LEFT SYMMETRIC POINTS FOR BIRKHOFF ORTHOGONALITY IN THE PREDUALS OF VON NEUMANN ALGEBRAS

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Abstract

In this paper, we give a complete description of left symmetric points for Birkhoff orthogonality in the preduals of von Neumann algebras. As a consequence, except for $\mathbb{C}$, ℓ_∞^2 and $M_2(\mathbb{C})$, there are no von Neumann algebras whose preduals have nonzero left symmetric points for Birkhoff orthogonality.

1. Introduction

In Banach space theory, there are various notions of generalised orthogonality relations defined by using norm (in)equalities. Among them, the orthogonality relation by means of nearest points is of particular importance. It is known as Birkhoff (–James) orthogonality and is defined as follows.

DEFINITION 1.1 (Birkhoff (–James) orthogonality). Let X be a Banach space and let $x, y \in X$. Then x is said to be *Birkhoff orthogonal* to y, denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{C}$.

A geometric meaning of $x \perp_B y$ is that x is the nearest point to 0 in the line $x + \lambda y$. If x is additionally a unit vector, then $x + \lambda y$ becomes a tangent line to the unit ball at x. Thus Birkhoff orthogonality is closely related to metric projections as well as support functionals for the unit ball.

As a basic property of Birkhoff orthogonality, $x \perp_B y$ implies that $\alpha x \perp_B \beta y$ for any scalars α, β. However, the relation '$\perp_B$' is not symmetric in general, that is, $x \perp_B y$ may not imply $y \perp_B x$. In fact, it is known that if Birkhoff orthogonality is symmetric in a Banach space X with $\dim X \geq 3$, then X is a Hilbert space (see, for example, [3]). We refer to [2] for further information about Birkhoff orthogonality.

Recently, instead of the global symmetry, some notions of local symmetry for Birkhoff orthogonality were introduced and studied.

DEFINITION 1.2 (Sain [11]). Let X be a Banach space and let $x \in X$. Then x is called a *left symmetric point* for Birkhoff orthogonality if $y \in X$ and $x \perp_B y$ imply that $y \perp_B x$.

DEFINITION 1.3 (Sain [11]). Let X be a Banach space and let $x \in X$. Then x is called a *right symmetric point* for Birkhoff orthogonality if $y \in X$ and $y \perp_B x$ imply that $x \perp_B y$.

These notions have been widely studied, for example, in [4, 6, 11–14]. Moreover, in [10], it was shown that left or right symmetric points for Birkhoff orthogonality in von Neumann algebras have some characteristic properties. Namely, a norm-one element A in a von Neumann algebra $\mathcal{R}$ is left symmetric for Birkhoff orthogonality if and only if $|A|$ is a central projection that is minimal in $\mathcal{R}$, while A is right symmetric if and only if it is an extreme point of the unit ball of $\mathcal{R}$.

Thus it is natural to ask whether similar characterisations for left or right symmetric points for Birkhoff orthogonality can be given in the preduals of von Neumann algebras.

The aim of this paper is to give, as a partial answer, a complete description of left symmetric points for Birkhoff orthogonality in the preduals of von Neumann algebras. As a consequence, except for $\mathbb{C}$, ℓ_∞^2 and $M_2(\mathbb{C})$, there are no von Neumann algebras whose preduals have nonzero left symmetric points for Birkhoff orthogonality.

2. Preliminaries

We start this section with the following useful characterisation of Birkhoff orthogonality.

LEMMA 2.1 (James [7]). *Let X be a Banach space and let x, y be nonzero elements of X. Then $x \perp_B y$ if and only if there exists an element f of X^* satisfying $\|f\| = 1$, $f(x) = \|x\|$ and $f(y) = 0$.*

For a Banach space X, let B_X denote the unit ball of X. Let $\mathcal{R}$ be a von Neumann algebra and let $\mathcal{R}_*$ be the predual of $\mathcal{R}$ (that is, the Banach space of all normal linear functionals on $\mathcal{R}$). For each element $\rho \in \mathcal{R}_*$ with $\|\rho\| = 1$, let

$$\{\rho\}^f = \{A \in B_{\mathcal{R}} : \rho(A) = 1\}.$$

We note that $\{\rho\}^f$ is a nonempty weak-operator closed face of $B_{\mathcal{R}}$.

