Proceedings of the Royal Society of Edinburgh, **153**, 262–274, *2023*
DOI:10.1017/prm.2021.83

Series expansion of Leray–Trudinger inequality

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(Received 17 June 2021; accepted 20 November 2021)

In this paper, we establish an infinite series expansion of Leray–Trudinger inequality, which is closely related with Hardy inequality and Moser Trudinger inequality. Our result extends early results obtained by Mallick and Tintarev [A. Mallick and C. Tintarev. An improved Leray-Trudinger inequality. Commun. Contemp. Math. 20 (2018), 17501034. OP 21] to the case with many logs. It should be pointed out that our result is about series expansion of Hardy inequality under the case $p = n$, which case is not considered by Gkikas and Psaradakis in [K. T. Gkikas and G. Psaradakis. Optimal non-homogeneous improvements for the series expansion of Hardy's inequality. Commun. Contemp. Math. doi:10.1142/S0219199721500310]. However, we can't obtain the optimal form by our method.

Keywords: Leray–Trudinger inequality; series expansion; improved Hardy inequality

2020 *Mathematics subject classification:* Primary: 26D15; 26E10; 35J20

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ containing the origin, $n \geqslant 2$ and $p > 1$, the classical p-Hardy inequality asserts that

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geqslant \left|\frac{n-p}{p}\right|^p \int_\Omega \frac{|u|^p}{|x|^p}\,\mathrm{d}x, \forall\, u \in C_0^\infty(\Omega), \tag{1.1}$$

with $\left|\frac{n-p}{p}\right|^p$ being the best constant and never achieved [**6**, **7**, **10**, **15**, **20**, **24**]. Many improvements of Hardy inequality can be obtained by adding the error term in the right side of (1.1) [**8**, **12**]. The first improvement was obtained by Brezis and Vazquez [**8**]. When $p = 2$, they have shown that (1.1) can be improved by adding subcritical Sobolev term $\int_\Omega |u|^q \,\mathrm{d}x (1 \leqslant q < 2^* = \frac{2n}{n-2})$. After that, Chaudhuri and Ramaswamy [**9**] improved inequality (1.1) by introducing a subcritical Hardy–Sobolev term $\int_\Omega \frac{|u|^q}{|x|^\beta}\,\mathrm{d}x \quad (0 \leqslant \beta < 2,\ 1 \leqslant q < 2^*_\beta := \frac{2(n-\beta)}{n-2})$ [**1**]. Later, Adimurthi, Chaudhuri and Ramaswamy [**2**] extended their results to general

L_p Hardy inequality for $2 \leqslant p < n$. In [11], Filippas and Tertikas pointed out that the critical Sobolev type improvement for $p = 2$ could be established by adding a logarithmic term. Their result is as follows.

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 3)$ containing the origin, $R_\Omega := \sup_{x\in\Omega} |x|$, then for any $u \in H_0^1(\Omega)$ and $R \geqslant R_\Omega$, there exists a constant $C_n > 0$ depending only on n, such that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x \geqslant C_n \left(\int_\Omega \left(|u|^{2^*} X_1^{1+\frac{2^*}{2}} \left(\frac{|x|}{R}\right)\right) \mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{1.2}$$

Here

$$X_1(t) = (1 - logt)^{-1}, \quad t \in (0, 1]. \tag{1.3}$$

Inequality (1.2) was sharp in the sense that $X_1^{1+\frac{2^*}{2}}$ cannot be replaced by a smaller power of X_1.

In [11], the authors also established the series expansion of Hardy inequality. Their results were extended to the following general L_p $(p \neq n)$ Hardy inequality [5].

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 3)$ containing the origin, $R_\Omega := \sup_{x\in\Omega} |x|$, then for any $u \in W_0^{1,p}(\Omega \setminus \{0\})$ and $R \geqslant R_\Omega$, there holds

$$\begin{aligned}\int_\Omega |\nabla u|^p \,\mathrm{d}x \geqslant{}& \left|\frac{n-p}{p}\right|^p \int_\Omega \frac{|u|^p}{|x|^p} \,\mathrm{d}x + \frac{p-1}{2p} \left|\frac{n-p}{p}\right|^{p-2} \\ &\times \sum_{i=1}^{\infty} \int_\Omega \frac{|u|^p}{|x|^p} X_1^2\left(\frac{|x|}{R}\right) X_2^2\left(\frac{|x|}{R}\right) \cdots X_i^2\left(\frac{|x|}{R}\right) \mathrm{d}x.\end{aligned} \tag{1.4}$$

Here

$$X_k(t) = X_1(X_{k-1}(t)), \quad k \geqslant 2. \tag{1.5}$$

In [11], the authors also proved the following series expansion of Hardy inequality for $p = 2$ with critical sobolev term.

