $\varphi_j$, $\psi_k = 0$ or 1) where here $\otimes$ denotes the Kroneker product. The Kroneker product $L \otimes M$ of matrices $L$, $M = (m_{ij})$ of orders $l$, $m$ respectively is the $lm$ order matrix formed by replacing each $m_{ij}$ of $M$ by the block $m_{ij}L$ (see [3]–other writers define the Kroneker product in the opposite way: they would denote $(m_{ij}L)$ by $M \otimes L$).

Clearly $\lambda$ is of order $1 \times \cdots \times 1 \times 2 \times \cdots \times 2 = 2^s$ so by 3.8(i) is irreducible. The two inequivalent choices for each $[\beta_i]$ give rise to all $2^r$ inequivalent irreducible representations of $\mathcal{C}$. Now since $\mathcal{C}$ is semi-simple, to within equivalence any representation $\mu$ of $\mathcal{C}$ may be formed from some family $\{\lambda_1, \ldots, \lambda_r\}$ of irreducible representations by defining $\mu(\gamma)$ to be the block-diagonal matrix

$$\begin{bmatrix} \lambda_1(\gamma) & & & \\ & \lambda_2(\gamma) & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix} \quad \text{for all } \gamma \text{ in } \mathcal{C}.$$

Any matrix representation $\mu'$ equivalent to $\mu$ is of course given by $\mu'(\gamma) = T^{-1}\mu(\gamma)T$ for some non-singular matrix $T$ over $F$.

Representations of $\mathcal{C}$ in the cases 3.6(ii), (iii) are formed similarly.

Now the class of $\{0, \pm 1\}$ matrices with just one non-zero entry per row and column is closed under the operations

(a) taking plus or minus ordinary products,

(b) taking Kroneker products,

(c) forming the block-diagonal matrix

$$\begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix}$$

from matrices $X_1$, $X_2$.

So using (3.2) and (3.5) the following can be deduced:

(3.10) THEOREM. *If $F$ is a field of type* III *($-1$ is not the sum of two squares) each representation of an SQC algebra $\mathcal{C}[m, (k_i), (\delta_{ij})]$ ($k_i = \pm 1$) on generators $(\alpha_i)$ is equivalent to a matrix representation in which each $\alpha_i$ corresponds to a $\{0, \pm 1\}$ matrix with just one non-zero entry in each row and column (an orthogonal $\{0, \pm 1\}$ matrix).*

## 4. Orders of systems

We now apply the theory of SQC algebras to the problem of determining, given $p_1 + 1, p_2, \ldots, p_K$ and $(\delta_{ij})$, the possible orders of $K$-systems $(X_i)$ of genus $(\delta_{ij})$ on $p_1 + 1, p_2, \ldots, p_K$ variables.

It has been shown that the existence of such a system of order $n$, whatever its type, implies the existence of an order $n$ representation of a certain real algebra $\mathcal{C}$ say on $m = p_1 + p_2 + \cdots + p_K$ generators (see 1.7). In fact $\mathcal{C}$ is the SQC

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algebra on generators $\alpha_{11}, \ldots, \alpha_{1p_1}, \alpha_{21}, \ldots, \alpha_{2p_2}, \ldots, \alpha_{K1}, \ldots, \alpha_{Kp_K}$ with defining relations

$$(4.1) \qquad \begin{aligned} &\alpha_{ij}^2 = (-1)^{\delta_{1i}}, \qquad \alpha_{ik}\alpha_{ij} = -\alpha_{ij}\alpha_{ik}, \qquad (j \neq k) \\ &\alpha_{jl}\alpha_{ik} = (-1)^{\delta_{1i}+\delta_{1j}+\delta_{ij}}\alpha_{ik}\alpha_{jl}, \qquad (i < j), \end{aligned}$$

where $\delta_{11}$ is to be interpreted as 1.

