The Frobenius action on local cohomology modules in mixed characteristic

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ABSTRACT

Heitmann's proof of the direct summand conjecture has opened a new approach to the study of homological conjectures in mixed characteristic. Inspired by his work and by the methods of almost ring theory, we discuss a normalized length for certain torsion modules, which was introduced by Faltings. Using the normalized length and the Frobenius map, we prove some results of local cohomology for local rings in mixed characteristic, which has an immediate implication for the subject of splinters studied by Singh.

1. Introduction

In this paper, we apply the Frobenius map to prove some results on local cohomology modules of local rings of mixed characteristic. These results were motivated by connections to the homological conjectures, in particular the direct summand conjecture which states the following.

CONJECTURE. Let R be a regular local ring and let $R \to S$ be a module-finite extension. Then R is a direct summand of S as an R-module.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Recall that an R-algebra T satisfying $\mathfrak{m}T \neq T$ is a (*balanced*) *big Cohen–Macaulay R-algebra* if every system of parameters of R is a regular sequence on T. If T is a big Cohen–Macaulay algebra, then the local cohomology modules $H_{\mathfrak{m}}^i(T)$ are zero for all $i < \dim R$. The importance of the notion of such algebras is contained in the following (see [Hoc75]).

PROPOSITION 1.1. *Let $(R, \mathfrak{m})$ be a complete local domain of arbitrary characteristic. If there exists a big Cohen–Macaulay R-algebra, then the direct summand conjecture holds.*

Let R^+ be the integral closure of R in the algebraic closure of the field of fractions of R. Then it is shown in [HH92] that R^+ is a big Cohen–Macaulay R-algebra if R has characteristic $p > 0$. In the mixed characteristic case, the following result was recently established by Heitmann (see [Hei02, Hei05]) to prove the direct summand conjecture in dimension 3.

THEOREM 1.2. *Let $(R, \mathfrak{m})$ be a three-dimensional complete local domain of mixed characteristic $p > 0$. Then $c^\epsilon \cdot H_{\mathfrak{m}}^2(R^+) = 0$ for any $c \in \mathfrak{m}$ and $\epsilon \in \mathbb{Q}, \epsilon > 0$.*

As a corollary, Hochster (see [Hoc02]) deduced the existence of big Cohen–Macaulay algebras.

THEOREM 1.3. *Let $(R, \mathfrak{m})$ be as in Theorem 1.2. Then there exists a big Cohen–Macaulay R-algebra in a weakly functorial sense.*

In connection with the above result, an interesting question is whether R^+ is 'almost' Cohen–Macaulay in higher dimension in the sense that the local cohomology modules $H^i_{\mathfrak{m}}(R^+)$ are 'v-almost zero' (we give a definition of this term using a valuation in Definition 2.7) for all $i < \dim R$. In fact, once this is answered positively, Heitmann's idea works everywhere to prove the direct summand conjecture in any dimension. We also remark that the direct summand conjecture is a theorem for all equicharacteristic rings (see [Hoc73] for the proof), while it is an open question in dimension greater than three in mixed characteristic. Now let $(R, \mathfrak{m})$ be a d-dimensional power series ring over a complete discrete valuation ring V of mixed characteristic $p > 0$ with perfect residue field. Let $R \to S$ be a module-finite extension of local domains. By Cohen structure theorem, every complete local ring is always module-finite over some power series ring. In particular, $R^+ = S^+$.

The structure of this paper is as follows. First we define a certain big ring R_∞ that is obtained as a flat colimit of R and then introduce a notion of *normalized length* $\lambda_\infty(M) \in \mathbb{R} \cup \{\infty\}$ for an $\mathfrak{m}$-torsion R_∞-module M, which was introduced by Faltings in his method of almost étale extensions (see [Fal02]). Then the aim of this paper is to investigate the following question.

Question 1. Let $\Phi : H^i_{\mathfrak{m}}(S) \otimes_R R_\infty \to H^i_{\mathfrak{m}}(S^+)$ be the map of local cohomology modules induced by the natural map $S \otimes_R R_\infty \to S^+$. Then is it true that $\lambda_\infty(\mathrm{Im}(\Phi)) = 0$ for all $i < d$?

