ON NULL-RECURRENT MARKOV CHAINS

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1. Introduction. Throughout this paper, the symbol $P = [P_{ij}]$ will represent the transition probability matrix of an irreducible, null-recurrent Markov process in discrete time. Explanation of this terminology and basic facts about such chains may be found in **(6**, ch. 15**)**. It is known **(3)** that for each such matrix P there is a unique (except for a positive scalar multiple) positive vector $Q = \{q_i\}$ such that $QP = Q$, or

$$q_j = \sum_i q_i P_{ij}; \tag{1}$$

this vector is often called the "invariant measure" of the Markov chain.

The first problem to be considered in this paper is that of determining for which vectors $U^{(0)} = \{\mu_i^{(0)}\}$ the vectors $U^{(n)}$ converge, or are summable, to the invariant measure Q, where $U^{(n)} = U^{(0)}P^n$ has components

$$\mu_j^{(n)} = \sum_i \mu_i^{(n-1)} P_{ij} = \sum_i \mu_i^{(0)} P_{ij}^{(n)}. \tag{2}$$

In § 2, this problem is attacked for general P. The main result is a negative one, and shows how to form $U^{(0)}$ for which $U^{(n)}$ will not be (termwise) Abel summable. As this negative result shows, the operators formed from P do not obey the mean ergodic theorem. (It is interesting to contrast this situation with the case when P is the stochastic matrix of an *ergodic* Markov chain **(6)**.) However, in § 3 more inclusive positive results are found for two special classes of matrices P.

The invariant measure may be used to form a stationary process as follows: let N_i^0 be independent Poisson random variables with respective means q_i, and suppose that at time 0, N_i^0 particles* are placed in state i of the Markov chain. Suppose that each particle thereafter moves according to the law of the chain independently of the others, and let N_i^n be the number of particles in state i at time n. Then for each n, the random variables N_i^n are independent, Poisson, and have means q_i. These facts are due to Derman **(4**, Theorem 2**)**. It is then natural to ask if there are not non-stationary processes associated with P which converge to this stationary process as a limit, and in fact Derman has already done so in **(4)**. The vectors A_n with components N_i^n form a Markov process which has an "invariant measure"; and the general

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*We shall speak of "moving particles" throughout without further apology. More exact statements may easily be supplied (as been done in **(4)**) at some cost in intuitive appeal.

theory of such processes together with other arguments, yields a variety of results. Here we will take a slightly different point of view.

Actually, the first problem described above is closely connected with the second in the following way: let N_i^0 be independent, non-negative integer-valued random variables with means $\mu_i^{(0)}$, and as in the construction of Derman's stationary process put N_i^0 particles into state i at time 0 and let the particles move independently. Then it is not hard to see that N_i^n, the number of particles in state i at time n, has mean $\mu_i^{(n)}$. Therefore the termwise convergence of $U^{(n)}$ is a necessary condition for the existence of Poisson limiting distributions for the N_i^n. In § 4, we give a sufficient condition that these limiting distributions exist and are Poisson; the condition roughly is that $U^{(n)}$ converge termwise and the variances of the N_i^0 are not too large. This theorem is closely related to Theorem 6 of **(4)**, and the proof is similar; however, our result, with the aid of the material of §§ 2 and 3, does apply to certain cases where the hypotheses of **(4)** are not satisfied. This sort of theorem has also been discussed by several other authors for the case of spatially homogeneous (sums of random variables) processes.

In § 5 a different sort of convergence is considered. Instead of putting all the particles into the various states of the Markov chain at time 0, they can be introduced continually into some fixed state and allowed to diffuse away from it. It is shown that this can always be done in such a way as to obtain convergence to Derman's stationary process. This sort of process was suggested to the author by F. Spitzer.

Finally, we observe that many of our results have analogues in the case of continuous time-parameter Markov chains. With the aid of recent work on such chains, mainly that of K. L. Chung, it appears that proofs quite similar to those in the discrete case can be given. These ideas will not be carried out in this paper.