Hence by (3.7, 3.8) the possible orders of the system are restricted to multiples of $\rho$, where $\rho = 2^s, 2^{s+1}, 2^{s+1}$ according as $\mathcal{C}$ has structure (3.7)(i), (ii), (iii) respectively (here $F$ is the reals, a field of type III). Call this number $\rho$ the *order number* of the family $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$.

Conversely let $n$ be any multiple of the order number of $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$. Then by (3.10) an order $n$ matrix representation of the algebra (4.1) can be constructed in which each $\alpha_{ij}$ is represented by a $\{0, \pm 1\}$ matrix, $E_{ij}$ say, with just one non-zero entry per row and column. The matrices $E_{ij}$ satisfy $E_{ij}^2 = (-1)^{\delta_{1i}}I$, $E_{ik}E_{ij} = -E_{ij}E_{ik}$ $(j \neq k)$, $E_{jl}E_{ik} = (-1)^{\delta_{1i}+\delta_{1j}+\delta_{ij}}E_{ik}E_{jl}$ $(i < j)$ and are orthogonal. It follows that $E_{ij}^T = (-1)^{\delta_{1i}}E_{ij}$ for all $i, j$. Hence since $\delta_{11} = 1$, all $E_{1j}$ are anti-symmetrical, so $I * E_{1j} = 0$. Also if $1 \leqslant j < k \leqslant p_i$, then $E_{ik}E_{ij}^T = (-1)^{\delta_{1i}}E_{ik}E_{ij} = -(-1)^{\delta_{1i}}E_{ij}E_{ik} = -E_{ij}E_{ik}^T$. It follows that $E_{ij} * E_{ik} = 0$ since $E_{ij}, E_{ik}$ have only one non-zero entry per row (as in the proof of 1.5).

Hence we may form designs $X_1 = x_{10}I + x_{11}E_{11} + \cdots + x_{1p_1}E_{1p_1}$, $X_2 = x_{21}E_{21} + \cdots + x_{2p_2}E_{2p_2}, \ldots, X_K = x_{K1}E_{K1} + \cdots + x_{Kp_K}E_{Kp_K}$ on the distinct variables $(x_{ij})$.

Now $E_{1j}I^T = -IE_{1j}^T$, $E_{ik}E_{ij}^T = -E_{ij}E_{ik}^T$ $(j \neq k)$, and $E_{ij}E_{ij}^T = (-1)^{\delta_{1i}}E_{ij}^2 = I$. It follows that each $X_i$ is orthogonal. Since for $i < j$, $E_{jl}E_{ik}^T = (-1)^{\delta_{1i}}E_{jl}E_{ik} = (-1)^{\delta_{1i}+\delta_{1i}+\delta_{1j}+\delta_{ij}}E_{ik}E_{jl} = (-1)^{\delta_{ij}}E_{ik}E_{jl}^T$, it follows that $X_jX_i^T = (-1)^{\delta_{ij}}X_iX_j^T$. So $(X_i)$ is a $K$-system of genus $(\delta_{ij})$, order $n$. Its type is $(1, 1, \ldots, 1; \cdots; 1, 1, \ldots, 1)$ so by (1.5) it is regular. We have shown:

(4.2) THEOREM. *Suppose $\rho$ is the order number of $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij})_{1 \leqslant i < j \leqslant K})$ $(p_i \geqslant 0, \delta_{ij} = 0$ or $1)$. Then any $K$-system of any type, genus $(\delta_{ij})$, on $p_1 + 1, p_2, \ldots, p_K$ variables has order a multiple of $\rho$. If $n$ is a multiple of $\rho$ there is a regular $K$-system of order $n$, type $(1, 1, \ldots, 1; \cdots; 1, 1, \ldots, 1)$, genus $(\delta_{ij})$ on $p_1 + 1, p_2, \ldots, p_k$ variables.*

Note that this theorem does not give information on which multiples of $\rho$ are the possible orders of $K$-systems (genus $(\delta_{ij})$, on $p_1 + 1, p_2, \ldots, p_K$ variables) for types other than $(1, 1, \ldots, 1; \cdots; 1, 1, \ldots, 1)$.