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \geqslant{}& \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x + \frac{1}{4} \sum_{i=1}^{k} \int_\Omega \frac{|u|^2}{|x|^2} \prod_{j=1}^{i} X_j^2\left(\frac{|x|}{R}\right) \mathrm{d}x \\ &+ C_n \left(\int_\Omega |u|^{2^*} \prod_{i=1}^{k+1} X_i^{1+\frac{2^*}{2}} \left(\frac{|x|}{R}\right) \mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{1.6}$$

The exponent $1 + \frac{2^*}{2}$ on X_{k+1} cannot be decreased.

Recently, Gkikas and Psaradakis [13] generalized inequality (1.6) to the general case $1 < p < n$ and $p > n$. When $1 < p < n$, by adding an optimally weighted critical Sobolev norm, they obtained the following results.

Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin, $n \geqslant 2$ and $1 < p < n$, $R_\Omega := \sup_{x\in\Omega} |x|$, there exist constants $C_n > 0$ depending only on n and

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$B := B(n, p) \geqslant 1$, such that for any $u \in W_0^{1,p}(\Omega)$, $R \geqslant BR_\Omega$ and $k \in \mathbb{N}$, there holds

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geqslant \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \,\mathrm{d}x + \frac{p-1}{2p}\left(\frac{n-p}{p}\right)^{p-2} \times \sum_{i=1}^{k} \int_\Omega \frac{|u|^p}{|x|^p} \prod_{j=1}^{i} X_j^2\left(\frac{|x|}{R}\right) \mathrm{d}x + C_n \left(\int_\Omega |u|^{p^*} \prod_{i=1}^{k+1} X_i^{1+\frac{p^*}{p}}\left(\frac{|x|}{R}\right) \mathrm{d}x\right)^{\frac{p}{p^*}}. \tag{1.7}$$

The exponent $1 + \frac{p^*}{p}$ on X_{k+1} cannot be decreased. When $p > n$, they established the series expansion of L_p Hardy inequality by adding the optimally weighted Hölder seminorm.

All the previous results we mentioned are concerning about the case $p \neq n$. When $p = n$, Hardy inequality can be stated as follows [2–4, 16].

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$, containing the origin, then for any $R \geqslant R_\Omega$ and $u \in W_0^{1,n}(\Omega)$, one has

$$\int_\Omega |\nabla u|^n \,\mathrm{d}x \geqslant \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n} X_1\left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{1.8}$$

Barbatis, Filippas and Tertikas [5] established the following series expansion of Hardy inequality for the case $p = n$.

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ containing the origin, then for any $R > R_\Omega$ and for all $u \in W_0^{1,n}(\Omega \setminus \{0\})$, one has

$$\int_\Omega |\nabla u|^n \,\mathrm{d}x \geqslant \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n} X_1^n\left(\frac{|x|}{R}\right) \mathrm{d}x + \frac{1}{2}\left(\frac{n-1}{n}\right)^{n-1} \times \sum_{i=2}^{\infty} \int_\Omega \frac{|u|^n}{|x|^n} X_1^n\left(\frac{|x|}{R}\right) X_2^2\left(\frac{|x|}{R}\right) \cdots X_i^2\left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{1.9}$$

In analogy with inequality (1.1), it is natural to ask whether similar critical Sobolev term can be added into inequality (1.8). Since the limit case of critical Sobolev inequality is Moser–Trudinger inequality [17, 18, 22, 23], the natural substitute of critical Sobolev term is some exponential function. Recently, Psaradakis and Spector [21] established the following Leray–Trudinger inequality.

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ containing the origin, then for any $\epsilon > 0$ and $R \geqslant R_\Omega$, there exist positive constants $A_{n,\epsilon}$ and B_n, such that for all $u \in W_0^{1,n}(\Omega)$ satisfying $I_1(u) \leqslant 1$, one has

$$\int_\Omega e^{A_{n,\epsilon}\left(|u| X_1^\epsilon\left(\frac{|x|}{R}\right)\right)^{\frac{n}{n-1}}} \,\mathrm{d}x \leqslant B_n \mathrm{vol}(\Omega), \tag{1.10}$$

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where $I_1(u)$ is defined by

$$I_1(u) := \int_\Omega |\nabla u|^n \,\mathrm{d}x - \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n} X_1^n \left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{1.11}$$

Moreover, inequality (1.10) failed for $\epsilon = 0$.