2. The convergence of $U^{(n)}$. The multiplications $U^{(0)}P^n$ will be well defined provided that

$$|\mu_i^{(0)}| \leqslant Mq_i, \tag{3}$$

and this condition will be assumed hereafter. It is also assumed that the $\mu_i^{(0)}$ are real.

THEOREM 1. *Suppose that for each i the sequence $\mu_i^{(n)}$ is Abel summable; call the limits μ_i. Then there is a constant α such that $\mu_i = \alpha q_i$ for all i.*

Proof. Without loss of generality we can assume that $\mu_i^{(0)} \geqslant 0$. For in any case, there is an M such that

$$\nu_i^{(0)} = \mu_i^{(0)} + Mq_i \geqslant 0,$$

and $\nu_i^{(n)}$ is summable to $\mu_i + Mq_i$; if these numbers are multiples of the q_i by a constant, so are the μ_i.

Now proceeding formally we have

$$\sum_i \mu_i P_{ij} = \sum_i P_{ij} \lim_{x\to 1-} \left\{(1-x)\sum_n \mu_i^{(n)} x^n\right\} = \lim_{x\to 1-} (1-x)\sum_i P_{ij} \sum_n \mu_i^{(n)} x^n$$

$$= \lim_{x\to 1-} (1-x) \sum_n x^n \sum_i \mu_i^{(n)} P_{ij} = \lim_{x\to 1-} \frac{1-x}{x} \sum_n \mu_j^{(n+1)} x^{n+1} = \mu_j.$$

Actually the exchanges of limits are justified; the first one since

$$0 \leqslant (1-x) \sum_n \mu_i^{(n)} x^n \leqslant M q_i \qquad \text{for all } x \in (0,1),$$

so that the sum over i is uniformly convergent, and the second since the terms summed are non-negative. Hence, μ_i are a solution of (1), and $\mu_i \geqslant 0$ since we assumed $\mu_i^{(0)} \geqslant 0$ for all i. The uniqueness of such solutions completes the proof.

We shall use (m) to denote the Banach space of bounded sequences of real numbers (sup norm) and (c) for the subspace of convergent sequences. This notation and other facts about Banach spaces used below may be found in **(1)**.

THEOREM 2. *Let $\{x_i\} \in (c)$, and let $\mu_i^{(0)} = x_i q_i$. Then for all i,*

$$\lim_n \mu_i^{(n)} = q_i \lim \{x_j\}. \tag{4}$$

Proof. We can assume that $x_i \to 0$, since if the limit is α we consider $\mu_i^{(0)} - \alpha q_i$ instead of $\mu_i^{(0)}$. If $\epsilon > 0$, by the hypothesis we can put

$$\mu_i^{(0)} = \nu_i^{(0)} + \omega_i^{(0)},$$

where $|\nu_i^{(0)}| \leqslant \frac{1}{2} q_i \epsilon$ and only a finite number of $\omega_i^{(0)}$ are different from 0. Now for each j

$$\omega_j^{(n)} = \sum_i \omega_i^{(0)} P_{ij}^{(n)} \to 0,$$

since $P_{ij}^{(n)} \to 0$ for each i, j **(6)**. Also, it is easy to see from (1) that

$$|\nu_j^{(n)}| = \left|\sum_i \nu_i^{(0)} P_{ij}^{(n)}\right| \leqslant \tfrac{1}{2}\epsilon q_j.$$

Hence for large n, $|\mu_i^{(n)}| \leqslant \epsilon q_i$ and so $\mu_i^{(n)} \to 0$ for every i. This and the remark above prove (4).

