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Closure of the Cone of Sums of $2d$-powers in Certain Weighted ℓ_1-seminorm Topologies

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Abstract. In a paper from 1976, Berg, Christensen, and Ressel prove that the closure of the cone of sums of squares $\sum \mathbb{R}[\underline{X}]^2$ in the polynomial ring $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$ in the topology induced by the ℓ_1-norm is equal to $\operatorname{Pos}([-1,1]^n)$, the cone consisting of all polynomials that are non-negative on the hypercube $[-1,1]^n$. The result is deduced as a corollary of a general result, established in the same paper, which is valid for any commutative semigroup. In later work, Berg and Maserick and Berg, Christensen, and Ressel establish an even more general result, for a commutative semigroup with involution, for the closure of the cone of sums of squares of symmetric elements in the weighted ℓ_1-seminorm topology associated with an absolute value. In this paper we give a new proof of these results, which is based on Jacobi's representation theorem from 2001. At the same time, we use Jacobi's representation theorem to extend these results from sums of squares to sums of $2d$-powers, proving, in particular, that for any integer $d \geq 1$, the closure of the cone of sums of $2d$-powers $\sum \mathbb{R}[\underline{X}]^{2d}$ in $\mathbb{R}[\underline{X}]$ in the topology induced by the ℓ_1-norm is equal to $\operatorname{Pos}([-1,1]^n)$.

1 Introduction

We denote the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$ by $\mathbb{R}[\underline{X}]$ for short. It was shown by Hilbert [12] that for $n \geq 2$ there are polynomials in $\mathbb{R}[\underline{X}]$ that are non-negative on all of $\mathbb{R}^n$ but are not in the cone $\sum \mathbb{R}[\underline{X}]^2$ consisting of sums of squares. The first explicit example was given by Motzkin [20]. Today, many examples are known, *e.g.*, see [6]. In [2] Berg, Christensen, and Jensen prove that, in the finest locally convex topology, $\sum \mathbb{R}[\underline{X}]^2$ is closed in $\mathbb{R}[\underline{X}]$; also see [22].

In marked contrast to this result, in [3] Berg, Christensen, and Ressel show that the closure of $\sum \mathbb{R}[\underline{X}]^2$ in $\mathbb{R}[\underline{X}]$ in the topology induced by the ℓ_1-norm is equal to $\operatorname{Pos}([-1,1]^n)$, the set of all polynomials in $\mathbb{R}[\underline{X}]$ that are non-negative on $[-1,1]^n$. In [3] the aforementioned result is established in the general context of commutative semigroups. In [4] and [5] the results in [3] are extended further to include commutative semigroups with involution and topologies induced by absolute values.

Let d be a positive integer and let $M \subseteq \mathbb{R}[\underline{X}]$ be a $\sum \mathbb{R}[\underline{X}]^{2d}$-module that is archimedean. In [13] Jacobi proves that any $f \in \mathbb{R}[\underline{X}]$ that is strictly positive on

$$K_M := \{\underline{x} \in \mathbb{R}^n \mid g(\underline{x}) \geq 0 \ \text{ for all } \ g \in M\}$$

belongs to M. Actually, Jacobi proves a more general version of this result that is valid for any commutative ring A with 1; see Section 2. There is special interest in the case

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where $d = 1$, $A = \mathbb{R}[\underline{X}]$ and M is finitely generated, because of the application to polynomial optimization in this case; see [16]. Jacobi's result in this case can be seen as a consequence of Putinar's criterion in [21].

In this paper we use Jacobi's theorem to give a new proof of the result of Berg, Christensen, and Ressel in [3] referred to above. At the same time we use Jacobi's theorem to extend this result, proving, for any integer $d \geq 1$, that the closure of $\sum \mathbb{R}[\underline{X}]^{2d}$ in $\mathbb{R}[\underline{X}]$ in the topology induced by the ℓ_1-norm is equal to $\mathrm{Pos}([-1,1]^n)$. As in [3–5], extensions of this result to absolute values on commutative semigroups with involution are also developed.

In Section 2 we provide necessary background. In Sections 3, 4, and 5 we explain how Jacobi's result can be exploited to prove the aforementioned results of Berg et al., and also how it can be used to generalize these results, replacing 2 by $2d$. Special attention is paid to the polynomial case, *i.e.,* the case where the semigroup in question is $(\mathbb{N}^n, +)$; see Section 3. In Section A, we explain how the simple proof of Jacobi's result in the case $d = 1$ given in [19, Theorem 5.4.4] can be extended to the case $d > 1$.

2 Background

Let A be a commutative ring with 1. For simplicity assume that $\mathbb{Q} \subseteq A$. Denote by X_A the set of all (unitary) ring homomorphisms $\alpha\colon A \to \mathbb{R}$. For $a \in A$, define $\widehat{a}\colon X_A \to \mathbb{R}$ by $\widehat{a}(\alpha) = \alpha(a)$. Give X_A the weakest topology making each $\widehat{a}$, $a \in A$ continuous. We have a ring homomorphism $\widehat{\ }\colon A \to \mathrm{Cont}(X_A, \mathbb{R})$ defined by $a \mapsto \widehat{a}$.[1]

Example 2.1 If $A = \mathbb{R}[\underline{X}]$, X_A is naturally identified with $\mathbb{R}^n$ via $\alpha \leftrightarrow \underline{x} := (\alpha(X_1), \dots, \alpha(X_n))$, and $\widehat{a}(\alpha) = a(\underline{x})$, *i.e.,* $\widehat{a}$ is the polynomial function on $\mathbb{R}^n$ associated with the polynomial a. Here one uses the fact that the only ring homomorphism from $\mathbb{R}$ to $\mathbb{R}$ is the identity map; see [19, Proposition 5.4.5].

We will be interested in the map $a \mapsto \widehat{a}|_{\mathcal{K}}$ from A to $\mathrm{Cont}(\mathcal{K}, \mathbb{R})$, where $\mathcal{K}$ is a subset of X_A. We record the following result.