Inequality (1.10) is closely related with Hardy inequality and Moser–Trudinger inequality. Subsequently, Mallick and Tintarev [**19**] extended inequality (1.10) to the following form:

Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ containing the origin, then for any $\beta \geqslant \frac{2}{n}$ and $R \geqslant R_\Omega$, there exist positive constants A_n and B_n, such that for any $0 < c < A_n$ and for all $u \in W_0^{1,n}(\Omega)$ satisfying $I_1(u) \leqslant 1$, one has

$$\int_\Omega e^{c\left(|u| X_2^\beta\left(\frac{|x|}{R}\right)\right)^{\frac{n}{n-1}}} \,\mathrm{d}x \leqslant B_n \mathrm{vol}(\Omega), \tag{1.12}$$

where $X_2(t) := X_1(X_1(t))$. Moreover, inequality (1.12) failed if $\beta < \frac{1}{n}$ for any $c > 0$.

The relationship of inequality (1.10) and inequality (1.12) motivates us to investigate whether inequality (1.12) can be improved to be series expansion. In this paper, we establish the following series expansion of Leray–Trudinger inequality. Our main result is as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin, $n \geqslant 2$ and $R_\Omega := \sup_{x\in\Omega} |x|$. Then for any $k \in \mathbb{N}$, $k \geqslant 1$ and $R \geqslant R_\Omega$, there exist constants $A(k, n)$ and $B(k, n)$, such that for any $0 < C < A(k, n)$ and $u \in W_0^{1,n}(\Omega)$ satisfying $I_k(u) \leqslant 1$, one has*

$$\int_\Omega e^{C\left(|u(x)| \prod_{i=2}^{k+1} X_i^{\frac{2}{n}}\left(\frac{|x|}{R}\right)\right)^{\frac{n}{n-1}}} \,\mathrm{d}x \leqslant B(k,n) Vol(\Omega), \tag{1.13}$$

where $I_1(u)$ is defined by (1.11) and for $k \geqslant 2$, $I_k(u)$ is defined by

$$I_k(u) := I_{k-1}(u) - \frac{1}{2}\left(\frac{n-1}{n}\right)^{n-1} \int_\Omega \frac{|u|^n}{|x|^n} X_1^n\left(\frac{|x|}{R}\right) X_2^2\left(\frac{|x|}{R}\right) \cdots X_k^2\left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{1.14}$$

Moreover, if replacing X_{k+1}^2 by X_{k+1}^β, one has that inequality (1.13) holds for any $\beta \geqslant \frac{2}{n}$.

Remark 1.2. When $k = 1$, inequality (1.13) becomes inequality (1.12). Hence our result extends early results obtained by Mallick and Tintarev [**19**] to series expansion form. However, in [**19**], they obtained that inequality (1.12) holds when $\beta \geqslant \frac{2}{n}$ and fails when $\beta < \frac{1}{n}$. Here we can't show that inequality (1.13) fails when $\beta < \frac{1}{n}$. Moreover, as we mentioned before, Gkikas and Psaradakis [**13**] obtained series optimal forms of Hardy inequality for $1 < p < n$ and $p > n$ but didn't consider $p = n$, our result is about this case. However, we can't obtain optimal forms by our method.

To prove the main result, we follow closely Trudinger's original proof (see [**14**]), which has been used in [**21**] and [**19**]. Our main steps are as follows. Firstly, we

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find a suitable function (2.6), which is a supersolution of some Laplace equation (lemma 2.5). By this function, we define corresponding transform to obtain L^q estimate (proposition 3.1). After that, we obtain the exponential integrability.

This paper is organized as follows. In § 2, we establish some important preliminaries. In § 3, we give the proof of theorem 1.1.

2. Preliminaries

In this section, we list some important preliminaries.

By the definition of $X_k(t)$(see (1.5)), we define

$$Y_k(t) := \prod_{i=2}^{k} X_i(t), \quad Z_k(t) := \sum_{i=2}^{k} Y_i(t), \ k = 2, 3, \cdots . \tag{2.1}$$

The following proposition is due to the derivative of X_k,Y_k and Z_k.

PROPOSITION 2.1. *For any $k \in \mathbb{N}$ and $k \geqslant 2$, one has*

$$\frac{d}{dt}\left(X_k^\beta(t)\right) = \frac{\beta}{t} X_1(t) Y_k(t) X_k^\beta(t); \tag{2.2}$$

$$\frac{d}{dt}\left(Y_k(t)\right) = \frac{1}{t} X_1(t) Y_k(t) Z_k(t); \tag{2.3}$$

$$\frac{d}{dt}\left(Z_k(t)\right) = \frac{1}{2t} X_1(t)\left(Z_k^2(t) + \sum_{i=2}^{k} Y_i^2(t)\right). \tag{2.4}$$

Proof. The first one is proved in [**13**], lemma 2.2. Since Y_k and Z_k are different from definition 2.1 appeared in [**13**]. We list the proof of (2.3) and (2.4) as follows.