By considering the corresponding SQC algebra we are now able, given $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$, to calculate the order number $\rho$, and produce suitable regular $K$-systems of order any multiple of $\rho$. In practice we may wish to produce tables of order numbers of $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$ for fixed $K$, $(\delta_{ij})$ and varying $p_i$. The following result is useful in this context.

(4.3) THEOREM. *If the order number of* $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$ *is* $\rho$, *then the order number of each of* $(p_1 + 9, p_2, \ldots, p_K, (\delta_{ij}))$, $(p_1 + 1, p_2 + 8, p_3, \ldots, p_K; (\delta_{ij})), \ldots, (p_1 + 1, p_2, \ldots, p_{K-1}, p_K + 8; (\delta_{ij}))$ *is* $2^4\rho$.

That is, increasing any $p_i$ by 8 multiplies the order number of 16. This means that from a table giving order numbers for the $8^K$ cases $0 \leqslant p_i \leqslant 7$, the order number for any other values of $(p_i)$ is readily calculated.

PROOF. Let $\mathcal{C} = [\alpha_1, \ldots, \alpha_m]$ be the SQC algebra (4.1) corresponding to $(p_1 + 1, p_2, \ldots, p_K; (\delta_{ij}))$ (where the notation of the generators has been simplified), and let $\mathcal{C}' = [\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_8]$ be the corresponding SQC algebra when $p_j$ say is increased by 8. From (4.1) it is clear that the $\beta_1, \ldots, \beta_8$ anti-commute with each other and either all square to 1 or all square to $-1$. Also each $\alpha_i$ either commutes with all of $\beta_1, \ldots, \beta_8$ or anticommutes with all of them. By (2.5)

$$\mathcal{C}' \cong \left[\alpha_1(\beta_1 \cdots \beta_8)^{\varepsilon_1}, \alpha_2(\beta_1 \cdots \beta_8)^{\varepsilon_2}, \ldots, \alpha_m(\beta_1 \cdots \beta_8)^{\varepsilon_m}\right] \otimes \left[\beta_1, \ldots, \beta_8\right],$$

where $\varepsilon_i = 0$ if $\alpha_i$ commutes with the $\beta_1, \ldots, \beta_8$, and $= 1$ otherwise, (since each $\alpha_i(\beta_1 \cdots \beta_8)^{\varepsilon_i}$ clearly commutes with each $\beta_j$).

Now $(\alpha_i(\beta_1 \cdots \beta_8)^{\varepsilon_i})^2 = \alpha_i^2$ and $\alpha_i(\beta_1 \cdots \beta_8)^{\varepsilon_i}$ commutes or anti-commutes with $\alpha_j(\beta_1 \cdots \beta_8)^{\varepsilon_j}$ according as $\alpha_i$ commutes or anti-commutes with $\alpha_j$. So $\mathcal{C}' \cong \mathcal{C} \otimes [\beta_1, \ldots, \beta_8]$. Decomposition yields:

$$\begin{aligned}
[\beta_1, \ldots, \beta_8] &\cong [\beta_1, \beta_2] \otimes [\beta_1\beta_2\beta_3, \beta_1\beta_2\beta_4] \otimes [\beta_1\beta_2\beta_3\beta_4\beta_5, \beta_1\beta_2\beta_3\beta_4\beta_6] \\
&\quad \otimes [\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7, \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_8] \\
&\cong \begin{cases} F_2 \otimes \mathbb{Q} \otimes F_2 \otimes \mathbb{Q} & (\text{if } \beta_i^2 = 1) \\ \mathbb{Q} \otimes F_2 \otimes \mathbb{Q} \otimes F_2 & (\text{if } \beta_i^2 = -1) \end{cases}, \text{ using (3.1)} \\
&\cong F_2 \otimes F_2 \otimes F_2 \otimes F_2 \quad (3.4) \\
&\cong F_{2^4}.
\end{aligned}$$

Hence $\mathcal{C}' \cong \mathcal{C} \otimes F_{16}$. So if the structure of $\mathcal{C}$ is $2^k\mathfrak{D} \otimes F_n$ ($\mathfrak{D}$ a division algebra) then the structure of $\mathcal{C}'$ is $2^k\mathfrak{D} \otimes F_{2^4n}$. The result follows.