A state ω of a von Neumann algebra $\mathcal{R}$ is said to be *faithful* if $\omega(A) > 0$ for each nonzero positive element $A \in \mathcal{R}$. This property has a characterisation in terms of $\{\omega\}^f$.

LEMMA 2.2. *Let $\mathcal{R}$ be a von Neumann algebra. Then ω is a faithful normal state of $\mathcal{R}$ if and only if $\{\omega\}^f = \{I\}$.*

Proof. Suppose that ω is a faithful state and that $\omega(A) = 1$ for some $A \in B_{\mathcal{R}}$. Let $H = 2^{-1}(A + A^*)$ and $K = -2^{-1}i(A - A^*)$. Then $\|H\| \leq 1$, $\|K\| \leq 1$ and $A = H + iK$. Since ω is positive, it follows from $\omega(H) + i\omega(K) = 1$ that $\omega(H) = 1$ and $\omega(K) = 0$. Moreover, $I - H \geq 0$ and $\omega(I - H) = 0$, so that $H = I$. By a characterisation of extreme points of the unit ball of a C^*-algebra (see, for example, [8, Theorem 7.3.1]), I is an extreme point of the unit ball of $\mathcal{R}$. Hence $A = A^* = I$.

Conversely, assume that $\{\omega\}^f = \{I\}$. Let $A \geq 0$ be such that $\omega(A) = 0$. It may be assumed that $\|A\| = 1$. Then $0 \leq I - A \leq I$ and $\omega(I - A) = 1$, which implies that $I - A \in \{\omega\}^f$. Thus $A = 0$. □

We also make use of the facial structure of von Neumann algebras given by Edwards and Rüttimann [5]; see also [1].

Theorem 2.3 (Edwards and Rüttimann [5]). *Let $\mathcal{R}$ be a von Neumann algebra and let F be a weak-operator closed proper face of $B_{\mathcal{R}}$. Then there exists a partial isometry V in $\mathcal{R}$ such that*

$$F = V + (I - VV^*)B_{\mathcal{R}}(I - V^*V).$$

3. Left symmetric points

We now proceed to study left symmetric points for Birkhoff orthogonality in the preduals of von Neumann algebras.

Lemma 3.1. *Let $\mathcal{R}$ be a von Neumann algebra and let ρ be an element of $\mathcal{R}_*$ with $\|\rho\| = 1$. If ρ is a left symmetric point for Birkhoff orthogonality in $\mathcal{R}_*$, then $\{\rho\}^f$ is a singleton, that is, there exists a unique extreme point V of $B_{\mathcal{R}}$ satisfying $\rho(V) = 1$.*

Proof. By Theorem 2.3, there exists a partial isometry $V \in \mathcal{R}$ such that

$$\{\rho\}^f = V + (I - VV^*)B_{\mathcal{R}}(I - V^*V).$$

Put $V^*V = E$ and $VV^* = F$ for short. If $(I - F)B_{\mathcal{R}}(I - E) \neq \{0\}$, then $C_{I-E}C_{I-F} \neq 0$, where C_{I-E} and C_{I-F} are the central carriers for $I - E$ and $I - F$, respectively. Using [8, Proposition 6.1.8], we have two nonzero projections $E_0 \leq I - E$ and $F_0 \leq I - F$ with $E_0 \sim F_0$. Let W be a partial isometry in $\mathcal{R}$ with $W^*W = E_0$ and $WW^* = F_0$. Then $W = (I - F)(F_0WE_0)(I - E) \in (I - F)B_{\mathcal{R}}(I - E)$. Take an arbitrary unit vector x satisfying $E_0x = x$, and let $\omega(A) = \langle (I - F)A(I - E)x, Wx \rangle$ for each $A \in \mathcal{R}$. Then $\|\omega\| = \omega(W) = 1$. Now let $\tau = 2^{-1}(\rho + \omega)$. Then $\|\tau\| = \tau(V + W) = 1$. Moreover, it follows from $V - W \in \{\rho\}^f$ and