Faltings investigated this question when the localization map $R[p^{-1}] \to S[p^{-1}]$ is étale. Our main result (see Theorem 3.6) gives a partial answer to this question without any assumptions on the module-finite extension $R \to S$. Question 1 is related to a notion of almost zero modules (see [Gab03] for the definition). We shall define a notion of v-almost zero modules below, which suffices for the study of local cohomology modules. The importance of Question 1 is contained in the following.

PROPOSITION 1.4. *If Question 1 has a positive answer for every module-finite extension $R \to S$, then R^+ is almost Cohen–Macaulay. In particular, the direct summand conjecture holds.*

The proof of this proposition follows from Proposition 2.15 together with Heitmann's idea to deduce the direct summand conjecture from the almost vanishing of local cohomology modules. Proposition 2.15 is essential in studying v-almost zero modules. The main point in working with S^+ is that the Frobenius endomorphism on S^+/pS^+ is surjective. Hence, it induces a unique ring isomorphism $\bar{F}_{S^+} : S^+/p^{1/p}S^+ \simeq S^+/pS^+$ and

$$H^i_{\mathfrak{m}}(S^+/p^{1/p}S^+) \simeq H^i_{\mathfrak{m}}(S^+/pS^+)^{[F]},$$

in which case the right-hand side is viewed as an R_∞-module via the Frobenius map. Then one applies Theorem 2.12 to study the above local cohomology map. To obtain non-trivial results, we need to restrict our attention to the finite length cohomology $H^k_{\mathfrak{m}}(S)$ together with an additional assumption that $H^{k-1}_{\mathfrak{m}}(S)$ is zero, in which case $\Phi : H^k_{\mathfrak{m}}(S) \otimes_R R_\infty \to H^k_{\mathfrak{m}}(S^+)$ is, at least, shown to be not injective if $H^k_{\mathfrak{m}}(S)$ is non-zero.

The second main theorem in this paper is to state a certain type of finiteness conditions under which the local cohomology modules $H^i_{\mathfrak{m}}(S_\infty)$ are v-almost zero for $i < d$ for some big normal domain S_∞ of mixed characteristic that contains both S and R_∞ and that satisfies the condition that the Frobenius endomorphism

$$F_{S_\infty} : S_\infty/pS_\infty \to S_\infty/pS_\infty$$

is surjective. We use normalized length to produce such conditions (see Theorem 3.4). At this point, as it is not clear as to the class of mixed characteristic rings on which the Frobenius map is surjective after killing p, we leave this issue in this paper. Finally, we point out that the idea of Faltings' work stems from an attempt to extend the classical Nagata–Zariski's purity theorem to its almost analogue.

2. Normalized length

Throughout this article, $(R, \mathfrak{m}, k)$ will be a complete regular local ring of mixed characteristic $p > 0$

$$V[[x_2, \ldots, x_d]]$$

such that (V, pV, k) is a complete discrete valuation ring with perfect residue field k of characteristic $p > 0$. Then we consider the direct system of regular local rings

$$R = R_0 \hookrightarrow R_1 \hookrightarrow \cdots \hookrightarrow R_e \hookrightarrow \cdots$$

where $R_e := V[p^{1/p^e}][[x_2^{1/p^e}, \ldots, x_d^{1/p^e}]]$ with maximal ideal $\mathfrak{m}_e$, and we let $R_\infty := \bigcup_{e \in \mathbb{N}} R_e$. Then the ring R_∞ possesses the following properties:

 (i) R_∞ is a henselian quasilocal ring of dimension d with unique maximal ideal $\mathfrak{m}_\infty$;

 (ii) R_∞ is faithfully flat and integral over R;

 (iii) the Frobenius endomorphism $F_{R_\infty} : R_\infty/pR_\infty \to R_\infty/pR_\infty$ is surjective.

To see this, note that the first two properties are stable under infinite ascending unions of rings, while the third follows from the surjectivity: $R_{e+1}/pR_{e+1} \twoheadrightarrow R_e/pR_e$ induced by the Frobenius map. For more on henselian local rings, see [Nag62].

