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ASYMPTOTIC DISTRIBUTION OF THE NUMBER AND SIZE OF PARTS IN UNEQUAL PARTITIONS

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An asymptotic formula is derived for the number of partitions of a large positive integer n into r unequal positive integer parts and maximal summand k. The number of parts has a normal distribution about its maximum, the largest summand an extreme-value distribution. For unrestricted partitions the two distributions coincide and both are extreme-valued. The problem of joint distribution of unrestricted partitions with r parts and largest summand k remains unsolved.

1. Introduction.

Let $q_r(n)$ denote the number of partitions of n into r unequal positive integer parts (unequal partitions for brevity). The asymptotic behaviour of $q_r(n)$ for fixed large n and variable r is known over a wide range of r[6], but in a form which is not very easy to handle. For applications it is better to have a simple expression which, although valid in a more restricted range, is nevertheless sufficiently extensive to include almost all partitions.

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It is well known (Erdös and Lehner [3], in a more precise form in [6]) that the maximum occurs very nearly at $r_0 = \frac{\log 2}{c}\sqrt{n}$ where

(1) $$c = \frac{\pi}{2\sqrt{3}} = 0.90689968\ldots ,$$

and the following is a fairly straight forward consequence of the main asymptotic formula of [6] :

THEOREM 1. *Let*

(2) $$\sigma = r - \frac{\log 2}{c}\sqrt{n} = o(n^{1/3}) .$$

Then asymptotically for large n

(3) $$q_r(n) \simeq \frac{1}{4n\sqrt{6\gamma}} \exp(2c\sqrt{n}) \exp\left(-\frac{c}{\gamma}\frac{\sigma^2}{\sqrt{n}}\right)$$

where

(4) $$\gamma = 1 - \left(\frac{\log 2}{c}\right)^2 = 0.41583918\ldots .$$

Hence the distribution about r_0 is Gaussian, with variance $\frac{\gamma}{2c}\sqrt{n}$. Note that

$$\sum_r q_r(n) \simeq \frac{1}{4n\sqrt{6\gamma}} \exp(2c\sqrt{n}) \int_{-\infty}^{\infty}\left(\exp - \frac{c}{\gamma}\frac{\sigma^2}{\sqrt{n}}\right) d\sigma$$

$$\simeq \frac{\exp(2c\sqrt{n})}{4.3^{1/4}\, n^{3/4}} = Q(n) ,$$

the well known asymptotic expression for the total number of unequal partitions. This shows that (3) is valid over almost all partitions. The symbol $\simeq$ will always mean that the quotient of the two sides tends to 1 when $n \to \infty$.

Next consider $Q_k(n)$, the number of those unequal partitions in which k is the largest summand. Erdös and Lehner have shown [3] that for almost all unequal partitions the largest summand is $k = \frac{\sqrt{n}}{c}\log\sqrt{n} + O(\sqrt{n}\,\omega(n))$ where $\omega(n)$ tends to infinity arbitrarily slowly. Using the generating function $x^k \prod_{\nu=1}^{k-1}(1+x^\nu) = \sum_n Q_k(n)x^n$ one

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can obtain by the circle method the following more specific result:

THEOREM 2. *Let* λ *be determined from the equation*

$$(5) \qquad \log \lambda = \frac{1}{2} \log n - \frac{c}{\sqrt{n}} k \; , \quad \lambda = \sqrt{n} \exp\left(- \frac{ck}{\sqrt{n}}\right) .$$

Then for large n *and for* $\lambda = O(n^{1/6})$, $1/\lambda = O(n^{1/6})$,

$$(6) \qquad Q_k(n) \simeq Q(n) \frac{\lambda}{\sqrt{n}} \exp\left(- \frac{\lambda}{c}\right) .$$

The result of Erdös and Lehner follows from here immediately (but not the other way round). Formula (6) represents a so called extreme-value distribution about $k_0 = \frac{\sqrt{n}}{c} \log \frac{\sqrt{n}}{c}$, with variance $2n$, see for example [1, p.930].