$$\begin{aligned}
\frac{d}{dt}\left(Y_k(t)\right) &= \sum_{j=2}^{k}\left(\frac{d}{dt}\left(X_j(t)\right) \prod_{i=2, i\neq j}^{k} X_i(t)\right) \\
&= \frac{1}{t} X_1(t) \sum_{j=2}^{k}\left(Y_j(t) X_j(t) \prod_{i=2, i\neq j}^{k} X_i(t)\right) \\
&= \frac{1}{t} X_1(t) Y_k(t) Z_k(t).
\end{aligned}$$

From the elementary identity

$$\begin{aligned}
2\sum_{i=2}^{k} Y_i Z_i &= 2\sum_{i,j=2; j\leqslant i}^{k} Y_i Y_j = 2\sum_{i,j=2; j<i}^{k} Y_i Y_j + 2\sum_{i=2}^{k} Y_i^2 \\
&= \left(\sum_{i=2}^{k} Y_i\right)^2 + \sum_{i=2}^{k} Y_i^2 = Z_k^2 + \sum_{i=2}^{k} Y_i^2,
\end{aligned}$$

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one has

$$\frac{d}{dt}\left(Z_k(t)\right) = \sum_{i=2}^{k} \frac{d}{dt}\left(Y_i(t)\right) = \frac{1}{t} X_1(t) \sum_{i=2}^{k} Y_i(t) Z_i(t)$$
$$= \frac{1}{2t} X_1(t) \left(Z_k^2(t) + \sum_{i=2}^{k} Y_i^2(t) \right).$$

Defining $Z_\infty(t) := \sum_{i=2}^{\infty} Y_k(t)$, it converges if and only if $t \in (0, 1)$, see [13].

Concerning $I_k(u)$, the following results hold, see [5]. □

Proposition 2.2 Theorem B [5]. *For any $k \in \mathbb{N}$, $R \geqslant R_\Omega$ and $u \in C_c^\infty(\Omega \setminus \{0\})$, one has*

$$I_k(u) \geqslant \frac{1}{2}\left(\frac{n-1}{n}\right)^{n-1} \int_\Omega \frac{|u|^n}{|x|^n} X_1^n\left(\frac{|x|}{R}\right) X_2^2\left(\frac{|x|}{R}\right) \cdots X_{k+1}^2\left(\frac{|x|}{R}\right) \mathrm{d}x.$$

The following lemma is a standard representation formula for smooth functions.

Lemma 2.3 [14], Lemma 7.14. *Let Ω be any open set in $\mathbb{R}^n$, $n \geqslant 2$, $u \in C_c^1(\Omega)$, then*

$$u(x) = \frac{1}{n w_n} \int_\Omega \frac{(x-y)\cdot \nabla u(y)}{|x-y|^n} \mathrm{d}y, \tag{2.5}$$

where w_n is the volume of unit ball in $\mathbb{R}^n$.

In [21], let $u(x) = X_1^{\frac{1-n}{n}}(\frac{|x|}{R}) v(x)$, the authors obtained the following lower bound of $I_1(u)$. That is,

$$I_1(u) \geqslant C_1(n) \int_\Omega |\nabla v|^n X_1^{1-n} \,\mathrm{d}x,$$

where $C_1(n) = \frac{1}{2^{n-1}-1}$. In the following, we are going to extend their result to arbitrary $k \in \mathbb{N}$. Precisely, we have

Theorem 2.4. *For any $R \geqslant R_\Omega$, $k \in \mathbb{N}$ and $k \geqslant 2$, set*

$$w_k(x) = X_1^{\frac{1-n}{n}}\left(\frac{|x|}{R}\right) X_2^{-\frac{1}{n}}\left(\frac{|x|}{R}\right) \cdots X_k^{-\frac{1}{n}}\left(\frac{|x|}{R}\right), \ x \in \Omega, \tag{2.6}$$

then for all $u \in C_c^\infty(\Omega \setminus \{0\})$, one has

$$I_k(u) \geqslant C_1(n) \int_\Omega |\nabla v|^n w_k^n \,\mathrm{d}x, \tag{2.7}$$

where v is defined by $u(x) := w_k(x) v(x)$.

In order to prove theorem 2.4, the following key lemma is needed.