$$\tau(V - W) = \tfrac{1}{2}(\rho(V - W) - \omega(W)) = 0$$

that $\rho \perp_B \tau$. However, since 1 is an extreme point of the unit disk $\mathbb{D}$ of $\mathbb{C}$, if $A \in B_{\mathcal{R}}$ and $\tau(A) = 1$, then $\rho(A) = \omega(A) = 1$. Hence $\tau \not\perp_B \rho$ by Lemma 2.1. This contradicts the left symmetry of ρ. Thus $(I - F)B_{\mathcal{R}}(I - E) = \{0\}$, that is, $\{\rho\}^f = \{V\}$ and V is an extreme point of $B_{\mathcal{R}}$. □

It is well known that each normal functional ρ on $\mathcal{R}$ (that is, an element of $\mathcal{R}_*$) can be represented as $\rho(A) = \omega(V^*A)$ for each A, where ω is a positive normal functional on $\mathcal{R}$ and V is a partial isometry that is an extreme point of $B_{\mathcal{R}}$ (see [8, Theorem 7.3.2]). Moreover, in that case, $\omega(A) = \rho(VA)$ for each A. In particular, $\|\rho\| = \|\omega\|$.

In what follows, for $\rho \in \mathcal{R}^*$ and $B \in \mathcal{R}$, we use the symbol ρB for the linear functional $A \mapsto \rho(BA)$. The fact mentioned in the preceding paragraph can be stated as *if $\rho \in \mathcal{R}_*$, then $\rho = \omega V^*$ and $\omega = \rho V$ for some positive $\omega \in \mathcal{R}_*$ and some partial isometry V.*

The following fact is well known. A proof was included in [10, Theorem 4.7].

LEMMA 3.2. *Let $\mathcal{R}$ be a von Neumann algebra and let V be an extreme point of $B_{\mathcal{R}}$. Then there exists an orthogonal pair P, Q of central projections with sum I such that $V^*VP = P$ and $VV^*Q = Q$.*

For a normal functional ρ on a von Neumann algebra $\mathcal{R}$, let $\rho^* \in \mathcal{R}_*$ be the normal functional defined by $\rho^*(A) = \overline{\rho(A^*)}$ for each $A \in \mathcal{R}$. We note that $\rho \mapsto \rho^*$ is a conjugate-linear isometry on $\mathcal{R}_*$ and that $(\rho^*)^* = \rho$.

LEMMA 3.3. *Let $\mathcal{R}$ be a von Neumann algebra and let ρ, τ be elements of $\mathcal{R}_*$. Then $\rho \perp_B \tau$ if and only if $\rho^* \perp_B \tau^*$.*

PROOF. Simply note that $\|\rho + \lambda\tau\| = \|(\rho + \lambda\tau)^*\| = \|\rho^* + \overline{\lambda}\tau^*\|$ for each λ.   □

The following is an immediate consequence of Lemma 3.3.

PROPOSITION 3.4. *Let $\mathcal{R}$ be a von Neumann algebra and let ρ be an element of $\mathcal{R}_*$. Then ρ is left symmetric for Birkhoff orthogonality in $\mathcal{R}_*$ if and only if ρ^* has the same property.*

We need a technical lemma concerning the number of mutually orthogonal nonzero projections in von Neumann algebras.