What can one say about the distribution of unequal partitions of n in which both the number of summands, r , and the largest summand, k , vary? This problem came up recently in the counting of spiral walks on a triangular lattice [7] where it was assumed that for every fixed r in a suitable neighbourhood of r_0 , the distribution is still given by (6) with $Q(n)$ replaced $q_r(n)$. That is, it was assumed that the distributions with respect to r and k are independent in a sufficiently extensive region which embraces almost all partitions. We shall prove the following more precise result which clearly contains both Theorems 1 and 2 as corollaries:

THEOREM 3. *Let* $q(n;\, r,\, k)$ *denote the number of those unequal partitions of* n *in which the number of summands is* r *and the size of the maximal summand is* k . *Then for large* n *and for*

$$(7) \quad \sigma = r - \frac{\log 2}{c} \sqrt{n} = o(n^{1/3}) \; , \quad \lambda = \sqrt{n}\, e^{-ck/\sqrt{n}} = O(n^{1/6}),\; 1/\lambda = O(n^{1/6}),$$

$$(8) \qquad q(n;r,k) \quad \frac{\lambda}{4\sqrt{6}\gamma\, n^{3/2}} \exp\left\{2c\sqrt{n} - \frac{\lambda}{c} - \frac{c\sigma^2}{\gamma\sqrt{n}}\right\} .$$

We shall only deal with the main asymptotic term; error terms can be obtained but they are fairly complicated. A similar problem arises of course with unrestricted partitions. The distribution of the number of parts and maximal summand for unrestricted partitions has been studied extensively by Szalay and Turán [5] and by Erdős and Szalay [4], but they never write down asymptotic expressions like (8), not even like (4) or (6). The latter can be obtained quite simply from the general asymptotic formula of [6]:

THEOREM 4. *Let* $p_k(n)$ *denote the number of unrestricted partitions of* n *into precisely* k *parts, or what is the same, in parts with largest summand* k . *Let* $c_0 = \pi/\sqrt{6}$, $\eta(n)$ *a positive function tending monotonically to* 0 , *and*

$$\mu = \frac{\sqrt{n}}{c_o}\, exp\left(-\frac{kc_o}{\sqrt{n}}\right), \qquad \mu + \frac{1}{\mu} \le \sqrt{n}\,\eta(n) .$$

Then

$$p_k(n) \approx P(n)\,\frac{\mu c_o}{\sqrt{n}}\, e^{-\mu}$$

where

$$P(n) \approx \frac{1}{4\sqrt{3}n}\, exp(2\, c_o\sqrt{n})$$

is the total number of unrestricted partitions of n .

We thus have an extreme-value distribution about $k_0 = \frac{\sqrt{n}}{c_o}\log\frac{\sqrt{n}}{c_o}$ (with variance n) for both the maximal summand and the number of parts. The two counting numbers of course coincide because of the one to one correspondence between partitions in k parts and conjugate partitions with largest summand k . For the same reason the joint distribution must necessarily be symmetric in k and r , but no analogy of Theorem 3 has been found for unrestricted partitions. The proof of Theorem 4 is omitted.

2. Proof of the asymptotic formula.

For fixed k consider

$$F_k(x,t) = (1+tx)(1+tx^2)\ldots(1+tx^k) = \sum_{n,r} Q(n;r,k)x^n t^r .$$

Clearly $Q(n;r,k)$ is the number of partitions of n into r unequal

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parts, each $\le k$. Hence

$$G_k(x,t) = F_k(x,t) - F_{k-1}(x,t) = tx^k F_{k-1}(x,t) = \Sigma\, q(n;r,k)x^n t^r$$

and so

$$q(n;r,k) = \frac{1}{2\pi i}\int dz \frac{1}{2\pi i}\int dw\, G_k(z,w)\, z^{-n-1} w^{-r-1}$$

(9)
$$= -\frac{1}{4\pi^2}\int dz \int dw \exp\left\{\sum_{\nu=1}^{k-1}\log\,(1+wz^\nu)\right\} z^{-(n-k+1)} w^{-r},$$

integrated over the product $C_w \times C_z$ of two circles

$$C_w : w = e^{-\alpha+i\phi},\ C_z : z = e^{-\beta+i\theta},\ -\pi < \phi \le \pi,\ -\pi < \theta \le \pi .$$