It may be remarked that the process of constructing an SQC algebra from a $K$-system $(X_i)$ on $(p_1 + 1, p_2, \ldots, p_K)$ variables seems to involve a certain lack of symmetry, in that while any of $p_2, \ldots, p_K$ could be zero (in which case in an obvious sense $(X_i)$ is equivalent to some $(K - 1)$ system), it would appear that

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$p_1 + 1$ cannot. However this restriction (on the number of variables in $X_1$) may be removed since in (1.6) $u_{10}$, $A_{10}$ can be replaced by $u_{ij}$, $A_{ij}$ respectively for any other $i, j$. Different SQC algebras (4.1) may arise in this way, but in the light of (4.2) it is clear that their irreducible representations would be of the same order. It also follows, in the light of (4.3), that the order number of $(0, p_2, \ldots, p_K; (\delta_{ij}))$ (obtained using an alternative to (1.6)) is $\frac{1}{16}$th the order number of $(8, p_2, \ldots, p_K; (\delta_{ij}))$, and (4.3) remains valid.

For convenience we define the order number of $(0, 0, \ldots, 0; (\delta_{ij}))$ to be $2^{-1}$. This is consistent with (4.3) since, for example, the order number of $(8, 0, \ldots, 0; (\delta_{ij}))$ is $2^3$ (that is, the minimal order of single orthogonal designs on 8 variables is 8, as is well-known: see [6]).

## 5. Product designs

We conclude by considering a particular genus of 3-systems–the product designs (1.4) which have proved of particular value in the theory of orthogonal designs (see [6], [8]). Here $K = 3$, $\delta_{12} = \delta_{13} = -1$, $\delta_{23} = 1$. Suppose we have a product design on $(p + 1, q, r)$ variables. Write $p = 2l(+1)$, $q = 2m(+1)$, $r = 2n(+1)$. For convenience we rename the generators in (4.1):

$$\mathcal{C} = \left[\alpha_1, \bar{\alpha}_1, \ldots, \alpha_l, \bar{\alpha}_l(,\bar{\bar{\alpha}}), \beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m\left(,\bar{\bar{\beta}}\right), \gamma_1, \bar{\gamma}_1, \ldots, \gamma_n, \bar{\gamma}_n(,\bar{\bar{\gamma}})\right]$$

$$\left(\text{where } \bar{\bar{\alpha}}, \bar{\bar{\beta}}, \bar{\bar{\gamma}} \text{ respectively is included when } p, q, r \text{ respectively is odd}\right)$$

$$= \left[\alpha_1, \bar{\alpha}_1, \ldots, \alpha_l, \bar{\alpha}_l, \beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m, \gamma_1, \bar{\gamma}_1, \ldots, \gamma_n, \bar{\gamma}_n(,\bar{\bar{\alpha}})\left(,\bar{\bar{\beta}}\right)(,\bar{\bar{\gamma}})\right].$$

Here every generator squares to $-1$, and all pairs anti-commute except that each $\beta$ commutes with each $\gamma$.