The theorem just proved seems far from what one might hope for, but it does provide a class of sequences which, *in any order*, when multiplied by the invariant measure yield a vector $U^{(0)}$ for which $U^{(n)}$ is convergent. Actually, the convergent sequences are the only ones with this property:

THEOREM 3. *For each P and each sequence $\{x_i\} \in (m)$ but which is not convergent, there is a permutation π of the positive integers such that if $U^{(0)}$ is formed using the rearrangement of $\{x_i\}$, that is,*

$$\mu_i^{(0)} = x_{\pi i} q_i,$$

then for some i, $\mu_i^{(n)}$ fails to be Abel summable.

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The proof rests on two lemmas, which may be of independent interest. The first is a much less precise version of the theorem.

LEMMA 1. *For each P, there exists a sequence $\{x_i\} \in (m)$ such that if $\mu_i^{(0)} = x_i q_i$ then for some i, $\mu_i^{(n)}$ is not Abel summable.*

Proof. Let $l(q)$ be the Banach space of sequences $\{y_i\}$ such that

$$||y|| = \sum_i |y_i| q_i < \infty .$$

With respect to the inner product

$$(y, x) = \sum_i y_i x_i q_i,$$

(m) is the conjugate space of $l(q)$. We can use the matrix P to define an operator T of norm one on $l(q)$:

$$\{Ty\}_i = \sum_j P_{ij} y_j.$$

It is easily verified that the operator T^* on (m) defined by

$$\{T^* x\}_j = \frac{1}{q_j} \sum_i x_i q_i P_{ij}$$

is the adjoint of T.

Let δ^k stand for the sequence whose kth term is one and the rest zero. Then for $|z| < 1$,

$$\left((1 - z) \sum_n z^n T^n \delta^j, \delta^i\right) = q_i (1 - z) \sum_n P_{ij}^{(n)} z^n$$

which has limit 0 as $z \to 1 -$. This means that for fixed j if the vectors $(1 - z) \sum_n z^n T^n \delta^j$ have a weak limit as $z \to 1 -$, that limit must be zero. But for any $z < 1$,

$$\left((1 - z) \sum_n z^n T^n \delta^j, 1\right) = q_j.$$

Therefore $(1 - z) \sum_n z^n T^n \delta^j$ does not converge weakly to zero, and hence has no weak limit. But the space $l(q)$ is weakly complete; we conclude that there exists a linear functional (that is, a sequence $\{x_i\} \in (m)$) such that

$$\left((1 - z) \sum_n z^n T^n \delta^j, x\right) = (1 - z) \sum_n z^n (\delta^j, T^{*n} x)$$
$$= (1 - z) \sum_n z^n \sum_i x_i q_i P_{ij}^{(n)} = (1 - z) \sum_n z^n \mu_j^{(n)}$$

does not have a limit as $z \to 1 -$. This is the assertion of the lemma.

Remark. We have actually proved that for each j, there is an $x \in (m)$ such that if $\mu_i^{(0)} = x_i q_i$, then the $\mu_j^{(n)}$ are not summable. It is not hard to add that there is an x with non-negative components such that for no j does $\mu_j^{(n)}$ form an Abel summable sequence; this is not needed in the proof of Theorem 3.

Lemma 2. *In the class of closed subspaces of* (m) *which are invariant under all reorderings,* (c) *is maximal.*

*Proof.** Let (d) be a closed subspace of (m) containing (c) and such that $\{x_i\} \in (d)$ implies $\{x_{\pi i}\} \in (d)$ for any permutation π of the integers. Suppose that (d) contains a sequence consisting only of 1's and 0's and with an infinite number of each. Then (d) contains all sequences containing only 1's and 0's.

A sequence will be called "simple" if it contains only finitely many different numbers. Under our assumptions, all simple sequences belong to (d), since such a sequence is a finite linear combination of sequences consisting of 1's and 0's. But any bounded sequence can be uniformly approximated by simple ones, so (d) must equal (m).