Here α,β can be any real numbers, but will be chosen so that the saddle point conditions

(10)
$$\sum_{\nu=1}^{k-1}\frac{\nu}{e^{\alpha+\nu\beta}+1} = n-k,\quad \sum_{\nu=1}^{k-1}\frac{1}{e^{\alpha+\nu\beta}+1} = r$$

be fulfilled, at least in suitable approximation. To achieve (10) write

(11)
$$\alpha = -\frac{2c\sigma}{\gamma\sqrt{n}},\quad \beta = \frac{c}{\sqrt{n}} + \frac{\sigma\log 2}{\gamma n}$$

where c,γ,σ are defined as in (1), (4) and (7). Then since $\sigma = o(n^{1/3})$,

(12)
$$1/\beta = \frac{\sqrt{n}}{c} - \frac{\sigma\log 2}{\gamma c^2} + \frac{\sigma^2\log^2 2}{\gamma^2 c^3\sqrt{n}} + o(1).$$

Defining further $u = \log(\sqrt{n}/\lambda)$ where λ is as in (5), we find from (7) and (1) that

(13)
$$u = \frac{1}{2}\log n - \log\lambda = ck/\sqrt{n}$$

and

(14)
$$k\beta = u + \frac{\sigma k\log 2}{n}.$$

The assumptions $\sigma = o(n^{1/3})$, $\lambda + 1/\lambda = O(n^{1/6})$ imply

(15)
$$u = O(\log n),\ e^{-u} = \lambda/\sqrt{n} = O(n^{-1/3}),\quad k = O(\sqrt{n}\log n),$$
$$\alpha = o(n^{-1/6}),$$

Using these and (7) we get from Euler-Maclaurin

$$\sum_{\nu=1}^{k-1} \frac{1}{e^{\alpha+\nu\beta}+1} = \frac{1}{\beta}\int_{\zeta}^{u} \frac{dt}{e^{\alpha+t}+1} + O(1)$$

$$= \frac{1}{\beta}\left[\log(1+e^{-u}) - \log(1+e^{-\alpha-u})\right] + O(1)$$

$$= \frac{\sqrt{n}}{c}\log 2 - \frac{\alpha\sqrt{n}}{2c} - \frac{\sigma}{\gamma}\frac{\log^2 2}{c^2} + O(n^{1/6}\log n)$$

$$= \frac{\sqrt{n}}{c}\log 2 + \frac{\sigma}{\gamma}\left[1 - \left(\frac{\log 2}{c}\right)^2\right] + O(n^{1/6}\log n)$$

$$= r + O(n^{1/6}\log n)$$

Similarly

$$\sum_{\nu=1}^{k-1} \frac{\nu}{e^{\alpha+\nu\beta}+1} = \frac{1}{\beta^2}\int_0^u \frac{t\,dt}{e^{\alpha+t}+1} - \frac{u}{2\beta(e^{\alpha+u}+1)} + O(1)$$

$$= \frac{1}{\beta^2}\left\{\frac{\pi^2}{12} - \alpha\log 2 + O(ue^{-u})\right\} + O(n^{1/6}\log n)$$

$$= n + O(n^{2/3}\log n) .$$

Since $k = O(\sqrt{n}\log n)$ we see that in consequence of definition (11) and our assumptions, (10) is replaced by

$$(16)\qquad \sum_{\nu=1}^{k-1} \frac{\nu}{e^{\alpha+\nu\beta}+1} = n-k+O(n^{2/3}\log n),\quad \sum_{\nu=1}^{k-1} \frac{1}{e^{\alpha+\nu\beta}+1} = r+O(n^{1/6}\log n) .$$

Both seem fairly crude approximations to (10) but they will suffice.