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Lemma 2.5. *For any $k \in \mathbb{N}$, the function w_k defined by* (2.6), *is a supersolution of the following Laplace equation*:

$$-\Delta_n w - \left(\left(\frac{n-1}{n}\right)^n + \frac{1}{2}\left(\frac{n-1}{n}\right)^{n-1} \sum_{i=2}^{k} Y_i^2\left(\frac{|x|}{R}\right)\right) X_1^n\left(\frac{|x|}{R}\right) \frac{|w|^{n-2}w}{|x|^n} = 0.$$

Proof. Let $A_k(x) = X_1(\frac{|x|}{R})(\frac{1-n}{n} - \frac{1}{n}Z_k(\frac{|x|}{R}))$, then by direct calculation, one has

$$\nabla w_k = w_k A_k \frac{x}{|x|^2}.$$

Hence,

$$\begin{aligned}-\Delta_n w_k &= -\operatorname{div}\left(|\nabla w_k|^{n-2}\nabla w_k\right) = -\operatorname{div}\left\{\frac{|w_k|^{n-2}w_k|A_k|^{n-2}A_k x}{|x|^n}\right\}\\ &= -\operatorname{div}\left\{\frac{|w_k|^{n-2}w_k x}{|x|^n}\right\}|A_k|^{n-2}A_k - \left\{\frac{|w_k|^{n-2}w_k x}{|x|^n}\right\}\cdot\nabla\left(|A_k|^{n-2}A_k\right).\end{aligned}$$

While

$$-\operatorname{div}\left\{\frac{|w_k|^{n-2}w_k x}{|x|^n}\right\} = (1-n)\frac{|w_k|^{n-2}w_k A_k}{|x|^n}$$

and

$$\nabla\left(|A_k|^{n-2}A_k\right) = (n-1)|A_k|^{n-2}\nabla A_k,$$

thus

$$\begin{aligned}-\Delta_n w_k &= \frac{|w_k|^{n-2}w_k|A_k|^{n-2}}{|x|^n}\left((1-n)A_k^2 - (n-1)x\cdot\nabla A_k\right)\\ &= \frac{|w_k|^{n-2}w_k}{|x|^n}X_1^n\left|\frac{1-n}{n} - \frac{Z_k}{n}\right|^{n-2}\left((1-n)\left(\frac{1-n}{n} - \frac{Z_k}{n}\right)^2\right.\\ &\quad\left. - (n-1)\left(\frac{1-n}{n} - \frac{Z_k}{n} - \frac{1}{2n}\left(Z_k^2 + \sum_{i=2}^{k} Y_i^2\right)\right)\right)\\ &= \frac{|w_k|^{n-2}w_k}{|x|^n}X_1^n\left(\frac{n-1}{n}\right)^{n-2}\left|1 + \frac{Z_k}{n-1}\right|^{n-2}\\ &\quad\times\left(\frac{(1-n)^2}{n^2} + \frac{(n-1)(2-n)}{n^2}Z_k + \frac{(n-1)(n-2)}{2n^2}Z_k^2 + \frac{n-1}{2n}\sum_{i=2}^{k} Y_i^2\right).\end{aligned}$$

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In order to prove the result, we should prove that

$$\begin{aligned}\left|1+\frac{Z_k}{n-1}\right|^{n-2}&\left(\frac{(1-n)^2}{n^2}+\frac{(n-1)(2-n)}{n^2}Z_k\right.\\&\left.+\frac{(n-1)(n-2)}{2n^2}Z_k^2+\frac{n-1}{2n}\sum_{i=2}^{k}Y_i^2\right)\\&\geqslant\left(\frac{n-1}{n}\right)^2+\frac{n-1}{2n}\sum_{i=2}^{k}Y_i^2.\end{aligned}\tag{2.8}$$

When $n=2$, inequality (2.8) naturally holds true. In the following, we just consider the case of $n>2$. Let $t=\frac{Z_k}{n}$, $h=\frac{1-n}{n}$ and $\lambda=\frac{n-1}{2n}\sum_{i=2}^{k}Y_i^2$, then inequality (2.8) can be written by

$$\left(h^2+\lambda\right)\left(1-\left|1-\frac{t}{h}\right|^{2-n}\right)+h(n-2)t+\frac{(n-2)(n-1)}{2}t^2\geqslant 0.\tag{2.9}$$

Consider function $g(x)=|1-x|^{2-n}$, by Taylor expansion at $x=0$ (see [**13**]), one has

$$g(x)=1+(n-2)x+\frac{(n-1)(n-2)}{2}x^2+\frac{n(n-1)(n-2)}{6}x^3+O(x^4).$$

Therefore, inequality (2.9) is equivalent to

$$-\frac{n-2}{h}\left(\frac{n(n-1)}{6}t^3+\lambda t\right)+O(\lambda t^2)\geqslant 0.\tag{2.10}$$

While inequality (2.10) holds since $n>2$ and $h<0$. Therefore, we complete our proof. □