Now suppose instead that (d) contains some sequence which is not convergent, say $\{y_i\}$. Then there must be two convergent subsequences, say $\{y_{n(i)}\}$ and $\{y_{m(i)}\}$, with limits α and β respectively, $\alpha \neq \beta \neq 0$. By adding two suitable convergent sequences and multiplying by a constant, we can see that $\{z_i\} \in (d)$, where

$$z_{n(i)} = 1, z_{m(i)} = 0, \qquad \text{and} \qquad z_j = \frac{y_j - \beta}{\alpha - \beta}$$

for other values of j. (d) then also contains $\{w_i\}$ where

$$w_{n(2i)} = 1, \ w_{n(2i-1)} = 0 = w_{m(i)}, \quad \text{and} \quad w_j = z_j$$

otherwise, since this sequence was obtained from $\{z_i\}$ by a rearrangement. Subtracting, we obtain a non-convergent sequence of 1's and 0's only which belongs to (d), and by the argument above we conclude that $(d) = (m)$, which completes the proof.

Proof of Theorem 3. The quantities $\mu_i^{(n)}$ are now defined by (2) using the matrix P under consideration. Let (e) be the class of all sequences $\{x_i\} \in (m)$ such that if $\mu_i^{(0)} = x_{\pi i} q_i$, then $\mu_j^{(n)}$ is Abel-summable for each j and for each permutation π of the integers. By definition, (e) is invariant under reorderings; by Theorem 2 $(e) \supset (c)$. It is not hard to see that (e) is a closed subspace of (m), but from Lemma 1 we know that $(e) \neq (m)$. Therefore by Lemma 2, $(e) = (c)$, which proves the theorem.

3. Special classes of P. In the previous section, dealing with arbitrary P having no intrinsic ordering of the states, we considered sequences giving rise in any ordering to convergent $\mu_j^{(n)}$. Here we shall look at certain Markov chains in which there is a natural order for the states. First consider a Markov process consisting of sums of independent, identically-distributed random variables X_n taking integer values, and let P be the matrix of transition probabilities. Then $q_j = 1$ for all j is a solution of (1); the sequences $\mu_j^{(0)}$ satisfying (3) are just those in (m). Let

*A discussion with Halsey Royden was very helpful in proving this lemma.

$$p_j = \Pr(X_n = j) \qquad \text{and} \qquad \phi(x) = \sum_{n=-\infty}^{\infty} e^{inx} p_n.$$

(In this section, the letter i will not be used for an index.)

THEOREM 4. *If*

$$\mu_j^{(0)} = \nu_j^{(0)} + \int_{-\pi}^{\pi} e^{ijx} dF(x), \tag{5}$$

where $\nu_j^{(0)} \to 0$ *as* $j \to \pm\infty$ *and* $F(x)$ *is of bounded variation, then* $\mu_k^{(n)}$ *is Cesàro summable for each* k. *If* $\gcd\{j : p_j = 0\} = 1$, *the sequences* $\mu_k^{(n)}$ *are convergent.*

Proof. In view of Theorem 2, we can assume the $\nu_j^{(0)}$ are zero. Then by (5),

$$\mu_k^{(n)} = \sum_j \mu_j^{(0)} P_{jk}^{(n)} = \int_{-\pi}^{\pi} \sum_j P_{jk}^{(n)} e^{ijx} dF(x).$$

But

$$P_{jk}^{(n)} = \Pr(X_1 + \ldots + X_n = k - j),$$

so that

$$\sum_j P_{jk}^{(n)} e^{i(k-j)x} = \phi^n(x).$$

Hence

$$\mu_k^{(n)} = \int_{-\pi}^{\pi} e^{ikx} \phi^{-n}(x) dF(x). \tag{6}$$

If $\gcd\{j : p_j = 0\} = 1$, $x = 0$ is the only point in $[-\pi, \pi]$ at which $|\phi(x)| = 1$, and so $\mu_k^{(n)}$ converges to the jump of $F(x)$ at 0. If the gcd $= d$, say, then $|\phi(x)| = 1$ only when e^{ix} is a dth root of unity, and Cesàro convergence follows from (6).