Let $\mathcal{M}$ denote the category of $\mathfrak{m}$-torsion R_∞-modules where $\mathfrak{m}$ is the maximal ideal of R. An R_∞-module M is said to be $\mathfrak{m}$-*torsion* if every element of M is annihilated by some power of $\mathfrak{m}$. Examples are local cohomology modules. Let us start with the definition of normalized length.

DEFINITION 2.1. We have the following.

 (i) Let $M \in \mathcal{M}$ be a finitely presented module. Then there is a finite presentation for M:

$$R_\infty^{\oplus m_1} \xrightarrow{\varphi} R_\infty^{\oplus m_0} \longrightarrow M \longrightarrow 0. \qquad (*)$$

For sufficiently large $n \in \mathbb{N}$, we have $a_{ij} \in R_n$ with $\varphi = (a_{ij})$, and for some finite R_n-module M_n we have

$$R_n^{\oplus m_1} \xrightarrow{\varphi} R_n^{\oplus m_0} \longrightarrow M_n \longrightarrow 0. \qquad (**)$$

Since R_∞ is flat over R_n, we may tensor R_∞ with $(**)$ over R_n to obtain

$$R_\infty^{\oplus m_1} \xrightarrow{\varphi} R_\infty^{\oplus m_0} \longrightarrow R_\infty \otimes_{R_n} M_n \longrightarrow 0,$$

which is equivalent to the presentation $(*)$, and thus $M \simeq R_\infty \otimes_{R_n} M_n$. Then define

$$\lambda_\infty(M) := \frac{1}{p^{dn}} \cdot \ell(M_n) \in \mathbb{R}.$$

 (ii) Let $M \in \mathcal{M}$ be a finitely generated module. Then define

$$\lambda_\infty(M) := \inf_{N \twoheadrightarrow M} \lambda_\infty(N) \in \mathbb{R}$$

where 'inf' is taken over all finitely presented modules N in $\mathcal{M}$ that map onto M.

 (iii) Let $M \in \mathcal{M}$ be an arbitrary module. Then define

$$\lambda_\infty(M) := \sup_{N \subset M} \lambda_\infty(N) \in \mathbb{R} \cup \{\infty\}$$

where 'sup' is taken over all finitely generated submodules N of M.

LEMMA 2.2. *The normalized length $\lambda_\infty(M)$ is well-defined for $M \in \mathcal{M}$.*

Proof. Let $M \in \mathcal{M}$ be a finitely presented module. Then one has to verify that $\lambda_\infty(M)$ is independent of n in the definition. Let $m \geqslant n$. Then since $R_n \to R_m$ is flat and $M_m \simeq R_m \otimes_{R_n} M_n$, we have

$$\lambda_\infty(M) = \frac{1}{p^{dn}} \cdot \ell(M_n) = \frac{1}{p^{dm}} \cdot \frac{1}{p^{d(n-m)}} \cdot \ell(M_n) = \frac{1}{p^{dm}} \cdot \ell(R_m \otimes_{R_n} M_n) = \frac{1}{p^{dm}} \cdot \ell(M_m),$$

which is the claim.

Next, one has to verify that Definition 2.1(ii) coincides with Definition 2.1(i) for finitely presented $M \in \mathcal{M}$. Let $N \in \mathcal{M}$ be a finitely presented module that maps onto M. Then it suffices to see that $\lambda_\infty(N) \geqslant \lambda_\infty(M)$ since M maps onto itself. This will be seen by taking a finite presentation $N_n \twoheadrightarrow M_n$ for $n \gg 0$ and hence $\ell(N_n) \geqslant \ell(M_n)$.

Finally, one needs to verify that Definition 2.1(iii) coincides with Definition 2.1(ii) for finitely generated $M \in \mathcal{M}$. Let $N \subset M$ be any finitely generated submodule. Let us choose finite sets of generators $\Sigma \subset N$ and $\Sigma' \subset M$ such that $\Sigma \subset \Sigma'$. If N_n and M_n denote the R_n-submodules generated by Σ and Σ', respectively, then $\ell(N_n) \leqslant \ell(M_n)$ for all $n \in \mathbb{N}$. Hence, the desired claim follows from Lemma 2.3 below. $\square$

We used the following lemma in the proof of Lemma 2.2.