Theorem 2.2 *Suppose A is an $\mathbb{R}$-algebra and $\mathcal{K}$ is a compact subset of X_A. Then*

(i) *the image of A in $\mathrm{Cont}(\mathcal{K}, \mathbb{R})$ is dense in the topology induced by the sup norm $\|\phi\| := \sup\{|\phi(\alpha)| \mid \alpha \in \mathcal{K}\}$;*

(ii) *if $L\colon A \to \mathbb{R}$ is an $\mathbb{R}$-linear map satisfying $L(\mathrm{Pos}(\mathcal{K})) \subseteq \mathbb{R}_{\geq 0}$, then there exists a unique positive Borel measure μ on $\mathcal{K}$ such that for all $a \in A$, $L(a) = \int \widehat{a}\, d\mu$.*

Here, $\mathrm{Pos}(\mathcal{K}) := \{a \in A \mid \widehat{a} \geq 0 \text{ on } \mathcal{K}\}$.

Proof (i) This is immediate from the Stone–Weierstrass Approximation Theorem.

(ii) L vanishes on the kernel of $\widehat{\ }|_{\mathcal{K}}$ so L induces an $\mathbb{R}$-linear map $L'\colon A' \to \mathbb{R}$, where A' is the image of A under $\widehat{\ }|_{\mathcal{K}}$. An application of the Hahn–Banach Theorem shows that L' extends to a positive $\mathbb{R}$-linear map $L''\colon \mathrm{Cont}(\mathcal{K}, \mathbb{R}) \to \mathbb{R}$. The density of A' in $\mathrm{Cont}(\mathcal{K}, \mathbb{R})$ implies the extension L'' of L' is unique. The existence and uniqueness of μ now follow using the Riesz Representation Theorem. ■

[1] We will abuse the notation occasionally, denoting $\widehat{a}(\alpha)$ by $a(\alpha)$ and $\widehat{a}$ by a.

Remark 2.3 (i) Theorem 2.2(ii) is well known. It is implicit, for example, in the proof of [15, Théorème 14]. In the special case $A = \mathbb{R}[\underline{X}]$ it is a consequence of Haviland's Theorem; see [10, 11]. See [19, Theorem 3.2.2] for a general result that includes Theorem 2.2(ii) and Haviland's Theorem as special cases.

(ii) Observe that the converse of Theorem 2.2(ii) holds trivially. If $L(a) = \int \widehat{a}\, d\mu$ for all $a \in A$, where μ is a positive Borel measure on $\mathcal{K}$, then $L(a) \geq 0$ for all $a \in \operatorname{Pos}(\mathcal{K})$.

We recall some basic terminology. A *preprime* of A is a subset P of A satisfying

$$P + P \subseteq P, \quad P \cdot P \subseteq P, \quad \text{and} \quad \mathbb{Q}_{\geq 0} \subseteq P.$$

P is said to be *generating* if $P - P = A$. If there exists a positive integer d such that $a^{2d} \in P$ for all $a \in A$, then P is called a *preordering*, more precisely, a *preordering of exponent* $2d$.[2] Denote by $\sum A^{2d}$ the set of all finite sums of $2d$-powers of elements of A. $\sum A^{2d}$ is the unique smallest preordering of A of exponent $2d$. The polynomial identity

$$n!X = \sum_{h=0}^{n-1} (-1)^{n-1-h} \binom{n-1}{h} \left[(X+h)^n - h^n \right],$$

see [9, p. 325], applied with $n = 2d$, shows that any preordering is generating.

A subset M of A is called a *P-module* if

$$M + M \subseteq M, \quad P \cdot M \subseteq M, \quad \text{and} \quad 1 \in M.$$

M is said to be *Archimedean* if for all $a \in A$ there exists $n \in \mathbb{N}$ such that $n + a \in M$. The *non-negativity set of M in X_A* is the subset $\mathcal{K}_M$ of X_A defined by

$$\mathcal{K}_M := \{\alpha \in X_A \mid \alpha(M) \subseteq \mathbb{R}_{\geq 0}\} = \{\alpha \in X_A \mid \widehat{a} \geq 0 \text{ at } \alpha \text{ for all } a \in M\}.$$

Remark 2.4 If M is Archimedean, then $\mathcal{K}_M$ is compact. This is well known. For each $a \in A$ there exists $n_a \in \mathbb{N}$ such that $n_a \pm a \in M$. The map $\alpha \mapsto (\alpha(a))_{a \in A}$ identifies $\mathcal{K}_M$ with a closed subset of the compact space $\prod_{a \in A} [-n_a, n_a]$.

Theorem 2.5 (Jacobi) *Suppose $M \subseteq A$ is an archimedean $\sum A^{2d}$-module of A for some integer $d \geq 1$. Then, for all $a \in A$,*

$$\widehat{a} > 0 \text{ on } \mathcal{K}_M \Longrightarrow a \in M.$$

Proof See [13, Theorem 4]. ∎

See [1, Hauptsatz] and [15, Théorème 12] for early variants of Theorem 2.5. See [7, Theorem 6.2] and [19, Theorem 5.4.4] for other proofs of Theorem 2.5 in the case $d = 1$. See Section A for the extension of the proof in [19, Theorem 5.4.4] to the case $d > 1$. See [18, Theorem 2.3] for an extension of Theorem 2.5.

[2] Preorderings of odd exponent are not interesting. If P is a preordering of odd exponent, then $P = P - P = A$.

Corollary 2.6 *Suppose A is an $\mathbb{R}$-algebra and $M \subseteq A$ is an archimedean $\sum A^{2d}$-module of A for some integer $d \geq 1$. If $L\colon A \to \mathbb{R}$ is an $\mathbb{R}$-linear map satisfying $L(M) \subseteq \mathbb{R}_{\geq 0}$, then there exists a unique positive Borel measure μ on $\mathcal{K}_M$ such that for all $a \in A$, $L(a) = \int \widehat{a}\, d\mu$.*

Proof Suppose $a \in A$, $a \geq 0$ on $\mathcal{K}_M$, and $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Then $a + \epsilon > 0$ on $\mathcal{K}_M$ so, by Jacobi's theorem, $a + \epsilon \in M$. Then $L(a + \epsilon) = L(a) + \epsilon L(1) \geq 0$. Since $\epsilon > 0$ is arbitrary, this implies $L(a) \geq 0$. This proves $L(\operatorname{Pos}(\mathcal{K}_M)) \subseteq \mathbb{R}_{\geq 0}$, so the result follows now, by Theorem 2.2(ii). ■

By a *topological $\mathbb{R}$-vector space* we mean an $\mathbb{R}$-vector space V equipped with a topology such that the addition and scalar multiplication are continuous. There is no requirement that the topology be Hausdorff. We are interested here in the case where the topology is defined by a seminorm. Such a topology is in particular locally convex. We record a version of the Hahn–Banach Separation Theorem.