The decomposition process yields:

$$\begin{aligned}
\mathcal{C} \cong & \left[\alpha_1, \bar{\alpha}_1\right] \otimes \left[\alpha_1\bar{\alpha}_1\alpha_2, \alpha_1\bar{\alpha}_1\bar{\alpha}_2\right] \otimes \cdots \\
& \otimes \left[\alpha_1\bar{\alpha}_1 \cdots \alpha_{l-1}\bar{\alpha}_{l-1}\alpha_l, \alpha_1\bar{\alpha}_1 \cdots \alpha_{l-1}\bar{\alpha}_{l-1}\bar{\alpha}_l\right] \\
& \otimes \left[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1, \alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\beta}_1\right] \otimes \cdots \\
& \otimes \left[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_{m-1}\bar{\beta}_{m-1}\beta_m,\right. \\
& \qquad\qquad \left.\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_{m-1}\bar{\beta}_{m-1}\bar{\beta}_m\right] \\
& \otimes \left[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1, \alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\gamma}_1\right] \otimes \cdots \\
& \otimes \left[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1\bar{\gamma}_1 \cdots \gamma_{n-1}\bar{\gamma}_{n-1}\gamma_n, \alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1\bar{\gamma}_1 \cdots \gamma_{n-1}\bar{\gamma}_{n-1}\bar{\gamma}_n\right]
\end{aligned}$$

$\otimes$either (i) no other factors ($p$ even, $q$ even, $r$ even) or

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(ii) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\bar{\alpha}}\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n]$ (odd, even, even) or

(iii) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\bar{\bar{\beta}}]$ (even, odd, even) or

(iv) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n\bar{\bar{\gamma}}]$ (even, even, odd) or

(v) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\bar{\bar{\beta}}]$
$\otimes[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n\bar{\bar{\gamma}}]$ (even, odd, odd) or

(vi) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\bar{\alpha}}\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n,$
$\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\bar{\bar{\beta}}]$ (odd, odd, even) or

(vii) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\bar{\alpha}}\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n,$
$\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n\bar{\bar{\gamma}}]$ (odd, even, odd) or

(viii) $[\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\bar{\bar{\alpha}}\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n,$
$\alpha_1\bar{\alpha}_1 \cdots \alpha_l\bar{\alpha}_l\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\bar{\bar{\beta}}]$
$\otimes[\beta_1\bar{\beta}_1 \cdots \beta_m\bar{\beta}_m\bar{\bar{\beta}}\gamma_1\bar{\gamma}_1 \cdots \gamma_n\bar{\gamma}_n\bar{\bar{\gamma}}]$ (odd, odd, odd)

$$\cong \mathbb{Q} \otimes F_2 \otimes \mathbb{Q} \otimes F_2 \otimes \cdots \otimes \begin{cases} \mathbb{Q} & (l \text{ odd}) \\ F_2 & (l \text{ even}) \end{cases}$$

$$\otimes \begin{cases} F_2 & (l \text{ odd}) \\ \mathbb{Q} & (l \text{ even}) \end{cases} \otimes \cdots \otimes \begin{Bmatrix} \mathbb{Q} & (l+m \text{ odd}) \\ F_2 & (l+m \text{ even}) \end{Bmatrix}$$

$$\otimes \begin{cases} F_2 & (l \text{ odd}) \\ \mathbb{Q} & (l \text{ even}) \end{cases} \otimes \cdots \otimes \begin{cases} \mathbb{Q} & (l+n \text{ odd}) \\ F_2 & (l+n \text{ even}) \end{cases}$$

(i) $\otimes$ no other factors

(ii) $\begin{cases} 2F & (l+m+n \text{ odd}) \\ \mathbb{C} & (l+m+n \text{ even}) \end{cases}$

(iii) $\begin{cases} 2F & (l+m \text{ odd}) \\ \mathbb{C} & (l+m \text{ even}) \end{cases}$

(iv) $\begin{cases} 2F & (l+n \text{ odd}) \\ \mathbb{C} & (l+n \text{ even}) \end{cases}$

(v) $\begin{cases} 4F & (l+m,\ l+n \text{ both odd}) \\ 2\mathbb{C} & (\text{otherwise}) \end{cases}$