LEMMA 2.3. *Let $M \in \mathcal{M}$ be a finitely generated module and let Σ be any system of generators of M. Let M_n denote the R_n-module generated by Σ. Then $M_n \otimes_{R_n} R_\infty \in \mathcal{M}$ is a finitely presented module that admits a surjection onto M, and*

$$\lambda_\infty(M) = \lim_{n \to \infty} \lambda_\infty(M_n \otimes_{R_n} R_\infty) = \lim_{n \to \infty} \frac{1}{p^{dn}} \cdot \ell(M_n).$$

Proof. Let Σ and Σ' be finite sets of generators of the module M. We denote by M_n and M_n' the R_n-modules generated by Σ and Σ', respectively. Then it is easy to see that $M_n = M_n'$ for sufficiently large $n > 0$. So $\ell(M_n) = \ell(M_n')$ for $n \gg 0$. By definition, it suffices to prove if N is a finitely presented module in $\mathcal{M}$ that surjects onto M, then there exists some $n \in \mathbb{N}$ such that the R_∞-module N admits a surjection onto $M_n \otimes_{R_n} R_\infty$ as we would get

$$\lambda_\infty(M) \leqslant \lambda_\infty(M_n \otimes_{R_n} R_\infty) \leqslant \lambda_\infty(N),$$

from which the conclusion easily follows.

Since N is finitely presented, there exists some $n \in \mathbb{N}$ such that $N \simeq N_n \otimes_{R_n} R_\infty$ for some finite R_n-module N_n. Then define M_n to be image of the composite map $N_n \hookrightarrow N_n \otimes_{R_n} R_\infty \twoheadrightarrow M$ and the module N obviously admits a surjection onto $M_n \otimes_{R_n} R_\infty$. $\square$

The following lemma follows from Definition 2.1(iii) of the definition.

LEMMA 2.4. *Let $M \in \mathcal{M}$ and let $N \subset M$ be its submodule. Then $\lambda_\infty(N) \leqslant \lambda_\infty(M)$.*

Remark 2.5. One can extend the class of rings for which the normalized length is well-defined as follows. For the definition of the length, we only use the fact that $R_e \hookrightarrow R_{e+1}$ is flat and of the constant rank p^d. It is therefore natural to start with a family of flat extensions of Noetherian local rings

$$R := R_0 \subset \cdots \subset R_n \subset R_{n+1} \subset \cdots \subset R_\infty$$

such that the rank $[R_{n+1} : R_n]$ stabilizes for $n \gg 0$ and $\sqrt{\mathfrak{m}_n R_{n+1}} = \mathfrak{m}_{n+1}$ for all $n \in \mathbb{N}$. Under this set-up, one can define the normalized length as in the case R is a complete regular local ring. We refer the reader to [Gab04] for more detail, where the normalized length is discussed for a more general class of rings.

Remark 2.6. From the definition, $\lambda_\infty(M)$ is finite for any finitely generated $M \in \mathcal{M}$. It is also true that if M is finitely presented in $\mathcal{M}$, then

$$\lambda_\infty(M) = 0 \iff M = 0,$$

which does not hold for finitely generated modules. For example, take $R_\infty/\mathfrak{m}_\infty \in \mathcal{M}$, which is not a finitely presented R_∞-module, and it is easy to see that $\lambda_\infty(R_\infty/\mathfrak{m}_\infty) = 0$.

This remark suggests the following definition.

Definition 2.7. Let M be an R_∞-module in $\mathcal{M}$, and let v be a valuation on R_∞ that is positive on the maximal ideal of R_∞. Then we say that:

(i) M has *almost finite length* if for any rational number $\epsilon > 0$, there exists an element $b \in R_\infty$ such that $\lambda_\infty(b \cdot M) < \infty$ and $v(b) < \epsilon$;

(ii) M is *v-almost zero* if for any $m \in M$ and rational number $\epsilon > 0$, there exists an element $b \in R_\infty$ such that $b \cdot m = 0$ and $v(b) < \epsilon$.

In Proposition 2.15, which is proved only for an $\mathfrak{m}$-adic valuation, we will relate the v-almost zero modules to the vanishing of the normalized length. The normalized length behaves additively on the short exact sequence of R_∞-modules in $\mathcal{M}$.