Theorem 2.7 *Suppose V is a topological $\mathbb{R}$-vector space, A and B are non-empty disjoint convex subsets of V and A is open in V. Then there exists a continuous linear map $L\colon V \to \mathbb{R}$ and $t \in \mathbb{R}$ such that $L(a) < t \leq L(b)$ for all $a \in A$ and for all $b \in B$. If B is a cone, we can choose $t = 0$.*

Proof See [14, Theorem 7.3.2] for the proof of the first assertion, [8, Theorem 2.4] for the proof of the second assertion. ■

Corollary 2.8 *Suppose V is a locally convex topological $\mathbb{R}$-vector space and C is a cone in V. The closure of C in V consists of all $v \in V$ satisfying $L(v) \geq 0$ for all continuous linear maps $L\colon V \to \mathbb{R}$ such that $L \geq 0$ on C.*

Proof This is immediate from Theorem 2.7. ■

3 Polynomial Case

Throughout, $\mathbb{N} := \{0, 1, \dots\}$, n denotes a fixed positive integer, $\underline{X}$ denotes the n-tuple of variables $X_1, \dots, X_n$, and $\underline{X}^s := X_1^{s_1} \dots X_n^{s_n}$, for $s = (s_1, \dots, s_n) \in \mathbb{N}^n$.

For any function $\phi\colon \mathbb{N}^n \to \mathbb{R}_{\geq 0}$, we define

$$\mathcal{K}_\phi := \{\underline{x} \in \mathbb{R}^n \mid |\underline{x}^s| \leq \phi(s) \text{ for all } s \in \mathbb{N}^n\}.$$

Fix an integer $d \geq 1$. We denote by $M_{\phi,2d}$ the $\sum \mathbb{R}[\underline{X}]^{2d}$-module of $\mathbb{R}[\underline{X}]$ generated by the elements $\phi(s) \pm \underline{X}^s$, $s \in \mathbb{N}^n$. $M_{\phi,2d}$ is Archimedean. This is a consequence of the fact that

$$\sum_s |f_s|\phi(s) + f = \sum_{f_s > 0} |f_s|\big(\phi(s) + \underline{X}^s\big) + \sum_{f_s < 0} |f_s|\big(\phi(s) - \underline{X}^s\big) \in M_{\phi,2d},$$

for any $f = \sum_s f_s \underline{X}^s \in \mathbb{R}[\underline{X}]$. Also, $\mathcal{K}_\phi$ is the non-negativity set of $M_{\phi,2d}$ in $\mathbb{R}^n$, so, by Jacobi's theorem, any $f \in \mathbb{R}[\underline{X}]$ strictly positive on $\mathcal{K}_\phi$ belongs to $M_{\phi,2d}$.[3]

[3]If one insists on $\mathcal{K}_\phi \neq \varnothing$ (equivalently, $-1 \notin M_{\phi,2d}$), it is necessary to assume that $\phi(0) \geq 1$.

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Definition 3.1 A function $\phi\colon \mathbb{N}^n \to \mathbb{R}_{\geq 0}$ is called an *absolute value* if

(i) $\phi(0) \geq 1$;
(ii) $\phi(s+t) \leq \phi(s)\phi(t)$ for all $s, t \in \mathbb{N}^n$.

Suppose now that ϕ is an absolute value. Denote by $\mathbb{R}[[\underline{X}]]$ the ring of formal power series in $X_1, \dots, X_n$ with coefficients in $\mathbb{R}$. For $f = \sum_s f_s \underline{X}^s \in \mathbb{R}[[\underline{X}]]$ define the ϕ-*seminorm* of f to be $\|f\|_\phi := \sum_s |f_s|\phi(s)$ and denote by $\mathbb{R}[[\underline{X}]]_\phi$ the subset of $\mathbb{R}[[\underline{X}]]$ consisting of all $f \in \mathbb{R}[[\underline{X}]]$ having finite ϕ-seminorm. Using

$$\|f+g\|_\phi \leq \|f\|_\phi + \|g\|_\phi, \quad \|rf\|_\phi = |r|\|f\|_\phi, \quad \text{and} \quad \|fg\|_\phi \leq \|f\|_\phi \|g\|_\phi$$

(these are easily verified), we see that $\mathbb{R}[[\underline{X}]]_\phi$ is a subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}[[\underline{X}]]$. It is the closure of $\mathbb{R}[\underline{X}]$ in the topology induced by the ϕ-seminorm.

Lemma 3.2 *Suppose* $r \in \mathbb{R}$, $s \in \mathbb{N}^n$, $r > \phi(s)$. *Then* $(r \pm \underline{X}^s)^{1/2d} \in \mathbb{R}[[\underline{X}]]_\phi$.

Proof We may assume that $s \neq 0$. Denote by $\sum_{i=0}^\infty a_i t^i$ the power series expansion of $f(t) = (r \pm t)^{1/2d}$ about $t = 0$, *i.e.*, $a_i = (f^{(i)}(0))/i!$. This has radius of convergence r so it converges absolutely for $|t| < r$. In particular, it converges absolutely for $t = \phi(s)$, *i.e.*, $\sum_{i=0}^\infty |a_i|\phi(s)^i < \infty$. Since $\phi(is) \leq \phi(s)^i$ for $i \geq 1$, this implies $\sum_{i=0}^\infty |a_i|\phi(is) < \infty$, *i.e.*, $(r \pm \underline{X}^s)^{1/2d} = \sum_{i=0}^\infty a_i \underline{X}^{is} \in \mathbb{R}[[\underline{X}]]_\phi$. ■

An important example of an absolute value, perhaps the most important one, is the constant function 1. If $\phi = 1$, then $\mathcal{K}_\phi = [-1, 1]^n$ and the ϕ-seminorm is the standard ℓ_1-norm $\|f\|_1 := \sum_s |f_s|$.