The proof of theorem 2.4. Setting $u(x)=w_k(x)v(x)$, from the following inequality (see [**13**, **21**])

$$|a+b|^p\geqslant|a|^p+C_1(p)|b|^p+p|a|^{p-2}a\cdot b,\quad\forall\ a,b\in\mathbb{R}^n,p\geqslant 2,\tag{2.11}$$

and integrating by parts, we deduce that

$$\begin{aligned}\int_\Omega|\nabla u|^n\,\mathrm{d}x&=\int_\Omega|v\nabla w_k+w_k\nabla v|^n\,\mathrm{d}x\\&\geqslant\int_\Omega\left[|v|^n\,|\nabla w_k|^n+C_1(n)|\nabla v|^n\,|w_k|^n+(\nabla|v|^n)\right.\\&\qquad\left.\cdot\left(|\nabla w_k|^{n-2}w_k\nabla w_k\right)\right]\,\mathrm{d}x\\&=C_1(n)\int_\Omega|\nabla v|^n|w_k|^n-\int_\Omega|u|^nw_k^{-1}|w_k|^{2-n}\Delta_n w_k\,\mathrm{d}x.\end{aligned}\tag{2.12}$$

Therefore, we obtain inequality (2.7) from lemma 2.5.

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3. Proof of theorem 1.1

In this section, we give the proof of theorem 1.1. Firstly, we prove the following L^q estimate.

PROPOSITION 3.1. *Let $u \in W_0^{1,n}(\Omega)$, for any $q > n$ and $R \geqslant R_\Omega$, we have*

$$\left(\int_\Omega \left|u(x)Y_{k+1}^{\frac{2}{n}}\left(\frac{|x|}{R}\right)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C(k,n)\left(1+\frac{q(n-1)}{n}\right)^{1-\frac{1}{n}+\frac{1}{q}} (\mathrm{vol}(\Omega))^{\frac{1}{q}} (I_k(u))^{\frac{1}{n}}, \tag{3.1}$$

where $C(k,\, n) = (\frac{1}{C_1(n)^{\frac{1}{n}}} + 2^{\frac{1}{n}} C'(k,\, n)(\frac{n}{n-1})^{\frac{n-1}{n}})\frac{1}{nw_n^{\frac{1}{n}}}$.

Proof. Let $u \in C_c^\infty(\Omega \setminus \{0\})$, we define $u(x) = w_k(x)v(x)$, then inequality (2.5) implies that

$$\begin{aligned}
&\left|u(x)Y_{k+1}^{\frac{2}{n}}\left(\frac{|x|}{R}\right)\right| \\
&= \left|v(x)X_1^{\frac{1-n}{n}}\left(\frac{|x|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|x|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|x|}{R}\right)\right| \\
&= \left|\frac{1}{nw_n}\int_\Omega \frac{(x-y)\cdot\nabla\left(v(y)X_1^{\frac{1-n}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)\right)}{|x-y|^n}\,\mathrm{d}y\right| \\
&\leqslant \frac{1}{nw_n}\int_\Omega \frac{|\nabla v(y)|\, X_1^{\frac{1-n}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)}{|x-y|^{n-1}}\,\mathrm{d}y \\
&\quad + \frac{1}{nw_n}\int_\Omega \frac{v(y)}{|x-y|^{n-1}}\left|\nabla\left(X_1^{\frac{1-n}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)\right)\right|\,\mathrm{d}y.
\end{aligned}$$

By proposition 2.1, we get

$$\begin{aligned}
&\left|\nabla\left(X_1^{\frac{1-n}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)\right)\right| \\
&\leqslant \frac{1}{|y|}X_1^{\frac{1}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right) \\
&\quad \times \left|\frac{n-1}{n} + \frac{1}{n}\sum_{i=2}^{k} Y_i\left(\frac{|y|}{R}\right) + \frac{2}{n}Y_{k+1}\left(\frac{|y|}{R}\right)\right| \\
&\leqslant C'(k,n)\frac{1}{|y|}X_1^{\frac{1}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right).
\end{aligned}$$

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Hence we deduce

$$\begin{aligned}\left|u(x)Y_{k+1}^{\frac{2}{n}}\left(\frac{|x|}{R}\right)\right| &\leqslant \frac{1}{nw_n}\int_\Omega \frac{|\nabla v(y)|w_k\left(\frac{|y|}{R}\right)}{|x-y|^{n-1}}\,\mathrm{d}y + \frac{1}{nw_n}C^{'}(k,n)\\ &\quad\times\int_\Omega \frac{|v(y)|}{|y||x-y|^{n-1}}X_1^{\frac{1}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)\mathrm{d}y\\ &:= \frac{1}{nw_n}\left(S(x)+C^{'}(k,n)T(x)\right),\end{aligned}$$

where

$$S(x)=\int_\Omega \frac{|\nabla v(y)|w_k\left(\frac{|y|}{R}\right)}{|x-y|^{n-1}}\,\mathrm{d}y,$$

and

$$T(x)=\int_\Omega \frac{|v(y)|}{|y||x-y|^{n-1}}X_1^{\frac{1}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right)\mathrm{d}y.$$