LEMMA 3.5. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If $I = E_1 + E_2 + E_3 + E_4 + E_5$ for an orthogonal family of nonzero projections $\{E_1, E_2, E_3, E_4, E_5\}$ in $\mathcal{R}$, then there are no nonzero left symmetric points for Birkhoff orthogonality in $\mathcal{R}_*$.*

PROOF. Let ρ be a nonzero normal functional on $\mathcal{R}$ that is left symmetric for Birkhoff orthogonality in $\mathcal{R}_*$. We may assume that $\|\rho\| = 1$. Then there is a normal state ω of $\mathcal{R}$ and an extreme point V of $B_{\mathcal{R}}$ such that $\rho = \omega V^*$ and $\omega = \rho V$. In particular, $1 = \omega(I) = \omega(V^*V)$. We also note that $\{\rho\}^f = \{V\}$ by Lemma 3.1. From this, ω is restricted to a faithful normal state of $E\mathcal{R}E$, where $E = V^*V$. Indeed, if $\|EAE\| \le 1$ and $\omega(EAE) = 1$, then $\rho(VEAE) = 1$ and $\{\rho\}^f = \{V\}$ imply that $V = VEAE = VAE$. Thus it follows that $E = EAE$. Now Lemma 2.2 works, and ω is faithful on $E\mathcal{R}E$. We recall that a von Neumann algebra is countably decomposable if and only if it has a faithful normal state (see [8, Exercise 7.6.46 (ii)] and its proof in [9]).

First, suppose that $E = E_1 + E_2 + E_3$ for mutually orthogonal nonzero projections $\{E_1, E_2, E_3\}$ in $E\mathcal{R}E$. We may assume that $\omega(E_1) \ge \omega(E_2) \ge \omega(E_3) > 0$. Since each von Neumann algebra $E_j\mathcal{R}E_j$ is also countably decomposable, there is a faithful normal

state ω_j on $E_j \mathcal{R} E_j$ for $j = 1, 2, 3$. In particular, $\omega'_j(A) = \omega_j(E_j A E_j)$ defines a normal state of $\mathcal{R}$. Let $\tau = 4^{-1}(\omega'_1 + \omega'_2) - 2^{-1}\omega'_3 \in \mathcal{R}_*$. Then

$$\|\tau V^*\| = (\tau V^*)(V E_1 + V E_2 - V E_3) = 1.$$

Moreover, $(\tau V^*)(V) = 0$ implies that $\rho \perp_B \tau V^*$. On the other hand, if A is an element of $\mathcal{R}$ satisfying $\|A\| = 1$ and $(\tau V^*)(A) = 1$, then $\omega'_1(V^*A) = \omega'_2(V^*A) = -\omega'_3(V^*A) = 1$. By Lemma 2.2, it follows that $E_1 V^* A E_1 = E_1$, $E_2 V^* A E_2 = E_2$ and $E_3 V^* A E_3 = -E_3$. Furthermore,

$$
\begin{aligned}
\|E_j x\|^2 \geq \|V^* A E_j x\|^2 &= \|E_j V^* A E_j x + (I - E_j) V^* A E_j x\|^2 \\
&= \|E_j A E_j x\|^2 + \|(I - E_j) V^* A E_j x\|^2 \\
&= \|E_j x\|^2 + \|(I - E_j) V^* A E_j x\|^2
\end{aligned}
$$

for each x and $j = 1, 2, 3$, which implies that $(I - E_j) V^* A E_j = 0$ for $j = 1, 2, 3$. Since $V^* A E_j = E_j V^* A E_j$, it follows that

$$
\begin{aligned}
\rho(A) = \omega(V^* A) = \omega(V^* A E) &= \omega(V^* A E_1 + V^* A E_2 + V^* A E_3) \\
&= \omega(E_1) + \omega(E_2) - \omega(E_3) > 0,
\end{aligned}
$$

which shows that $\tau V^* \not\perp_B \rho$, which is a contradiction.

Next, we consider the functional ρ^* which is also left symmetric by Proposition 3.4. Note that $\{\rho^*\}^f = \{V^*\}$. Put $F = V V^*$. As shown above, the identity $F = F_1 + F_2 + F_3$ never holds for mutually orthogonal nonzero projections $\{F_1, F_2, F_3\}$ in $F \mathcal{R} F$.