(vi) $\begin{cases} \mathbb{Q} & (l+m+n,\ l+m \text{ both even}) \\ F_2 & (\text{otherwise}) \end{cases}$

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$$\text{(vii)} \begin{cases} \mathbb{Q} & (l+m+n,\ l+n \text{ both even}) \\ F_2 & (\text{otherwise}) \end{cases}$$

$$\text{(viii)} \begin{cases} \mathbb{C} \otimes F_2 & (m+n \text{ odd}) \\ 2\mathbb{Q} & (m+n,\ l+m+n,\ l+m \text{ all even}) \\ 2F_2 & (\text{otherwise}) \end{cases}$$

with (i), (ii), . . . , (viii) as above.

Thus the structures of $\mathcal{C}$ for each $p, q, r$ can be obtained, giving the order numbers for each $(p+1, q, r)$.

(5.1) EXAMPLE. We find the possible orders of product designs on (2, 2, 4) variables and construct such a design of minimal order.

$$\begin{aligned}
\mathcal{C} &= \left[\beta_1, \bar{\beta}_1, \gamma_1, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_2, \bar{\bar{\alpha}}\right] \\
&\cong \left[\beta_1, \bar{\beta}_1\right] \otimes \left[\gamma_1, \bar{\gamma}_1\right] \otimes \left[\gamma_1\bar{\gamma}_1\gamma_2, \gamma_1\bar{\gamma}_1\bar{\gamma}_2\right] \otimes \left[\bar{\bar{\alpha}}\beta_1\bar{\beta}_1\gamma_1\bar{\gamma}_1\gamma_2\bar{\gamma}_2\right] \\
&\cong \mathbb{Q} \otimes \mathbb{Q} \otimes F_2 \otimes 2F \\
&\cong \left[\beta_1\bar{\gamma}_1, \bar{\beta}_1\right] \otimes \left[\bar{\beta}_1\gamma_1, \bar{\gamma}_1\right] \otimes \left[\gamma_1\bar{\gamma}_1\gamma_2, \gamma_1\bar{\gamma}_1\bar{\gamma}_2\right] \otimes \left[\bar{\bar{\alpha}}\beta_1\bar{\beta}_1\gamma_1\bar{\gamma}_1\gamma_2\bar{\gamma}_2\right] \text{ (using 3.4)} \\
&\cong F_2(=\mathbb{Q}_{1,-1}) \otimes F_2(=\mathbb{Q}_{1,-1}) \otimes F_2(=\mathbb{Q}_{1,1}) \otimes 2F(=\mathbb{C}_1) \\
&\cong 2F_{2^3}.
\end{aligned}$$

So the order number is $2^3 = 8$ (3.8(i)), and the possible orders of product designs (1.3) on (2, 2, 4) variables are multiples of 8. Such a product design of order 8 may be constructed as follows:

$$\begin{aligned}
\pm\bar{\bar{\alpha}} &= (\beta_1\bar{\gamma}_1) \otimes (\bar{\beta}_1\gamma_1) \otimes (\gamma_1\bar{\gamma}_1\gamma_2, \gamma_1\bar{\gamma}_1\bar{\gamma}_2) \otimes (\bar{\bar{\alpha}}\beta_1\bar{\beta}_1\gamma_1\bar{\gamma}_1\gamma_2\bar{\gamma}_2) \\
&\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes (1) \\
&= \begin{bmatrix} & & & & 0 & 0 & 0 & -1 \\ & & \bigcirc & & 0 & 0 & -1 & 0 \\ & & & & 0 & -1 & 0 & 0 \\ & & & & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & \bigcirc & \\ 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \end{bmatrix} = E_1 \text{ say}
\end{aligned}$$

(by (3.1)).