Then for $q>n$, one has

$$\left\|uY_{k+1}^{\frac{2}{n}}\right\|_{L^q(\Omega)} \leqslant \frac{1}{nw_n}\left(\|S\|_{L^q(\Omega)}+C^{'}(k,n)||T||_{L^q(\Omega)}\right). \tag{3.2}$$

Define r by $\frac{1}{n}+\frac{1}{r}=1+\frac{1}{q}$. In order to estimate $\|S\|_{L^q(\Omega)}$, we write

$$\begin{aligned}\frac{|\nabla v(y)|w_k\left(\frac{|y|}{R}\right)}{|x-y|^{n-1}} &= \left(\frac{1}{|x-y|^{(n-1)r}}\right)^{\frac{1}{r}-\frac{1}{q}}\left(|\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right)\right)^{\frac{1}{n}-\frac{1}{q}}\\ &\quad\times\left(\frac{|\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right)}{|x-y|^{(n-1)r}}\right)^{\frac{1}{q}},\end{aligned}$$

and define

$$h_r(x):=\int_\Omega \frac{1}{|x-y|^{(n-1)r}}\,\mathrm{d}y.$$

Then by Hölder's inequality, we get

$$S(x)\leqslant h_r(x)^{\frac{1}{r}-\frac{1}{q}}\left(\int_\Omega |\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right)\mathrm{d}y\right)^{\frac{1}{n}-\frac{1}{q}}\left(\int_\Omega \frac{|\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right)}{|x-y|^{(n-1)r}}\,\mathrm{d}y\right)^{\frac{1}{q}}.$$

Integrating $S(x)$ and using Tonelli' Theorem, one has

$$\begin{aligned}\|S\|_{L^q(\Omega)} &\leqslant \|h_r\|_{L^\infty(\Omega)}^{1-\frac{1}{n}} \left(\int_\Omega |\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right) \mathrm{d}y\right)^{\frac{1}{n}-\frac{1}{q}} \\ &\quad \times \left(\int_\Omega |\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right) h_r(y)\,\mathrm{d}y\right)^{\frac{1}{q}} \\ &\leqslant \|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}} \left(\int_\Omega |\nabla v(y)|^n w_k^n\left(\frac{|y|}{R}\right) \mathrm{d}y\right)^{\frac{1}{n}}. \end{aligned} \tag{3.3}$$

From theorem 2.4, we get

$$\|S\|_{L^q(\Omega)} \leqslant \frac{1}{(C_1(n))^{\frac{1}{n}}}\|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}}(I_k(u))^{\frac{1}{n}}. \tag{3.4}$$

To estimate $\|T\|_{L^q(\Omega)}$, we use similar steps. Firstly, we write that

$$\begin{aligned}&\frac{|v(y)|}{|y||x-y|^{n-1}}X_1^{\frac{1}{n}}\left(\frac{|y|}{R}\right)Y_k^{\frac{1}{n}}\left(\frac{|y|}{R}\right)X_{k+1}^{\frac{2}{n}}\left(\frac{|y|}{R}\right) \\ &= \left(\frac{1}{|x-y|^{(n-1)r}}\right)^{\frac{1}{r}-\frac{1}{q}} \left(\frac{|v(y)|^n}{|y|^n}X_1\left(\frac{|y|}{R}\right)Y_k\left(\frac{|y|}{R}\right)X_{k+1}^2\left(\frac{|y|}{R}\right)\right)^{\frac{1}{n}-\frac{1}{q}} \\ &\times \left(\frac{|v(y)|^n}{|y|^n|x-y|^{(n-1)r}}X_1\left(\frac{|y|}{R}\right)Y_k\left(\frac{|y|}{R}\right)X_{k+1}^2\left(\frac{|y|}{R}\right)\right)^{\frac{1}{q}}\end{aligned}$$

Applying Hölder's inequality and taking the L^q-norm of the both sides, we obtain

$$\begin{aligned}\|T\|_{L^q(\Omega)} &\leqslant \|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}} \left(\int_\Omega \frac{|v(y)|^n}{|y|^n}X_1\left(\frac{|y|}{R}\right)Y_k\left(\frac{|y|}{R}\right)X_{k+1}^2\left(\frac{|y|}{R}\right)\mathrm{d}y\right)^{\frac{1}{n}} \\ &= \|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}} \left(\int_\Omega \frac{|u(y)|^n}{|y|^n}X_1^n\left(\frac{|y|}{R}\right)Y_k^2\left(\frac{|y|}{R}\right)X_{k+1}^2\left(\frac{|y|}{R}\right)\mathrm{d}y\right)^{\frac{1}{n}}.\end{aligned}$$