Since V is an extreme point of $B_\mathcal{R}$, by Lemma 3.2, there exists an orthogonal pair P, Q of central projections with sum I such that $EP = P$ and $FQ = Q$. In particular, $P \leq E$ and $Q \leq F$. If $I = G_1 + G_2 + G_3 + G_4 + G_5$ for some orthogonal family of nonzero projections $\{G_1, G_2, G_3, G_4, G_5\}$, then $G_j P \neq 0$ or $G_j Q \neq 0$ for $j = 1, 2, 3, 4, 5$. Therefore

$$\sharp\{j \in \{1, 2, 3, 4, 5\} : G_j P \neq 0\} \geq 3, \quad \text{or}$$
$$\sharp\{j \in \{1, 2, 3, 4, 5\} : G_j Q \neq 0\} \geq 3.$$

However, then either E or F must be the sum of three or more nonzero subprojections. This is impossible. Thus, if $\{E_\lambda\}_{\lambda \in \Lambda}$ is an orthogonal family of nonzero projections in $\mathcal{R}$, then $\sharp \Lambda \leq 4$. This completes the proof. $\qquad \square$

We use the preceding lemma to prove the main result which gives a complete description of left symmetric points for Birkhoff orthogonality in the preduals of von Neumann algebras.

THEOREM 3.6. *Let $\mathcal{R}$ be a von Neumann algebra. If $\mathcal{R}_*$ has nonzero left symmetric points for Birkhoff orthogonality, then one of the following three statements holds.*

(i) *$\mathcal{R} = \mathbb{C}$, and each element of $\mathcal{R}_* = \mathbb{C}$ is left symmetric for Birkhoff orthogonality in $\mathcal{R}_*$.*

(ii) $\mathcal{R} = \ell_\infty^2$, *and a nonzero* $\rho \in \mathcal{R}_* = \ell_1^2$ *is left symmetric for Birkhoff orthogonality in* $\mathcal{R}_*$ *if and only if* $\|\rho\|^{-1}\rho \in \{(a,b) \in \mathbb{C}^2 : |a| = |b| = 1/2\}$.

(iii) $\mathcal{R} = M_2(\mathbb{C})$, *and a nonzero* $\rho \in \mathcal{R}_* = (M_2(\mathbb{C}), \|\cdot\|_1)$ *is left symmetric for Birkhoff orthogonality in* $\mathcal{R}_*$ *if and only if* $\|\rho\|^{-1}\rho \in \{A \in M_2(\mathbb{C}) : \sigma_1 = \sigma_2 = 1/2\}$, *where* σ_1, σ_2 *are the singular values of* A.

PROOF. Suppose that $\mathcal{R}_*$ has nonzero symmetric points. Lemma 3.5 guarantees that if $\{E_\lambda\}_{\lambda \in \Lambda}$ is an orthogonal family of nonzero projections in $\mathcal{R}$, then $\sharp\Lambda \le 4$. Considering the type decomposition for $\mathcal{R}$, we know that $\mathcal{R}$ only has the type I portion. More precisely, $\mathcal{R}$ must be the finite direct sum of type I_n algebras with $n \le 4$. Moreover, the centre $\mathcal{C}$ of $\mathcal{R}$ is at most four dimensional. To see this, it is enough to consider a maximal orthogonal family of minimal projections in $\mathcal{C}$. As a consequence, the whole algebra $\mathcal{R}$ is finite dimensional. This information allows us to improve Lemma 3.5.

Let ρ be a nonzero left symmetric point for Birkhoff orthogonality in $\mathcal{R}_*$. Then $\{\rho\}^f = \{U\}$ for a unitary operator $U \in \mathcal{R}$ since the set of extreme points of the unit ball of a finite (dimensional) von Neumann algebra coincides with the unitary group. As in the first two paragraphs of the proof of Lemma 3.5, $I = U^*U$ cannot be the sum of three or more nonzero projections.