Similarly

$$\begin{aligned}
\pm\beta_1 &= (\beta_1\bar{\gamma}_1) \otimes (\bar{\gamma}_1) \otimes 1 \otimes 1 \\
&\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1) \quad = F_1 \text{ say,}
\end{aligned}$$

$$\begin{aligned}
\pm\bar{\beta}_1 &= (\bar{\beta}_1)\otimes 1\otimes 1\otimes 1\\
&\to \begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes(1) &&= F_2 \text{ say},\\
\pm\gamma_1 &= (\bar{\beta}_1)\otimes(\bar{\beta}_1\gamma_1)\otimes 1\otimes 1\\
&\to \begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes(1) &&= G_1 \text{ say},\\
\pm\bar{\gamma}_1 &= 1\otimes(\bar{\gamma}_1)\otimes 1\otimes 1\\
&\to \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes(1) &&= G_2 \text{ say},\\
\pm\gamma_2 &= (\bar{\beta}_1)\otimes(\bar{\beta}_1\gamma_1\bar{\gamma}_1)\otimes(\gamma_1\bar{\gamma}_1\gamma_2)\otimes 1\\
&\to \begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes(1) &&= G_3 \text{ say},\\
\pm\bar{\gamma}_2 &= (\bar{\beta}_1)\otimes(\bar{\beta}_1\gamma_1\bar{\gamma}_1)\otimes(\gamma_1\bar{\gamma}_1\bar{\gamma}_2)\otimes 1\\
&= \begin{pmatrix}0&-1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes(1) &&= G_4 \text{ say}.
\end{aligned}$$

Set $X = x_0 I + x_1 E_1$, $Y = y_1 F_1 + y_2 F_2$, $Z = z_1 G_1 + \cdots + z_4 G_4$. Then $(X, Y, Z)$ is the product design:

$$\begin{bmatrix}
x_0 & \bar{y}_2\bar{z}_4 & \bar{z}_2 & \bar{y}_1\bar{z}_1 & 0 & \bar{z}_3 & 0 & \bar{x}_1\\
y_2 z_4 & x_0 & \bar{y}_1 z_1 & \bar{z}_2 & z_3 & 0 & \bar{x}_1 & 0\\
z_2 & y_1\bar{z}_1 & x_0 & \bar{y}_2 z_4 & 0 & \bar{x}_1 & 0 & z_3\\
y_1 z_1 & z_2 & y_2\bar{z}_4 & x_0 & \bar{x}_1 & 0 & \bar{z}_3 & 0\\
0 & \bar{z}_3 & 0 & x_1 & x_0 & \bar{y}_2 z_4 & \bar{z}_2 & \bar{y}_1\bar{z}_1\\
z_3 & 0 & x_1 & 0 & y_2\bar{z}_4 & x_0 & \bar{y}_1 z_1 & \bar{z}_2\\
0 & x_1 & 0 & z_3 & z_2 & y_1\bar{z}_1 & x_0 & \bar{y}_2\bar{z}_4\\
x_1 & 0 & \bar{z}_3 & 0 & y_1 z_1 & z_2 & y_2 z_4 & x_0
\end{bmatrix}$$

(superimposing $X$, $Y$, $Z$ for convenience, and writing $\bar{x}_1$ for $-x_1$, etc).

Similarly each of the $8^3 = 512$ cases $p, q, r$ (mod 8) can be considered. A suitable computer program (see [5]) quickly produces the appended tables of order numbers for product designs $(p + 1, q, r)$, in which the entries are the powers to base 2 of the order numbers.

The case $p + 1 = 0$ reduces in effect to a table of order numbers for amicable pairs and, if two of $p + 1, q, r$ are zero, order numbers of single orthogonal designs result.

From this table order numbers for $p + 1, q, r > 7$ can quickly be deduced, using (4.3).