Returning now to the evaluation of the repeated integral (9) in the neighbourhood of $\phi = 0$, $\theta = 0$ write

$$(17)\qquad t = e^{-\alpha+i\phi},\quad z = e^{-\beta+i\theta},\quad |\phi| \le n^{-1/5},\quad |\theta| \le n^{-5/7} .$$

The integrals over the complementary arcs $n^{-1/5} < |\phi| \le \pi$, $n^{-5/7} < |\theta| \le \pi$ are negligible compared with the dominant part (17); this can be seen just as in [6] or in Andrews [2], chapter 6 and the estimates need not be repeated here.

The integrand of (9) over the range (17) then becomes

$$(18)\qquad -\exp\left\{\sum_{\nu=1}^{k-1}\log(1+e^{-\alpha-\nu\beta+i\phi+i\nu\theta}) - i(r-1)\phi - i(n-k)\theta\right\} d\phi d\theta .$$

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Here

$$\sum_{\nu=1}^{k-1} \log(1+e^{-\alpha-\nu\beta+i\phi+i\nu\theta}) = \sum \log\left\{1+e^{-\alpha-\nu\beta}(1+i(\phi+\nu\theta)-\tfrac{1}{2}(\phi+\nu\theta)^2 + O(|\phi+\nu\theta|^3)\right\}$$

$$= \sum \log(1+e^{-\alpha-\nu\beta}) + \sum \log\left\{1+\frac{i}{e^{\alpha+\nu\beta}+1}(\phi+\nu\theta) - \frac{1}{2}\frac{(\phi+\nu\theta)^2}{e^{\alpha+\nu\beta}+1} + O\left(\frac{|\phi+\nu\theta|^3}{e^{\alpha+\nu\beta}+1}\right)\right\}$$

$$= \sum \log(1+e^{-\alpha-\nu\beta}) + i\sum \frac{\phi}{e^{\alpha+\nu\beta}+1} + i\sum\frac{\nu\theta}{e^{\alpha+\nu\beta}+1}$$

$$- \frac{1}{2}\sum \frac{e^{\alpha+\nu\beta}}{(e^{\alpha+\nu\beta}+1)^2}(\phi+\nu\theta)^2 + O\left(\frac{|\phi|^3}{\beta}+\frac{|\theta|^3}{\beta^4}\right)$$

$$= \sum \log(1+e^{-\alpha-\nu\beta}) + ir\phi + i(n-k)\theta + O(|\phi|\, n^{1/6}\log n) + O(|\theta|n^{2/3}\log n)$$

$$- \frac{1}{2}\sum \frac{e^{\alpha+\nu\beta}}{(e^{\alpha+\nu\beta}+1)^2}(\phi+\nu\theta)^2 + O(n^{-1/10}+n^{-1/7})$$

by (16). All summations go from 1 to $k-1$. But $n^{1/6}\phi = O(n^{-1/30})$, $n^{2/3} = O(n^{-1/21})$, hence the expression in (18) is equal to

$$(19)\qquad - \exp\left\{\sum_{\nu=1}^{k-1}\log(1+e^{-\alpha-\nu\beta}) - \frac{1}{2}\sum_{\nu=1}^{k-1}\frac{e^{\alpha+\nu\beta}}{(e^{\alpha+\nu\beta}+1)^2}(\phi+\nu\theta)^2 + o(1)\right\} d\phi d\theta\ .$$

Summarising from (9) and (19)

$$q(n;r,k) \simeq \frac{1}{4\pi^2}e^{\alpha r+\beta(n-k)}\exp\left\{\sum_{\nu=1}^{k-1}\log(1+e^{-\alpha-\nu\beta})\right\}.$$

$$\int_{-n^{-1/5}}^{n^{-1/5}} d\phi \int_{-n^{-5/7}}^{n^{-5/7}} \exp\left\{\frac{1}{2\beta}\int_0^u \frac{e^{\alpha+t}}{(e^{\alpha+t}+1)^2}(\phi+\frac{t}{\beta}\theta)^2 dt\right\} d\theta\ .$$