We recall that the finite dimensional algebra $\mathcal{R}$ can be identified with the direct sum of type I factors. However, by the preceding paragraph, the candidates of summands are only $\mathbb{C}$ and $M_2(\mathbb{C})$. Thus $\mathcal{R}$ has one of the following forms: $\mathbb{C}$, $M_2(\mathbb{C})$, $\mathbb{C} \oplus \mathbb{C}(= \ell_\infty^2)$. In the other cases (that is, $\mathbb{C} \oplus M_2(\mathbb{C})$ and $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$), the identity I can be the sum of three or more projections.

Now, let $\mathcal{R} = \mathbb{C}$. Then $\mathcal{R}_* = \mathbb{C}$. If a, b are nonzero complex numbers, then $a \perp_B b$ if and only if $ab = 0$. Consequently, Birkhoff orthogonality itself is symmetric in $\mathbb{C}$ and therefore (i) holds.

In the case of $\mathcal{R} = \ell_\infty^2$, the predual $\mathcal{R}_*$ is identified with ℓ_1^2. Namely, if $\rho \in \mathcal{R}_*$ is identified with (a,b), then $\rho(c,d) = ac + bd$ for each $(c,d) \in \ell_\infty^2$. Suppose that $\rho = (a,b)$ is a nonzero left symmetric point for Birkhoff orthogonality in $\mathcal{R}_*$. We may assume that $\|\rho\| = 1$, that is, $|a| + |b| = 1$. To see $|a| = |b| = 1/2$, suppose to the contrary that $|a| \ne |b|$. Since $\{\rho\}^f$ is a singleton, we note that $ab \ne 0$. Let $\text{sgn}(c) = |c|/c$ for each nonzero complex number c. Then $(\text{sgn}(a), \text{sgn}(b)) \in \{\rho\}^f$. Putting

$$\tau = \left(\frac{a}{2|a|}, -\frac{b}{2|b|} \right)$$

yields $\|\tau\|_1 = 1$ and $\rho \perp_B \tau$ since $\tau(\text{sgn}(a), \text{sgn}(b)) = 0$. However, $\max\{|c|, |d|\} \le 1$ and $\tau(c,d) = 1$ imply that $ac/|a| = -bd/|b| = 1$, which implies that $c = \text{sgn}(a)$ and $d = -\text{sgn}(b)$. It follows that $\rho(c,d) = |a| - |b| \ne 0$, and hence we obtain $\tau \not\perp_B \rho$, which is a contradiction. Thus $|a| = |b|$ is necessary.

Conversely, suppose that $\rho = (a,b)$ and that $|a| = |b| = 1/2$. In this case, $\text{sgn}(a)a = \text{sgn}(b)b = 1/2$. If $\rho(c,d) = 1$, then $c/\text{sgn}(a) = d/\text{sgn}(b) = 1$, that is, $c = \text{sgn}(a)$ and $d = \text{sgn}(b)$. This shows that $\{\rho\}^f = \{(\text{sgn}(a), \text{sgn}(b))\}$. If $\tau = (a', b') \in \mathcal{R}_*$ and $\rho \perp_B \tau$, then $\tau(\text{sgn}(a), \text{sgn}(b)) = \text{sgn}(a)a' + \text{sgn}(b)b' = 0$. From this, it follows that

$$\tau = \left(a', -\frac{\text{sgn}(a)a'}{\text{sgn}(b)} \right).$$

We note that $\|\tau\| = 2|a'|$. Since

$$\tau(\mathrm{sgn}(a), -\mathrm{sgn}(b)) = 2\mathrm{sgn}(a)a',$$
$$\rho(\mathrm{sgn}(a), -\mathrm{sgn}(b)) = |a| - |b| = 0,$$

one obtains $\tau \perp_B \rho$ and therefore ρ is left symmetric.