Another type of Markov chain with intrinsic ordering of the states is a random walk. Karlin and McGregor have shown (7) that for every random walk on the non-negative integers there is a non-decreasing function $\psi(x)$ such that

$$P_{jk}^{(n)} = q_k \int_{-1}^{1} x^n Q_j(x) Q_k(x) d\psi(x), \tag{7}$$

where as usual q_j are a solution of (1) and the $Q_j(x)$ are the orthogonal polynomials of the measure $d\psi(x)$, normalized so that $Q_j(1) = 1$.

THEOREM 5. *If, in the case of a null-recurrent random walk,*

$$\mu_j^{(0)} = q_j\{y_j + \int_{-1}^{1} Q_j(x) dF(x)\} \tag{8}$$

where $F(x)$ *is of bounded variation and* $y_j \to 0$, *then* $\mu_j^{(2n)}$ *converges for each* j. *The limits of* $\mu_j^{(n)}$ *exist if and only if* $F(x)$ *is continuous at* -1; *in any case the Cesàro limits exist.*

Proof. As before, it is enough in view of Theorem 2 to consider the case when $y_j = 0$ for all j. Using (7) we have

$$\mu_k^{(n)} = \lim_{r\to 1-} \sum_j \mu_j^{(0)} r^j P_{jk}^{(n)} = \lim_{r\to 1-} q_k \int_{-1}^{1} x^n Q_k(x) \sum_j \mu_j^{(0)} r^j Q_j(x) d\psi(x). \tag{9}$$

Putting $n = 0$ and taking note of (8) gives for each k

$$q_k \int_{-1}^{1} Q_k(x) dF(x) = \lim_{r\to 1-} q_k \int_{-1}^{1} Q_k(x) dF_r(x),$$

where $dF_r(x) = \sum_j \mu_j^{(0)} r^j Q_j(x) d\psi(x)$; this implies that for every polynomial $f(x)$,

$$\lim_{r\to 1-} \int_{-1}^{1} f(x) dF_r(x) = \int_{-1}^{1} f(x) dF(x).$$

Hence (9) can be rewritten as

$$\mu_k^{(n)} = q_k \int_{-1}^{1} x^n Q_k(x) dF(x)$$

from which the conclusions of the theorem are obvious.

4. Convergence to Derman's stationary process. In this section we assume that at time 0, there are N_j^0 particles in state j of the Markov chain, where the random variables N_j^0 are independent and

$$\mu_j^{(0)} = E(N_j^0),\ m_j = E[N_j^0(N_j^0 - 1)].$$

The particles then evolve independently of each other according to the law of the chain, and the number in state j at time n is called N_j^n. It is easy to see from (2) that $E(N_j^n) = \mu_j^{(n)}$. It is also not hard to verify (Derman's computations in **(4)** do it) that if the N_j^0 are all Poisson distributed, so are the N_j^n. In this case the question of the existence of limiting distributions as $n \to \infty$ reduces to the existence of limits of the sequences $\mu_j^{(n)}$.

We shall investigate a more general situation:

THEOREM 6. *Suppose that for some k, the moments of the random variables N_j^0 satisfy*

$$\lim_{n\to\infty} \max_j \mu_j^{(0)} P_{jk}^{(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_j m_j (P_{jk}^{(n)})^2 = 0. \tag{10}$$

Suppose also that $\mu_k = \lim \mu_k^{(n)}$ exists. Then as $n \to \infty$, the distribution of N_k^n approaches a Poisson distribution with mean μ_k.