Using the conclusion of proposition 2.2, one has

$$\|T\|_{L^q(\Omega)} \leqslant 2^{\frac{1}{n}}\left(\frac{n}{n-1}\right)^{\frac{n-1}{n}} \|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}} (I_k(u))^{\frac{1}{n}}. \tag{3.5}$$

Thus using the following estimate ([**19**],(3.4))

$$\|h_r\|_{L^\infty(\Omega)}^{\frac{1}{r}} \leqslant w_n^{1-\frac{1}{n}}\left(1+\frac{(n-1)q}{n}\right)^{1-\frac{1}{n}+\frac{1}{q}} \mathrm{vol}(\Omega)^{\frac{1}{q}}, \tag{3.6}$$

we get

$$\left\|uY_{k+1}^{\frac{2}{n}}\right\|_{L^q(\Omega)} \leqslant C(k,n)\left(1+\frac{(n-1)q}{n}\right)^{1-\frac{1}{n}+\frac{1}{q}} \mathrm{vol}(\Omega)^{\frac{1}{q}}(I_k(u))^{\frac{1}{n}}, \tag{3.7}$$

where $C(k,\,n)$ is defined by $C(k,\,n)=(\frac{1}{C_1(n)^{\frac{1}{n}}}+2^{\frac{1}{n}}C'(k,\,n)(\frac{n}{n-1})^{\frac{n-1}{n}})\frac{1}{nw_n^{\frac{1}{n}}}$. Thus, we complete the proof of proposition 3.1. □

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In the following, we prove theorem 1.1.

Proof. Let $u \in W_0^{1,n}(\Omega)$ such that $I_k(u) \leqslant 1$. Applying proposition 3.1 with $q = \frac{ns}{n-1}$, $s \in \{n, n+1, \cdots\}$, we have

$$\int_\Omega \left| u(x) Y_{k+1}^{\frac{2}{n}} \left(\frac{|x|}{R} \right) \right|^{\frac{ns}{n-1}} \mathrm{d}x \leqslant (C(k,n))^{\frac{ns}{n-1}} \operatorname{vol}(\Omega)(1+s)^{1+s}.$$

Given $C > 0$, multiplying both sides by $\frac{C^s}{s!}$ and adding from n to m ($m \geqslant n$), it yields

$$\int_\Omega \sum_{s=n}^{m} \frac{1}{s!} \left[C \left| u(x) Y_{k+1}^{\frac{2}{n}} \left(\frac{|x|}{R} \right) \right|^{\frac{n}{n-1}} \right]^s \mathrm{d}x$$
$$\leqslant \sum_{s=n}^{m} \left(C \left(C(k,n) \right)^{\frac{n}{n-1}} \right)^s \operatorname{vol}(\Omega) \frac{(1+s)^{1+s}}{s!}.$$

Clearly, the right side above inequality converges as $m \to \infty$ if and only if

$$C < \frac{1}{e \left(C(k,n) \right)^{\frac{n}{n-1}}}. \tag{3.8}$$

While each term of the finite sum

$$S = \int_\Omega \sum_{s=0}^{n-1} \frac{1}{s!} \left[C \left| u(x) Y_{k+1}^{\frac{2}{n}} \left(\frac{|x|}{R} \right) \right|^{\frac{n}{n-1}} \right]^s \mathrm{d}x \tag{3.9}$$

is bounded by a constant depending only on k, n due to Hölder inequality. And so, there exist constants $A(k, n)$ and $B(k, n)$ such that for any $0 < C < A(k, n)$, there has

$$\int_\Omega \sum_{s=0}^{\infty} \frac{1}{s!} \left[C \left| u(x) Y_{k+1}^{\frac{2}{n}} \left(\frac{|x|}{R} \right) \right|^{\frac{n}{n-1}} \right]^s \mathrm{d}x \leqslant B(k,n) \operatorname{vol}(\Omega). \tag{3.10}$$

The left side inequality (3.10) is the power series expansion of $e^{C\left|u(x)Y_{k+1}^{\frac{2}{n}}(\frac{|x|}{R})\right|^{\frac{n}{n-1}}}$. Thus, theorem 1.1 is valid, and for $\beta > \frac{2}{n}$, the result is also valid because of $X_{k+1}(\frac{|x|}{R}) < 1$. □

Acknowledgments

The first author was supported by National Natural Science Foundation of China (No. 11601173 and No. 61772223).

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