Finally, assume that $\mathcal{R} = M_2(\mathbb{C})$. For each $A \in M_2(\mathbb{C})$, let $\|A\|_1 = \sigma_1 + \sigma_2$, where σ_1, σ_2 are the singular values of A. In this case, $\mathcal{R}_*$ is isometrically isomorphic to $(M_2(\mathbb{C}), \|\cdot\|_1)$. Moreover, if $\rho \in \mathcal{R}_*$ is identified with $A \in M_2(\mathcal{R})$, then $\rho(B) = \mathrm{tr}(AB)$ for each B. Now let ρ be an element of $\mathcal{R}_*$ identified with $A \in M_2(\mathbb{C})$. Suppose that $\|A\|_1 = 1$ and that the singular values σ_1, σ_2 of A are mutually distinct. Let U, V be unitary matrices satisfying $UAV = \mathrm{diag}(\sigma_1, \sigma_2)$. Then we have $\rho(VU) = \mathrm{tr}(AVU) = 1$. Let $A' = U^*\mathrm{diag}(1/2, -1/2)V^*$ and let $\tau(B) = \mathrm{tr}(A'B)$ for each B. Then $\|A'\|_1 = \|\tau\| = 1$ and $\tau(VU) = \mathrm{tr}(A'VU) = 0$, which implies that $\rho \perp_B \tau$. On the other hand, if $\|B\| \leq 1$ and $\tau(B) = 1$, then

$$\mathrm{tr}(A'B) = \mathrm{tr}(U^*\mathrm{diag}(1/2, -1/2)V^*B) = \mathrm{tr}(\mathrm{diag}(1/2, -1/2)V^*BU^*) = 1.$$

Putting $V^*BU^* = C = (c_{ij})$ yields $2^{-1}(c_{11} - c_{22}) = 1$. Since $|c_{ij}| \leq \|V^*BU^*\| \leq 1$ for each i, j, it follows that $c_{11} = -c_{22} = 1$, and hence $V^*BU^* = \mathrm{diag}(1, -1)$. This shows that $\rho(B) = \mathrm{tr}(AV\mathrm{diag}(1, -1)U) = \sigma_1 - \sigma_2 \neq 0$. As a consequence, $\tau \not\perp_B \rho$, that is, ρ is not left symmetric. In other words, if $\rho = A$ is a left symmetric point for Birkhoff orthogonality in $\mathcal{R}_*$, then A only has the singular value $1/2$ of multiplicity two.

For the converse, let $A \in M_2(\mathbb{C})$ be such that $\sigma_1 = \sigma_2 = 1/2$, and let $\rho(B) = \mathrm{tr}(AB)$ for each B. Suppose that $\rho \perp_B \tau$, where $\tau \in \mathcal{R}_*$ is identified with A'. We may assume that $\|\tau\| = \|A'\|_1 = 1$. By Lemma 2.1, there exists a $B \in M_2(\mathbb{C})$ such that $\mathrm{tr}(AB) = 1$ and $\mathrm{tr}(A'B) = 0$. Let U, V be unitaries such that $UAV = \mathrm{diag}(1/2, 1/2) = I/2$. Then, as in the argument about V^*BU^* in the preceding paragraph, we have $B = VU$. From this, $\mathrm{tr}(A'VU) = 0$. We take a pair of unitaries U', V' satisfying $U'A'V' = \mathrm{diag}(\sigma'_1, \sigma'_2)$, where σ'_1, σ'_2 are the singular values of A'. It follows that $\mathrm{tr}(AVU) = \mathrm{tr}(U'^*\mathrm{diag}(\sigma'_1, \sigma'_2)V'^*VU) = \mathrm{tr}(\mathrm{diag}(\sigma'_1, \sigma'_2)V'^*VUU'^*) = 0$. We shall show that $\mathrm{tr}(V'^*VUU'^*) = 0$. Put $V'^*VUU'^* = W = (w_{ij})$. Since W is also unitary,

$$0 = \left\langle W\begin{pmatrix} 1 \\ 0 \end{pmatrix}, W\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix}, \begin{pmatrix} w_{12} \\ w_{22} \end{pmatrix} \right\rangle = w_{11}\overline{w_{12}} + w_{21}\overline{w_{22}}.$$