Proposition 2.8. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $\mathfrak{m}$-torsion R_∞-modules. Then*

$$\lambda_\infty(M) = \lambda_\infty(M') + \lambda_\infty(M'').$$

Proof. In the case where all relevant modules are either finitely presented or finitely generated, the claim follows easily by the definition of the normalized length together with Lemma 2.3. So let us consider the general case. Since the image of every finitely generated submodule of M in M'' is finitely generated, it suffices to consider the case in which both M and M'' are finitely generated by the definition of normalized length for arbitrary modules. Let us denote by M_n an R_n-submodule of M generated by any fixed generators of the R_∞-module M. Let $M'_n := M' \cap M_n$ and let M''_n be the image of M_n in M''. Then we have a short exact sequence: $0 \to M'_n \to M_n \to M''_n \to 0$.

For a finite set of elements of M', the R_n-submodule $\widetilde{M'_n}$ generated by them is contained in some M'_n and satisfies that $\ell(\widetilde{M'_n}) \leqslant \ell(M'_n)$ for $n \gg 0$. Then since M and M'' are finitely generated, it follows from Lemma 2.3 that

$$\lambda_\infty(M') = \lim_{n \to \infty} \limsup_{\widetilde{M'_n} \subset M'} \frac{1}{p^{dn}} \ell(\widetilde{M'_n}) \leqslant \lim_{n \to \infty} \frac{1}{p^{dn}} \ell(M'_n) = \lambda_\infty(M) - \lambda_\infty(M''),$$

which is equivalent to the inequality

$$\lambda_\infty(M) - \lambda_\infty(M'') \geqslant \lambda_\infty(M').$$

On the other hand, since $M'' = \lim_{\to n} M''_n \otimes_{R_n} R_\infty$, for any $\epsilon > 0$ there exists sufficiently large $n > 0$ such that

$$\lambda_\infty(M'') \geqslant \lambda_\infty(M''_n \otimes_{R_n} R_\infty) - \epsilon.$$

Let us consider the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M'_n \otimes_{R_n} R_{n+m} & \longrightarrow & M_n \otimes_{R_n} R_{n+m} & \longrightarrow & M''_n \otimes_{R_n} R_{n+m} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & M'_{n+m} & \longrightarrow & M_{n+m} & \longrightarrow & M''_{n+m} & \longrightarrow & 0
\end{array}
$$

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Since $\mathrm{Coker}(g) = 0$, there is an epimorphism $\mathrm{Ker}(h) \twoheadrightarrow \mathrm{Coker}(f) = M'_{n+m}/R_{n+m} \cdot M'_n$, which follows from the snake lemma. Let $\gamma = \ell(\mathrm{Ker}(h))$. Then there follows that

$$\gamma = \ell(M''_n \otimes_{R_n} R_{n+m}) - \ell(M''_{n+m}) = p^{dm} \cdot \ell(M''_n) - \ell(M''_{n+m}).$$

It also follows by our assumption that

$$\lambda_\infty(M'') \geqslant \frac{\ell(M''_n)}{p^{dn}} - \epsilon.$$

We can put all inequalities obtained above together to obtain that

$$\frac{\gamma}{p^{d(n+m)}} \leqslant \lambda_\infty(M'') - \frac{\ell(M''_{n+m})}{p^{d(n+m)}} + \epsilon \leqslant \epsilon.$$

On the other hand, since $\{\lambda_\infty(M''_{n+m} \otimes_{R_{n+m}} R_\infty)\}_{m \in \mathbb{N}}$ is a decreasing sequence in $\mathbb{Q}$, we get the inequality $\lambda_\infty(M'') \leqslant \lambda_\infty(M''_{n+m} \otimes_{R_{n+m}} R_\infty)$ for all $m \in \mathbb{N}$. Now we have $\gamma \leqslant \epsilon \cdot p^{d(n+m)}$. This inequality together with an epimorphism $\mathrm{Ker}(h) \twoheadrightarrow \mathrm{Coker}(f)$ yields that

$$\ell(M'_{n+m}) \leqslant \ell(R_{n+m} \cdot M'_n) + \epsilon \cdot p^{d(n+m)}$$

and, therefore,

$$\lambda_\infty(M) - \lambda_\infty(M'') \leqslant \lim_{n \to \infty} \frac{1}{p^{dn}} \cdot \ell(M'_n) \leqslant \lim_{m \to \infty} \frac{1}{p^{d(n+m)}} \cdot \ell(R_{n+m} \cdot M'_n) + \epsilon$$
$$= \lambda_\infty(R_\infty \cdot M'_n) + \epsilon \leqslant \lambda_\infty(M') + \epsilon.$$