Theorem 3.3 *Suppose* ϕ *is an absolute value on* $\mathbb{N}^n$ *and* $f \in \mathbb{R}[\underline{X}]$, $f > 0$ *on* $\mathcal{K}_\phi$. *Then* $f \in \sum \mathbb{R}[[\underline{X}]]_\phi^{2d}$.

Proof For each real $\delta > 0$ consider the function $\phi + \delta\colon \mathbb{N}^n \to \mathbb{R}_{\geq 0}$ defined by

$$(\phi + \delta)(s) := \phi(s) + \delta.$$

Since $\bigcap_{\delta > 0} \mathcal{K}_{\phi+\delta} = \mathcal{K}_\phi$, each $\mathcal{K}_{\phi+\delta}$ is compact and $f > 0$ on $\mathcal{K}_\phi$, there exists $\delta > 0$ such that $f > 0$ on $\mathcal{K}_{\phi+\delta}$. The $\sum \mathbb{R}[\underline{X}]^{2d}$-module $M_{\phi+\delta,2d}$ of $\mathbb{R}[\underline{X}]$ generated by the elements $\phi(s) + \delta \pm \underline{X}^s$, $s \in \mathbb{N}^n$ is archimedean. By Jacobi's theorem, $f \in M_{\phi+\delta,2d}$. By Lemma 3.2, $(\phi(s) + \delta \pm \underline{X}^s)^{1/2d} \in \mathbb{R}[[\underline{X}]]_\phi$ for each $s \in \mathbb{N}^n$. ■

Corollary 3.4 *For any absolute value* ϕ *on* $\mathbb{N}^n$ *the closure of the cone* $\sum \mathbb{R}[\underline{X}]^{2d}$ *in* $\mathbb{R}[\underline{X}]$ *in the topology induced by the* ϕ*-seminorm is* $\operatorname{Pos}(\mathcal{K}_\phi)$.

Proof The inclusion ($\subseteq$) follows from continuity of the evaluation map $f \mapsto f(\underline{x})$, for $\underline{x} \in \mathcal{K}_\phi$, which follows in turn from the fact that $|f(\underline{x}) - g(\underline{x})| \leq \|f - g\|_\phi$, for $\underline{x} \in \mathcal{K}_\phi$. To prove ($\supseteq$), suppose $f \in \mathbb{R}[\underline{X}]$, $f \geq 0$ on $\mathcal{K}_\phi$ and $\epsilon > 0$. Then $f + \frac{\epsilon}{2} > 0$ on $\mathcal{K}_\phi$, so there exists $f_1, \dots, f_m \in \mathbb{R}[[\underline{X}]]_\phi$ such that $f + \frac{\epsilon}{2} = f_1^{2d} + \cdots + f_m^{2d}$, by Theorem 3.3. Take $g = g_1^{2d} + \cdots + g_m^{2d}$, where $g_i \in \mathbb{R}[\underline{X}]$ is such that $\|f_i^{2d} - g_i^{2d}\|_\phi \leq \frac{\epsilon}{2m}$, $i = 1, \dots, m$. Then $g \in \sum \mathbb{R}[\underline{X}]^{2d}$, $\|f - g\|_\phi \leq \epsilon$. ■

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Corollary 3.5 *Suppose* $L\colon \mathbb{R}[\underline{X}] \to \mathbb{R}$ *is linear,* $L(p^{2d}) \geq 0$ *for all* $p \in \mathbb{R}[\underline{X}]$*, and there exists an absolute value* ϕ *and a constant* $C > 0$ *such that* $|L(\underline{X}^s)| \leq C\phi(s)$ *for all* $s \in \mathbb{N}^n$*. Then there exists a unique positive Borel measure* μ *on the set* $\mathcal{K}_\phi$ *such that* $L(f) = \int f\, d\mu$ *for all* $f \in \mathbb{R}[\underline{X}]$*.*

Proof The hypothesis implies that $|L(f) - L(g)| \leq C\|f - g\|_\phi$, so L is continuous. Fix $f \in \operatorname{Pos}(\mathcal{K}_\phi)$. Fix $\epsilon > 0$. By Corollary 3.4, there exists $g \in \sum \mathbb{R}[\underline{X}]^{2d}$ such that $\|f - g\|_\phi \leq \epsilon$, so $|L(f) - L(g)| \leq C\epsilon$. Since $L(g) \geq 0$, this implies that $L(f) \geq -C\epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $L(f) \geq 0$. The conclusion follows, by Theorem 2.2(ii). ■

Remark 3.6 (i) In the case $d = 1$ Corollary 3.5 is well known. It can be obtained by applying [4, Theorem 4.2.5] to the semigroup $(\mathbb{N}^n, +)$ equipped with the identity involution; see [17, Theorem 2.2]. At the same time, the proof given here is new, even in the case $d = 1$.

(ii) The converse of Corollary 3.5 holds. If $L(f) = \int f\, d\mu$ where μ is a positive Borel measure on $\mathcal{K}_\phi$, then $L(p^{2d}) \geq 0$ for all $p \in \mathbb{R}[\underline{X}]$ and $|L(\underline{X}^s)| \leq C\phi(s)$, where $C := \mu(\mathcal{K}_\phi)$. This is clear.

(iii) We have proved Corollary 3.5 from Corollary 3.4 using Theorem 2.2(ii). One can also prove Corollary 3.4 from Corollary 3.5 using Corollary 2.8. In this way, Corollary 3.4 and Corollary 3.5 can be seen to carry exactly the same information.

(iv) Corollary 3.4 extends [3, Theorem 9.1].

(v) In [17], Lasserre and Netzer use [4, Theorem 4.2.5] to prove that for ϕ equal to the constant function 1 and for any $f \in \operatorname{Pos}(\mathcal{K}_\phi)$ and any real $\epsilon > 0$, and any integer $k \geq 1$ sufficiently large (depending on ϵ and f),

$$f + \epsilon\left(1 + \sum_{i=1}^{n} X_i^{2k}\right) \in \sum \mathbb{R}[\underline{X}]^2.$$

It is not clear how to extend this result with $\sum \mathbb{R}[\underline{X}]^2$ replaced by $\sum \mathbb{R}[\underline{X}]^{2d}$.