This shows that $(w_{12}, w_{22}) = \overline{w_{11}}^{-1}w_{22}(-\overline{w_{21}}, \overline{w_{11}})$ unless $w_{11} = w_{22} = 0$. If $w_{11} = w_{22} = 0$, then $\mathrm{tr}(W) = 0$. In the case of $w_{11}w_{22} \neq 0$, since $|w_{11}|^2 + |w_{21}|^2 = 1$ and $|w_{12}|^2 + |w_{22}|^2 = 1$, we have $|w_{11}| = |w_{22}| > 0$. It follows from $\mathrm{tr}(\mathrm{diag}(\sigma'_1, \sigma'_2)W) = \sigma'_1 w_{11} + \sigma'_2 w_{22} = 0$ that $\sigma'_1 = \sigma'_2 = 1/2$. Thus $0 = \mathrm{tr}(\mathrm{diag}(\sigma'_1, \sigma'_2)W) = \mathrm{tr}(W)/2$, that is, $\mathrm{tr}(W) = 0$, as desired. Now, $\tau(V'U') = \mathrm{tr}(A'V'U') = 1$ and

$$\rho(V'U') = \mathrm{tr}(AV'U') = \mathrm{tr}(U^*V^*V'U')/2$$
$$= \mathrm{tr}(U'U^*V^*V')/2 = \overline{\mathrm{tr}(V'^*VUU'^*)}/2 = 0$$

show that $\tau \perp_B \rho$. Hence ρ is a left symmetric point for Birkhoff orthogonality in $\mathcal{R}_*$. □

We conclude this paper with the following open problem.

PROBLEM 3.7. Can we characterise right symmetric points for Birkhoff orthogonality in the preduals of von Neumann algebras?

References

[1] C. A. Akemann and G. K. Pedersen, 'Facial structure in operator algebra theory', *Proc. Lond. Math. Soc. (3)* **64** (1992), 418–448.

[2] J. Alonso, H. Martini and S. Wu, 'On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces', *Aequationes Math.* **83** (2012), 153–189.

[3] D. Amir, *Characterizations of Inner Product Spaces* (Birkhäuser Verlag, Basel, 1986).

[4] L. Arambašić and R. Rajić, 'On symmetry of the (strong) Birkhoff–James orthogonality in Hilbert C^*-modules', *Ann. Funct. Anal.* **7** (2016), 17–23.

[5] C. M. Edwards and G. T. Rüttimann, 'On the facial structure of the unit balls in a JBW^*-triple and its predual', *J. Lond. Math. Soc. (2)* **38** (1988), 317–332.

[6] P. Ghosh, D. Sain and K. Paul, 'On symmetry of Birkhoff–James orthogonality of linear operators', *Adv. Oper. Theory* **2** (2017), 428–434.

[7] R. C. James, 'Orthogonality and linear functionals in normed linear spaces', *Trans. Amer. Math. Soc.* **61** (1947), 265–292.

[8] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. II, Advanced Theory*, Pure and Applied Mathematics, 100 (Academic Press, Orlando, FL, 1986).

[9] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. IV, Special Topics. Advanced Theory–An Exercise Approach* (Birkhäuser, Boston, MA, 1992).

[10] N. Komuro, K.-S. Saito and R. Tanaka, 'Symmetric points for (strong) Birkhoff orthogonality in von Neumann algebras with applications to preserver problems', *J. Math. Anal. Appl.* **463** (2018), 1109–1131.

[11] D. Sain, 'Birkhoff–James orthogonality of linear operators on finite dimensional Banach spaces', *J. Math. Anal. Appl.* **447** (2017), 860–866.

[12] D. Sain, P. Ghosh and K. Paul, 'On symmetry of Birkhoff–James orthogonality of linear operators on finite-dimensional real Banach spaces', *Oper. Matrices* **11** (2017), 1087–1095.

[13] A. Turnšek, 'On operators preserving James' orthogonality', *Linear Algebra Appl.* **407** (2005), 189–195.

[14] A. Turnšek, 'A remark on orthogonality and symmetry of operators in $\mathcal{B}(\mathcal{H})$', *Linear Algebra Appl.* **535** (2017), 141–150.

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