But from (13) it is seen that u goes to ∞ like $\log n$ throughout the whole range of λ and we can replace u by ∞ in the t-integral, also $\alpha = O(n^{-1/6})$ by 0. Furthermore

$$\phi^2/\beta \simeq \frac{1}{c}n^{1/10}\ ,\quad \phi\theta/\beta^2 \simeq \frac{1}{c^2}n^{3/35}\ ,\quad \theta^2/\beta^3 \simeq \frac{1}{c^3}n^{1/14}$$

at the upper integration limits of ϕ and θ and we can replace these

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limits by infinity in the asymptotic expression. Hence

$$(20) \qquad q(n;r,k) \simeq \frac{1}{4\pi^2} e^{\alpha r+\beta(n-k)} \exp\left\{ \sum_{\nu=1}^{k-1} \log(1+e^{-\alpha-\nu\beta}) \right\} .$$

$$\int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} \exp\left\{- \frac{1}{2\beta} \int_0^{\infty} \frac{e^t}{(e^t+1)^2} (\phi+ \frac{t}{\beta}\,\theta)^2 dt \right\} d\theta .$$

To evaluate the double integral note that

$$A = \frac{1}{2\beta} \int_0^{\infty} \frac{e^t}{(e^t+1)^2}\, dt = \frac{1}{4\beta} , \quad B = \frac{1}{2\beta^2} \int_0^{\infty} \frac{te^t}{(e^t+1)^2}\, dt = \frac{1}{2\beta^2} \log 2 ,$$

$$C = \frac{1}{2\beta^3} \int_0^{\infty} \frac{t^2 e^t}{(e^t+1)^2}\, dt = \frac{1}{\beta^3} \frac{\pi^2}{12} .$$

Thus

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(A\phi^2+2B\phi\theta+C\theta^2)} d\phi d\theta = \frac{\pi}{\sqrt{AC-B^2}} = 2\beta^2 \left[\frac{1}{12} - (\frac{\log 2}{\pi})^2\right]^{-1/2}$$

$$\simeq \frac{12c^2}{\sqrt{3\gamma}\, n} = \frac{2}{\sqrt{3\gamma}\, n}$$

and

$$(21) \qquad q(n;r,k) \quad \frac{1}{4\sqrt{3\gamma}\, n} \exp\left\{ \sum_{\nu=1}^{k-1} \log(1+e^{-\alpha-\nu\beta}) + \alpha r + \beta(n-k)\right\} .$$

It remains to evaluate the expression in the exponent. Once more by Euler-Maclaurin

$$\sum_{\nu=1}^{k-1} \log(1+e^{-\alpha-\nu\beta}) = \frac{1}{\beta} \int_0^{u} \log(1+e^{-\alpha-t}) dt - \frac{1}{2} \log(1+e^{-\alpha}) - \frac{1}{2}\log(1+e^{-\alpha-u}) + o(1)$$

$$= \frac{1}{\beta}\left\{ \int_0^{\infty} \log(1+e^{-t}) dt - \int_0^{\alpha} \log(1+e^{-t}) dt - \int_u^{\infty} \log(1+e^{-\alpha-t}) dt\right\}$$

$$- \frac{1}{2} \log 2 + o(1)$$

$$= \frac{1}{\beta} (\frac{\pi^2}{12} - \alpha \log 2 + \frac{\alpha^2}{4} - e^{-u}) - \frac{1}{2} \log 2 + o(1)$$

$$= c\sqrt{n} + \frac{\sigma}{\gamma} \log 2 + \frac{c\sigma^2}{\gamma\sqrt{n}} - \frac{\lambda}{c} - \frac{1}{2} \log 2 + o(1)$$

and by (7), (11), (12)

$$\alpha r + \beta(n-k) = -\frac{2c\sigma}{\gamma\sqrt{n}}\left(\sigma + \frac{\log 2}{c}\sqrt{n}\right) + c\sqrt{n} + \frac{\sigma}{\gamma}\log 2 - \log\frac{\sqrt{n}}{\lambda} + o(1) .$$

Substituting these into (21) we finally obtain

$$q(n;r,k) \simeq \frac{\lambda}{4\sqrt{6\gamma}\, n^{3/2}} \exp\left(2c\sqrt{n} - \frac{\lambda}{c} - \frac{c\sigma^2}{\gamma\sqrt{n}}\right) ,$$

that is, expression (8).

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