Since ϵ can be taken arbitrarily small, $\lambda_\infty(M) - \lambda_\infty(M'') \leqslant \lambda_\infty(M')$, and hence the claim follows. $\qquad\square$

PROPOSITION 2.9. *Let N be an R_∞-module that admits an injection into some finitely presented module $M \in \mathcal{M}$. If $\lambda_\infty(N) = 0$, then $N = 0$.*

Proof. As $N = 0$ if and only if every finitely generated submodule of N is zero, we may harmlessly assume N is finitely generated. Let us consider the exact sequence

$$0 \to N \to M \to M/N \to 0.$$

Then the quotient module M/N is finitely presented with $\lambda_\infty(M) = \lambda_\infty(M/N)$. However, since M is finitely presented, there exists an R_n-submodule $M_n \subset M$ such that $R_\infty \otimes_{R_n} M_n \simeq M$. Let M'_n be the image of M_n under the surjection $M \twoheadrightarrow M/N$. Then it can be shown that $R_\infty \otimes_{R_n} M'_n \simeq M/N$ by computing the finite presentation of M/N. Then we must have $\ell(M_n) = \ell(M'_n)$, which is the case only when $M_n = M'_n$ and $M = M/N$. Hence $N = 0$, which is the desired result. $\qquad\square$

DEFINITION 2.10. Let M be any R_∞-module such that $p \cdot M = 0$. Then $M^{[F]}$ denotes the R_∞-module which is isomorphic to M as an abelian group and has the module structure given by $a \cdot m := a^p m =: m \cdot a$ for $a \in R_\infty$ and $m \in M^{[F]}$.

Note that $M \mapsto M^{[F]}$ defines the identity functor on the underlying abelian groups and we give the bimodule structure on the modules via the Frobenius map.

Example 2.11. One expects that it would be very useful if the Frobenius-like maps could be defined for rings of mixed characteristic. We now give an example. Let M be an $R_\infty/p^{1/p}R_\infty$-module that admits a Frobenius map F in the following sense: the map $F : M \to M$ is a homomorphism of abelian groups. As an R_∞-module map, let $F(am) = a^p F(m)$ for $a \in R_\infty$, $m \in M$, and hence $F((a+b)m) = (a+b)^p F(m) = a^p F(m) + b^p F(m)$. Now one has the following map:

$$M \otimes_{R_\infty} R_\infty^{[F]} \to M^{[F]} \; : \; m \otimes a \mapsto F(m)a,$$

which is well-defined as $p^{1/p}$ annihilates the module M by assumption and is called the *relative Frobenius map.*

We are now ready to prove the following 'Frobenius pull-back formula', the importance of which is expressed by studying certain torsion modules, in particular local cohomology modules, that admit a Frobenius action.

THEOREM 2.12. *Let M be an $\mathfrak{m}$-torsion R_∞/pR_∞-module. Then*

$$\lambda_\infty(M^{[F]}) = \frac{1}{p^d} \cdot \lambda_\infty(M).$$

Proof. Let us start with the case where M is finitely presented in $\mathcal{M}$, and let

$$R_\infty^{\oplus m_1} \xrightarrow{\varphi} R_\infty^{\oplus m_0} \longrightarrow M \longrightarrow 0$$

be the presentation for M such that φ is defined over R_n for some $n \geqslant 0$. Since the functor $M \mapsto M^{[F]}$ is exact and $p \cdot M = 0$ by assumption, there follows the commutative diagram

$$
\begin{array}{ccc}
(R_\infty/p^{1/p}R_\infty)^{\oplus m_0} & \longrightarrow & (R_\infty/p^{1/p}R_\infty)^{\oplus m_1} \\
F^{m_0} \downarrow \wr & & F^{m_1} \downarrow \wr \\
((R_\infty/pR_\infty)^{[F]})^{\oplus m_0} & \longrightarrow ((R_\infty/pR_\infty)^{[F]})^{\oplus m_1} & \longrightarrow M^{[F]} \longrightarrow 0
\end{array}
$$