(vi) In [8, Theorem 4.6] and [8, Theorem 4.10] Ghasemi, Kuhlmann, and Samei prove analogs of [3, Theorem 9.1] for the ℓ_p-norms

$$\|f\|_p := \left(\sum_{s\in\mathbb{N}^n} |f_s|^p\right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_\infty := \sup\{|f_s| \mid s \in \mathbb{N}^n\}$$

and for certain weighted versions of the ℓ_p-norms. Replacing [3, Theorem 9.1] by Corollary 3.4 in these proofs, one verifies that these results carry over word-for-word with $\sum \mathbb{R}[\underline{X}]^2$ replaced by $\sum \mathbb{R}[\underline{X}]^{2d}$.

4 General Case

Our goal in this section is to extend Corollary 3.4 and Corollary 3.5 to arbitrary commutative semigroups with involution; see Theorem 4.3 and Corollary 4.4.

As in [4,5], we work with a commutative $*$-semigroup $S = (S, \cdot, 1, *)$ with neutral element 1 and involution $*$. The involution $*\colon S \to S$ satisfies

$$(st)^* = s^* t^*, \quad (s^*)^* = s, \quad \text{and} \quad 1^* = 1.$$

We denote by $\mathbb{C}[S]$ the semigroup ring of S with coefficients in $\mathbb{C}$. Elements of $\mathbb{C}[S]$ have the form $f = \sum_{s\in S} f_s s$ (finite sum), $f_s \in \mathbb{C}$. $\mathbb{C}[S]$ has the structure of a $\mathbb{C}$-algebra with involution. Addition, scalar multiplication, and multiplication are defined by

$$f + g = \sum (f_s + g_s)s, \quad zf = \sum (zf_s)s, \quad fg = \sum_{s,t} f_s g_t st = \sum_u \Big(\sum_{st=u} f_s g_t \Big) u.$$

The involution is defined by $f^* = \sum \overline{f_s} s^*$. An element $f \in \mathbb{C}[S]$ is said to be *symmetric* if $f^* = f$, *i.e.*, if $f_{s^*} = \overline{f_s}$ for all $s \in S$. We denote the $\mathbb{R}$-algebra consisting of all symmetric elements of $\mathbb{C}[S]$ by A_S. Clearly

$$\mathbb{C}[S] = A_S \oplus iA_S.$$

As an $\mathbb{R}$-vector space A_S is generated by the elements $s + s^*$ and $i(s - s^*)$, $s \in S$. If the involution on S is the identity, *i.e.*, $s^* = s$ for all $s \in S$, then $A_S = \mathbb{R}[S]$, the semigroup ring of S with coefficients in $\mathbb{R}$.

A *semicharacter* of S is a function $\alpha\colon S \to \mathbb{C}$ satisfying the following:

(a) $\alpha(1) = 1$;
(b) $\alpha(st) = \alpha(s)\alpha(t)$ for all $s, t \in S$;
(c) $\alpha(s^*) = \overline{\alpha(s)}$ for all $s \in S$.

We denote by S' the set of all semicharacters of S. Semicharacters α of S correspond bijectively to $*$-algebra homomorphisms $\alpha\colon \mathbb{C}[S] \to \mathbb{C}$ via $\alpha(f) := \sum_{s\in S} f_s \alpha(s)$. In turn, $*$-algebra homomorphisms $\alpha\colon \mathbb{C}[S] \to \mathbb{C}$ correspond bijectively to ring homomorphisms $\alpha\colon A_S \to \mathbb{R}$ via $\alpha(f + gi) = \alpha(f) + \alpha(g)i$. In this way, S' and X_{A_S} are naturally identified.

For any function $\phi\colon S \to \mathbb{R}_{\geq 0}$ define

$$\mathcal{K}_\phi := \left\{ \alpha \in S' \mid |\alpha(s)| \leq \phi(s) \ \text{ for all } \ s \in S \right\}.$$

Fix an integer $d \geq 1$. Denote by $M_{\phi,2d}$ the $\sum A_S^{2d}$-module of A_S generated by the elements

$$\phi(s)^2 - ss^*, \quad 2\phi(s) \pm (s + s^*), \quad \text{and} \quad 2\phi(s) \pm i(s - s^*), \quad s \in S.$$

Lemma 4.1

(i) *$M_{\phi,2d}$ is Archimedean.*
(ii) *The non-negativity set of $M_{\phi,2d}$ in S' is $\mathcal{K}_\phi$.*

Proof (i) The elements $s + s^*$, $i(s - s^*)$ generate A_S as an $\mathbb{R}$-vector space and $2\phi(s) \pm (s + s^*)$, $2\phi(s) \pm i(s - s^*) \in M_{\phi,2d}$, so $M_{\phi,2d}$ is Archimedean.

(ii) For $\alpha \in S'$,

$$|\alpha(s)| \leq \phi(s) \iff \alpha(s)\overline{\alpha(s)} \leq \phi(s)^2 \iff \phi(s)^2 - ss^* \geq 0 \text{ at } \alpha.$$

Also, using the inequality $\sqrt{a^2 + b^2} \geq \max\{|a|, |b|\}$,

$$|\alpha(s)| \leq \phi(s) \Longrightarrow \Big|\frac{\alpha(s) + \overline{\alpha(s)}}{2}\Big| \leq \phi(s) \iff 2\phi(s) \pm (s + s^*) \geq 0 \text{ at } \alpha,$$

and

$$|\alpha(s)| \leq \phi(s) \Longrightarrow \Big|\frac{\alpha(s) - \overline{\alpha(s)}}{2i}\Big| \leq \phi(s) \iff 2\phi(s) \pm i(s - s^*) \geq 0 \text{ at } \alpha. \qquad \blacksquare$$

A function $\phi\colon S \to \mathbb{R}_{\geq 0}$ is called an *absolute value* if

(a) $\phi(1) \geq 1$;
(b) $\phi(st) \leq \phi(s)\phi(t)$ for all $s, t \in S$;
(c) $\phi(s^*) = \phi(s)$ for all $s \in S$.

Suppose that ϕ is an absolute value on S. For $f = \sum_s f_s s \in \mathbb{C}[S]$ define the ϕ-*seminorm* of f to be $\|f\|_\phi := \sum_s |f_s|\phi(s)$. One checks easily that

$$\|f + g\|_\phi \leq \|f\|_\phi + \|g\|_\phi, \quad \|zf\|_\phi = |z|\|f\|_\phi,$$
$$\|fg\|_\phi \leq \|f\|_\phi\|g\|_\phi, \quad \text{and} \quad \|f^*\|_\phi = \|f\|_\phi,$$

so the addition, scalar multiplication, multiplication, and conjugation in the semigroup algebra $\mathbb{C}[S]$ are continuous in the topology induced by the ϕ-seminorm.