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## Appendix

*Table of order numbers (indices base 2) of product designs* ($p + 1, q, r$)

$p + 1 = 0$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | −1 | 0 | 1 | 2 | 2 | 3 | 3 | 3 |
| 1 | 0 | 0 | 1 | 2 | 3 | 3 | 4 | 4 |
| 2 | 1 | 1 | 1 | 2 | 3 | 4 | 4 | 5 |
| 3 | 2 | 2 | 2 | 2 | 3 | 4 | 5 | 5 |
| 4 | 2 | 3 | 3 | 3 | 3 | 4 | 5 | 6 |
| 5 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 6 |
| 6 | 3 | 4 | 4 | 5 | 5 | 5 | 5 | 6 |
| 7 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |

$p + 1 = 1$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 |
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 2 | 2 | 2 | 2 | 2 | 3 | 4 | 5 | 5 |
| 3 | 2 | 2 | 2 | 2 | 3 | 4 | 5 | 5 |
| 4 | 3 | 3 | 3 | 3 | 4 | 5 | 6 | 6 |
| 5 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |
| 6 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |
| 7 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |

$p + 1 = 2$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 5 |
| 2 | 2 | 3 | 3 | 3 | 3 | 4 | 5 | 6 |
| 3 | 3 | 3 | 3 | 3 | 4 | 5 | 6 | 6 |
| 4 | 3 | 3 | 3 | 4 | 5 | 6 | 6 | 7 |
| 5 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 6 | 3 | 4 | 5 | 6 | 6 | 7 | 7 | 7 |
| 7 | 4 | 5 | 6 | 6 | 7 | 7 | 7 | 7 |

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$p + 1 = 3$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 5 |
| 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 5 |
| 2 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 6 |
| 3 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |
| 4 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 6 | 4 | 4 | 5 | 6 | 7 | 7 | 8 | 8 |
| 7 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 |

$p + 1 = 4$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 2 | 3 | 3 | 3 | 3 | 4 | 5 | 6 |
| 1 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 6 |
| 2 | 3 | 4 | 4 | 5 | 5 | 5 | 5 | 6 |
| 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |
| 4 | 3 | 4 | 5 | 6 | 6 | 7 | 7 | 7 |
| 5 | 4 | 4 | 5 | 6 | 7 | 7 | 8 | 8 |
| 6 | 5 | 5 | 5 | 6 | 7 | 8 | 8 | 9 |
| 7 | 6 | 6 | 6 | 6 | 7 | 8 | 9 | 9 |

$p + 1 = 5$

| $r =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q = 0$ | 3 | 3 | 3 | 3 | 4 | 5 | 6 | 6 |
| 1 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |
| 2 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |
| 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 |
| 4 | 4 | 5 | 6 | 6 | 7 | 7 | 7 | 7 |
| 5 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 |
| 6 | 6 | 6 | 6 | 6 | 7 | 8 | 9 | 9 |
| 7 | 6 | 6 | 6 | 6 | 7 | 8 | 9 | 9 |

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$p+1=6$

| $r=$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q=0$ | 3 | 3 | 3 | 4 | 5 | 6 | 6 | 7 |
| 1 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 2 | 3 | 4 | 5 | 6 | 6 | 7 | 7 | 7 |
| 3 | 4 | 5 | 6 | 6 | 7 | 7 | 7 | 7 |
| 4 | 5 | 6 | 6 | 7 | 7 | 7 | 7 | 8 |
| 5 | 6 | 6 | 7 | 7 | 7 | 7 | 8 | 9 |
| 6 | 6 | 7 | 7 | 7 | 7 | 8 | 9 | 10 |
| 7 | 7 | 7 | 7 | 7 | 8 | 9 | 10 | 10 |

$p+1=7$

| $r=$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $q=0$ | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 1 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 2 | 4 | 4 | 5 | 6 | 7 | 7 | 8 | 8 |
| 3 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 |
| 4 | 6 | 6 | 7 | 7 | 7 | 7 | 8 | 9 |
| 5 | 6 | 6 | 7 | 7 | 7 | 7 | 8 | 9 |
| 6 | 7 | 7 | 8 | 8 | 8 | 8 | 9 | 10 |
| 7 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 |

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