Remark. For each k such that

$$\lim_{n\to\infty} \max_j q_j P_{jk}^{(n)} = 0, \tag{11}$$

condition (10) holds provided the simpler conditions

$$|\mu_j^{(0)}| \leqslant M q_j \quad \text{and} \quad |m_j| \leqslant M q_j^2 \quad \text{for all } j \tag{12}$$

are satisfied. For most interesting types of Markov chains, (11) holds for all k. However, an example of a chain with a state not satisfying (11) can be constructed along the lines of Example 4 of **(4)**.

Proof of Theorem 6. The proof is based on the use of generating functions. Let

$$f_k^{(n)}(x) = E[x^{N_k^n}],$$

and notice that

$$N_k^n = \sum_j N_{jk}^n,$$

where $N_{jk}{}^n$ is the number of particles which are in state j at time 0 and in state k at time n. For given n and k the $N_{jk}{}^n$ are independent, and $N_{jk}{}^n$ is the sum of $N_j{}^0$ independent random variables equal to 1 or 0 with probabilities $P_{jk}{}^{(n)}$ and $1 - P_{kj}{}^{(n)}$. Combining these facts gives

$$f_k^{(n)}(x) = \prod_j f_j^{(0)}(1 - P_{jk}^{(n)} + P_{jk}^{(n)}x). \tag{13}$$

Now under our assumptions,

$$f_j^{(0)}[1 - P_{jk}^{(n)}(1-x)] = 1 - \mu_j^{(0)}P_{jk}^{(n)}(1-x) + \tfrac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1-x)]^2 \tag{14}$$

where $0 \leqslant \theta_j \leqslant 1$. Substituting (14) in (13), taking the logarithm, and using the estimate

$$u - 1 \geqslant \log u \geqslant u - 1 - \frac{(u-1)^2}{u}$$

for $0 < u < 1$ yields

$$\left|\log f_k^{(n)}(x) - \sum_j (x-1)\mu_j^{(0)}P_{jk}^{(n)}\right| \leqslant \tfrac{1}{2}\sum_j \theta_j m_j[P_{jk}^{(n)}(1-x)]^2 + \sum_j \frac{\{(x-1)\mu_j^{(0)}P_{jk}^{(n)} + \frac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1-x)]^2\}^2}{1 - \mu_j^{(0)}P_{jk}^{(n)}(1-x) + \frac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1-x)]^2}. \tag{15}$$

For n large, the first bounding term is arbitrarily small because of the second part of (10). The denominators of the second term are uniformly bounded away from 0 for large n, again by (10), and, further, it follows that the sum of the numerators in the second term approaches 0 as n increases. All these estimates hold uniformly in x for $0 \leqslant x \leqslant 1$. Since it was also assumed that $\mu_k{}^{(n)} \to \mu_k$, an additional estimate in (15) yields the conclusion that

$$\lim_{n\to\infty} \log f_k^{(n)}(x) = (x-1)\mu_k$$

uniformly for $0 \leqslant x \leqslant 1$, and the theorem follows.

Remark. Theorems of a similar sort have been proved for spatially-homogeneous processes by Maruyama **(8)** and (more generally) by Dobrusin **(5)**. A theorem (similar to Maruyama's) for the case of discrete time and states is found in **(9)**; a very similar result can be deduced from Theorems 2 and 6 of the present paper. Using in addition Theorem 3 allows a generalization, which overlaps with some of the results of **(5)**.*

5. Another type of process. In this section we suppose that a special state of P, say 0, has been selected and that X_n particles are put in state 0 at each time n. As before, the X_n are independent, and each particle, once introduced, independently moves subject to the law of the Markov chain. Again let N_i^n denote the number of particles in state i at time n; let $E(X_n) = a_n$. It is not hard to see that

$$E(N_i^n) = \mu_i^{(n)} = \sum_{l=0}^{n} a_l P_{0i}^{(n-l)}. \tag{16}$$

First we shall study convergence properties of the $\mu_i^{(n)}$, and then give a theorem analogous to that of the last section on the convergence of the distributions of N_i^n.