Lemma 4.2 *Let $r \in \mathbb{R}$, $f \in A_S$, $r > \|f\|_\phi$. Then, for each real $\epsilon > 0$, there exists $g \in A_S$ such that $\|(r + f) - g^{2d}\|_\phi < \epsilon$.*

Proof Consider the $\mathbb{R}$-algebra homomorphism $\tau\colon \mathbb{R}[X] \to A_S$ defined by $X \mapsto f$ and consider the absolute value ϕ' on $(\mathbb{N}, +)$ defined by $\phi'(i) = \|f^i\|_\phi$. Applying Lemma 3.2 we see that $(r + X)^{1/2d} \in \mathbb{R}[[X]]_{\phi'}$. Combining this with the density of $\mathbb{R}[X]$ in $\mathbb{R}[[X]]_{\phi'}$ and the continuity of the multiplication in the topology induced by the ϕ'-seminorm, there exists $h \in \mathbb{R}[X]$ such that $\|r+X-h^{2d}\|_{\phi'} < \epsilon$. Take $g = \tau(h)$. Since $\tau(r + X - h^{2d}) = r + f - g^{2d}$ and $\|\tau(p)\|_\phi \leq \|p\|_{\phi'}$, for all $p \in \mathbb{R}[X]$, this completes the proof. $\blacksquare$

Theorem 4.3 *Suppose ϕ is an absolute value on a commutative semigroup S with involution and d is any positive integer. Then the closure of the cone $\sum A_S^{2d}$ in A_S in the topology induced by the ϕ-seminorm is equal to* $\operatorname{Pos}(\mathcal{K}_\phi)$.

Proof Since $\sum A_S^{2d} \subseteq \operatorname{Pos}(\mathcal{K}_\phi)$ and $\operatorname{Pos}(\mathcal{K}_\phi)$ is closed, one inclusion is clear. The fact that $\operatorname{Pos}(\mathcal{K}_\phi)$ is closed comes from the fact that each $\alpha \in \mathcal{K}_\phi$, viewed as a ring homomorphism $\alpha\colon A_S \to \mathbb{R}$ in the standard way, satisfies $|\alpha(f)| \leq \|f\|_\phi$ for all $f \in A_S$, so α is continuous for each $\alpha \in \mathcal{K}_\phi$, and $\operatorname{Pos}(\mathcal{K}_\phi) = \bigcap_{\alpha \in \mathcal{K}_\phi} \alpha^{-1}(\mathbb{R}_{\geq 0})$.

For the other inclusion, we must show if $f \in \operatorname{Pos}(\mathcal{K}_\phi)$ and $\epsilon > 0$, there exists $g \in \sum A_S^{2d}$ such that $\|f-g\|_\phi \le \epsilon$. Note that $f + \frac{\epsilon}{2}$ is strictly positive at each $\alpha \in \mathcal{K}_\phi$ so, by Lemma 4.1 and Jacobi's theorem,

$$f + \frac{\epsilon}{2} = \sum_{i=0}^{k} g_i m_i,$$

where $g_i \in \sum A_S^{2d}$, $i = 0, \dots, k$, $m_0 = 1$, and

$$m_i \in \{\phi(s)^2 - ss^*,\ 2\phi(s) \pm (s+s^*),\ 2\phi(s) \pm i(s-s^*) \mid s \in S\}, \quad i = 1, \dots, k.$$

Choose $\delta > 0$ so that $(\sum_{i=1}^{k} \|g_i\|_\phi)\delta \le \frac{\epsilon}{2}$. By Lemma 4.2 there exists $h_i \in A_S$ such that $\|\frac{\delta}{2} + m_i - h_i^{2d}\|_\phi \le \frac{\delta}{2}$, hence $\|m_i - h_i^{2d}\|_\phi \le \delta$, $i = 1, \dots, k$. Take $g = g_0 + \sum_{i=1}^{k} g_i h_i^{2d}$. Then $g \in \sum A_S^{2d}$, and

$$\|f-g\|_\phi = \Big\| \sum_{i=1}^{k} g_i m_i - \sum_{i=1}^{k} g_i h_i^{2d} - \frac{\epsilon}{2} \Big\| \le \sum_{i=1}^{k} \|g_i\|_\phi \|m_i - h_i^{2d}\|_\phi + \frac{\epsilon}{2} \le \epsilon. \quad \blacksquare$$

Corollary 4.4 *Let S be a commutative semigroup with involution and let d be a positive integer. Let $L\colon \mathbb{C}[S] \to \mathbb{C}$ be a $*$-linear mapping such that $L(p^{2d}) \ge 0$ for all $p \in A_S$ and suppose there exists an absolute value ϕ on S and a constant $C > 0$ such that $|L(s)| \le C\phi(s)$ for all $s \in S$. Then there exists a unique positive borel measure μ on $\mathcal{K}_\phi$ such that $L(f) = \int \widehat{f}\, d\mu$ for each $f \in \mathbb{C}[S]$.*

Here, $\widehat{f}\colon S' \to \mathbb{C}$ is defined by $\widehat{f}(\alpha) := \alpha(f)$ for all $\alpha \in S'$; equivalently, if $f = g + ih$, $g, h \in A_S$, then $\widehat{f} := \widehat{g} + i\widehat{h}$.

Proof $*$-linear mappings $L\colon \mathbb{C}[S] \to \mathbb{C}$ correspond bijectively to $\mathbb{R}$-linear mappings $L\colon A_S \to \mathbb{R}$, the correspondence being given by $L(f+gi) = L(f) + L(g)i$. The hypothesis implies that $|L(f) - L(g)| \le C\|f-g\|_\phi$, so L is continuous. Fix $f \in \operatorname{Pos}(\mathcal{K}_\phi)$. Fix $\epsilon > 0$. By Theorem 4.3, there exists $g \in \sum A_S^{2d}$ such that $\|f-g\|_\phi \le \epsilon$, so $|L(f) - L(g)| \le C\epsilon$. Since $L(g) \ge 0$, this implies $L(f) \ge -C\epsilon$. Since $\epsilon > 0$ is arbitrary, this implies that $L(f) \ge 0$. The conclusion follows, by Theorem 2.2(ii). $\blacksquare$

Remark 4.5 For $p \in \mathbb{C}[S]$, $p = q + ir$, $q, r \in A_S$, $pp^* = (q+ir)(q-ir) = q^2 + r^2$. Thus, for $L\colon \mathbb{C}[S] \to \mathbb{C}$ $*$-linear, $L(p^2) \ge 0$ for all $p \in A_S \Leftrightarrow L(pp^*) \ge 0$ for all $p \in \mathbb{C}[S] \Leftrightarrow L$ is positive (semi)definite, terminology as in [3–5]. Consequently, Corollary 4.4 generalizes and provides another proof of what is proved in [3, Corollary 2.5] and [4, Theorem 4.2.5].

5 Berg–Maserick Result

In this section we relax the requirement that an absolute value satisfies $\phi(1) \ge 1$. If $\phi(1) < 1$, then, since $\phi(s) = \phi(s1) \le \phi(s)\phi(1)$ for all $s \in S$, ϕ is identically zero. Then $\|\cdot\|_\phi$ is also identically zero, so the topology on $\mathbb{C}[S]$ is the trivial one

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and the closure of $\sum A_S^{2d}$ in A_S is A_S. At the same time, $\mathcal{K}_\phi = \varnothing$ so $\operatorname{Pos}(\mathcal{K}_\phi) = A_S$. Consequently, Theorem 4.3 and Corollary 4.4 continue to hold in this more general situation.

We explain how the Berg–Maserick result [5, Theorem 2.1] can be deduced as a consequence of Corollary 4.4. See Corollary 5.3.

A *weak absolute value* on S is a function $\phi\colon S \to \mathbb{R}_{\geq 0}$ satisfying

$$\phi(ss^*) \leq \phi(s)^2 \text{ for all } s \in S.$$

Replacing s by s^*, we see that $\phi(ss^*) \leq \phi(s^*)^2$, so

$$\phi(ss^*) \leq \min\{\phi(s)^2, \phi(s^*)^2\} \text{ for all } s \in S,$$

for any weak absolute value ϕ on S.

For any weak absolute value ϕ on S, define $\phi'\colon S \to \mathbb{R}_{\geq 0}$ by

$$\phi'(s) = \inf\Big\{ \prod_{i=1}^{k} \min\big\{ \phi(s_i), \phi(s_i^*) \big\} \;\Big|\; k \geq 1,\ s_1, \dots, s_k \in S,\ s = s_1 \cdots s_k \Big\}.$$

Lemma 5.1 *Let ϕ be a weak absolute value on S. Then*

(i) *ϕ' is an absolute value (possibly $\phi' \equiv 0$);*

(ii) *if $L\colon \mathbb{C}[S] \to \mathbb{C}$ is $*$-linear and positive semidefinite and there exists $C > 0$ such that $|L(s)| \leq C\phi(s)$ for all $s \in S$, then $|L(s)| \leq C\phi'(s)$ for all $s \in S$;*

(iii) *$\mathcal{K}_\phi = \mathcal{K}_{\phi'}$.*

Proof (i) This is clear.

(ii) It suffices to show that

$$|L(s_1 \cdots s_k)| \leq C \prod_{i=1}^{k} \min\big\{ \phi(s_i), \phi(s_i^*) \big\} \qquad \text{for all } s_1, \dots, s_k \in S.$$

Since $|L(s)| \leq C\phi(s)$ and $|L(s)| = |\overline{L(s)}| = |L(s^*)| \leq C\phi(s^*)$, the result is clear when $k = 1$. Suppose now that $k \geq 2$. We make use of the Cauchy–Schwarz inequality for the inner product

$$\langle f, g \rangle := L(fg^*), \qquad f, g \in \mathbb{C}[S].$$

This implies, in particular, that

$$|L(st^*)|^2 \leq L(ss^*)L(tt^*) \qquad \text{for all } s, t \in S.$$

Using this we obtain

$$\begin{aligned} |L(s_1 \cdots s_k)|^2 &\leq L(s_1 s_1^*) L(s_2 s_2^* \cdots s_k s_k^*) \leq C\phi(s_1 s_1^*) C \prod_{i=2}^{k} \phi(s_i s_i^*) \\ &= C^2 \prod_{i=1}^{k} \phi(s_i s_i^*) \leq C^2 \prod_{i=1}^{k} \min\big\{ \phi(s_i)^2, \phi(s_i^*)^2 \big\}. \end{aligned}$$

(the second inequality by induction on k). The result follows by taking square roots.

(iii) Since $\phi'(s) \leq \phi(s)$ for all $s \in S$, the inclusion $\mathcal{K}_{\phi'} \subseteq \mathcal{K}_\phi$ is clear. For the other inclusion, note that each $\alpha \in S'$ is positive semidefinite, so $\mathcal{K}_\phi \subseteq \mathcal{K}_{\phi'}$ by (ii). ∎

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Corollary 5.2 *Suppose ϕ is a weak absolute value on S. Then the closure of $\sum A_S^2$ in A_S in the topology induced by the ϕ-seminorm $\|f\|_\phi := \sum |f_s|\phi(s)$ is equal to $\operatorname{Pos}(\mathcal{K}_\phi)$.*

Proof Denote the closure of $\sum A_S^2$ in A_S in the topology induced by the ϕ-seminorm by $\overline{\sum A_S^2}^{\phi}$. By Lemma 5.1(ii), an $\mathbb{R}$-linear map $L\colon A_S \to \mathbb{R}$ non-negative on $\sum A_S^2$ is continuous in the topology induced by $\|\cdot\|_\phi$ if and only if it is continuous in the topology induced by $\|\cdot\|_{\phi'}$. It follows, using Corollary 2.8, that $\overline{\sum A_S^2}^{\phi} = \overline{\sum A_S^2}^{\phi'}$. By Lemma 5.1(3), $\mathcal{K}_\phi = \mathcal{K}_{\phi'}$ so $\operatorname{Pos}(\mathcal{K}_\phi) = \operatorname{Pos}(\mathcal{K}_{\phi'})$. The result now follows using Lemma 5.1(i) and Theorem 4.3. ■

Corollary 5.3 *Suppose $L\colon \mathbb{C}[S] \to \mathbb{C}$ is $*$-linear and positive semidefinite and there exists a weak absolute value ϕ on S and a constant $C > 0$ such that $|L(s)| \le C\phi(s)$ for all $s \in S$. Then there exists a unique positive borel measure μ on $\mathcal{K}_\phi$ such that $L(f) = \int \widehat{f}\, d\mu$ for each $f \in \mathbb{C}[S]$.*

Proof In view of Lemma 5.1, this is immediate from Corollary 4.4. ■

Since the argument in Lemma 5.1(ii) makes essential use of the Cauchy–Schwarz inequality, it seems unlikely that Corollaries 5.2 and 5.3 extend to the case $d > 1$.