THEOREM 7. *If for some value of k, the sequence $\mu_k^{(n)}$ is Abel summable to sum μ_k, then $\mu_i^{(n)}$ is summable for all i to μ_i, say, and there is a constant α such that $\mu_i = \alpha q_i$ for all i.*

Proof. Let $U_i(x) = \sum_n \mu_i^{(n)} x^n$, $A(x) = \sum_n a_n x^n$, and $P_{ij}(x) = \sum_n P_{ij}^{(n)} x^n$. From (16),

$$U_i(x) = A(x) P_{0i}(x). \tag{17}$$

From this and the hypothesis of Abel summability of $\mu_k^{(n)}$ we obtain

$$A(x) \sim \frac{\mu_k}{(1-x)P_{0k}(x)}$$

as $x \to 1-$. Therefore for any i,

$$U_i(x) \sim \frac{\mu_k}{(1-x)} \frac{P_{0i}(x)}{P_{0k}(x)}.$$

But it follows from Doeblin's ratio theorem **(2)** that

$$\lim_{x \to 1-} \frac{P_{0i}(x)}{P_{0k}(x)} = \frac{q_i}{q_k}.$$

Therefore $\{\mu_i^{(n)}\}$ is Abel summable to $q_i \mu_k / q_k$, which proves the theorem.

THEOREM 8. *For each P, there exists a monotone sequence of positive numbers a_n such that $\{u_k^{(n)}\}$ converges for every k.*

*This result obtained by combining our theorems 3 and 6, specialized to the case of the coin-tossing process, may be compared with the example at the end of §1 of **(4)**. It can be seen that Derman's results do not contain ours, or vice versa.

Proof. Let us define the a_n by supposing that

$$A(x) = \frac{1}{(1-x)P_{00}(x)}.$$

It is easy to verify that $a_n \downarrow 0$; in fact

$$a_n = f_{00}^{(n+1)} + f_{00}^{(n+2)} + \ldots,$$

where $f_{ij}^{(n)}$ are the first passage probabilities for P. In view of (17), this definition of $A(x)$ implies that $\mu_0^{(n)} = 1$ for all n. Now if $k \neq 0$, let ${}_0P_{0k}^{(n)}$ be the probability of a transition from state 0 to state k in n steps during which state 0 is not revisited **(2)**. It follows that if $k \neq 0$,

$$P_{0k}(x) = P_{00}(x)\,{}_0P_{0k}(x),$$

where ${}_0P_{0k}(x)$ is the generating function of the ${}_0P_{0k}^{(n)}$. Hence

$$u_k(x) = \frac{P_{0k}(x)}{(1-x)P_{00}(x)} = \frac{1}{1-x}\,{}_0P_{0k}(x).$$

But $\sum_n {}_0P_{0k}^{(n)} < \infty$ **(2)**, which implies that $\lim \mu_k^{(n)} = \mu_k$ exists.

Finally we study the distribution of N_k^n; define

$$m_l = E(X_l(X_l - 1)).$$

THEOREM 9. *Let $\{a_n\}$ be a sequence with the property specified in Theorem* 8, *and suppose that $m_n \leqslant Ma_n^2$. Then for each k, N_k^n is asymptotically Poisson distributed; the asymptotic means are proportional to the q_k.*

Proof. Let $g_l(x)$ be the generating function of X_l, and again let $f_k^{(n)}(x)$ be that of N_k^n. Then

$$f_k^{(n)}(x) = \prod_{l=0}^{n} g_l[1 - (1-x)P_{0k}^{(n-l)}].$$

We perform upon this generating function very much the same sort of estimate which was used in the proof of Theorem 6. The result is that

$$\lim_{n\to\infty} \log f_k^{(n)}(x) = (x-1)\lim_{n\to\infty}\sum_{l=0}^{n} a_l P_{0k}^{(n-l)} = (x-1)\mu_k$$

uniformly for $x \in (0, 1)$; the theorem follows.

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