A Appendix

Let A be a commutative ring with 1. For simplicity assume that $\mathbb{Q} \subseteq A$. In what follows, P is assumed to be a preprime of A, and $M \subseteq A$ is a P-module.

We explain how the simple proof of Jacobi's theorem in the case $d = 1$ found in [19, Theorem 5.4.4] can be extended to $d > 1$. We use the following lemma.

Lemma A.1 *Suppose M is Archimedean, $a \in A$, $t \in P$, $at - 1 \in M$. Let n be a positive integer that is even. Then for any sufficiently large $k \in \mathbb{Q}$,*

$$k^n(a+r) - 1 - (k-t)^n(a+r) \in M$$

for each non-negative $r \in \mathbb{Q}$.

Proof Since

$$\begin{aligned} k^n(a+r) - 1 - (k-t)^n(a+r) &= -1 - \sum_{i=1}^{n} \binom{n}{i} k^{n-i}(-t)^i(a+r) \\ &= -1 + \binom{n}{1} k^{n-1} - \sum_{i=2}^{n} \binom{n}{i} k^{n-i}(-t)^i a \\ &\quad + \binom{n}{1} k^{n-1}(at-1) + rt \sum_{i=1}^{n} \binom{n}{i} k^{n-i}(-t)^{i-1} \end{aligned}$$

and $at - 1 \in M$, it suffices to show that

$$\sum_{i=1}^{n} \binom{n}{i} k^{n-i}(-t)^{i-1} \in M \tag{A.1}$$

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and

$$-1+\binom{n}{1}k^{n-1}-\sum_{i=2}^{n}\binom{n}{i}k^{n-i}(-t)^{i}a\in M \tag{A.2}$$

for sufficiently large k. Since

$$\sum_{i=1}^{n}\binom{n}{i}k^{n-i}(-t)^{i-1}=k^{n-2}\Big[\binom{n}{1}k-\binom{n}{2}t\Big]+k^{n-4}t^{2}\Big[\binom{n}{3}k-\binom{n}{4}t\Big]+\cdots,$$

and M is Archimedean, so $\binom{n}{1}k-\binom{n}{2}t$, $\binom{n}{3}k-\binom{n}{4}t,\dots$ belong to M for k sufficiently large; (A.1) is clear, for k sufficiently large.

Regarding (A.2), write

$$\binom{n}{3}k^{n-3}t^{3}a=\binom{n}{3}k^{n-3}t^{2}(k+ta)-\binom{n}{3}k^{n-2}t^{2},$$
$$-\binom{n}{4}k^{n-4}t^{4}a=\binom{n}{4}k^{n-4}t^{2}(k^{2}-t^{2}a)-\binom{n}{4}k^{n-2}t^{2},\ \text{etc.},$$

choosing k so large that $k+ta$, k^2-t^2a, etc., belong to M. We are reduced to showing that

$$-1+\binom{n}{1}k^{n-1}-\binom{n}{2}k^{n-2}t^{2}a-\binom{n}{3}k^{n-2}t^{2}-\binom{n}{4}k^{n-2}t^{2}-\cdots\in M \tag{A.3}$$

for k sufficiently large. Dividing through by k^{n-2}, using the fact that M is Archimedean and $\binom{n}{1}k-\frac{1}{k^{n-2}}\geq\binom{n}{1}k-1$ if $k\geq 1$, we see that this is true, *i.e.,* (A.3) does indeed hold for k sufficiently large. ■

Proof of Theorem 2.5 We argue as in [19, Theorem 5.4.4]. Let $P=\sum A^{2d}$. Set $M_1:=M-aP$. Since $M\subseteq M_1$, M_1 is Archimedean. The assumption $\alpha(a)>0$ for all $\alpha\in\mathcal{K}_M$ implies that $\mathcal{K}_{M_1}=\varnothing$ and, as noted earlier, P is generating, so, by [19, Corollary 5.4.1], $-1\in M_1$. Thus $-1=s-at$, $s\in M$, $t\in P$, so $at-1=s\in M$. Apply Lemma A.1 with $n=2d$ to conclude that

$$k^{2d}(a+r)-1-(k-t)^{2d}(a+r)\in M$$

for each $r\in\mathbb{Q}_{\geq 0}$, for any k sufficiently large. Dividing by k^{2d}, we see that

$$a+r\in M\ \Rightarrow\ a+r-\frac{1}{k^{2d}}\in M.$$

Iterating, we obtain eventually that $a+r\in M$ for some negative $r\in\mathbb{Q}$, so $a=(a+r)+(-r)\in M$. ■

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Recall that a preprime P of A is *torsion* if for all $a \in A$ there exists $n \geq 1$ such that $a^n \in P$, and P is *weakly torsion* if for all $a \in A$ there exists rational $r > 0$ and $n \geq 1$ such that $(r + a)^n \in P$. Clearly, for any preprime P,

$$P \text{ is a preordering} \implies P \text{ is torsion} \implies P \text{ is weakly torsion.}$$

It is proved in [18, Lemma 2.4] that any preprime that is weakly torsion is generating. Moreover, one has the following extension of Jacobi's result.

Theorem A.2 *Suppose $M \subseteq A$ is an Archimedean P-module, P a weakly torsion preprime of A. Then, for any $a \in A$,*

$$\widehat{a} > 0 \text{ on } \mathcal{K}_M \implies a \in M.$$

Proof See [18, Theorem 2.3]. ∎

Unfortunately, it is not clear how Lemma A.1 can be applied to give a proof of Theorem A.2.

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