where the vertical maps are induced by the ring isomorphism

$$\bar{F}_{R_\infty} : R_\infty/p^{1/p}R_\infty \simeq (R_\infty/pR_\infty)^{[F]}$$

which is induced by the Frobenius map. Then, using the above diagram together with the ring isomorphism $R_{n+1}/p^{1/p}R_{n+1} \simeq (R_n/pR_n)^{[F]}$ induced by the Frobenius, we have the following exact sequence

$$(R_{n+1}/p^{1/p}R_{n+1})^{m_0} \longrightarrow (R_{n+1}/p^{1/p}R_{n+1})^{m_1} \longrightarrow M_n^{[F]} \longrightarrow 0$$

which in turn gives an R_∞-module isomorphism $R_\infty \otimes_{R_{n+1}} M_n^{[F]} \simeq M^{[F]}$. On the other hand, since the residue class field of R is assumed to be perfect, we have

$$\ell(M_n^{[F]}) = \ell(M_n)$$

and, therefore,

$$\lambda_\infty(M^{[F]}) = \frac{1}{p^{d(n+1)}} \cdot \ell(M_n^{[F]}) = \frac{1}{p^d} \cdot \frac{1}{p^{dn}} \cdot \ell(M_n) = \frac{\lambda_\infty(M)}{p^d},$$

which completes the case of finitely presented modules.

Next let us assume that M is finitely generated in $\mathcal{M}$. Let N be a finitely presented module in $\mathcal{M}$ that admits a surjection $N \twoheadrightarrow M$. Then this map factors as $N \twoheadrightarrow N/pN \twoheadrightarrow M$ by assumption and we deduce that

$$\lambda_\infty(N) \geqslant \lambda_\infty(N/pN) \geqslant \lambda_\infty(M).$$

Therefore, we can replace N with N/pN to prove the theorem. Then since $N^{[F]}$ belongs to $\mathcal{M}$ and is a finitely presented module that maps onto $M^{[F]}$, we get $\lambda_\infty(N) = p^d \cdot \lambda_\infty(N^{[F]}) \geqslant p^d \cdot \lambda_\infty(M^{[F]})$ and thus $\lambda_\infty(M) \geqslant p^d \cdot \lambda_\infty(M^{[F]})$.

Conversely, let N' be a finitely presented module that maps onto $M^{[F]}$. Then there is an R_n-module N_n' such that $N' = R_\infty \otimes_{R_n} N_n'$. Now take $u_1, \ldots, u_s$ to be a set of generators of $M^{[F]}$ and define the R_{n-1}-module

$$M_n := R_{n-1}u_1 + \cdots + R_{n-1}u_s \subset M^{[F]}.$$

Then $M_n^{[F]}$ can be viewed as an R_n-module via the Frobenius $F : R_n/pR_n \to R_{n-1}/pR_{n-1}$, and the module $M_n^{[F]}$ is finitely generated over R_n. Hence, we get the surjective map of R_∞-modules

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as follows

$$N' = R_\infty \otimes_{R_n} N'_n \twoheadrightarrow R_\infty \otimes_{R_n} M_n^{[F]} \twoheadrightarrow M^{[F]}.$$

Now we claim that there is an R_∞-module isomorphism:

$$R_\infty \otimes_{R_n} M_n^{[F]} \simeq (R_\infty \otimes_{R_n} M_{n+1})^{[F]}.$$

For a proof, let

$$(R_{n-1}/pR_{n-1})^{\oplus m_0} \to (R_{n-1}/pR_{n-1})^{\oplus m_1} \to M_n \to 0$$

be the presentation of the R_{n-1}-module M_n. Then the presentation of M_{n+1} can be obtained by replacing R_{n-1} by R_n in the above presentation. Consider the following commutative diagram

$$
\begin{array}{ccccccc}
R_\infty \otimes_{R_n} ((R_{n-1}/pR_{n-1})^{[F]})^{\oplus m_0} & \longrightarrow & R_\infty \otimes_{R_n} ((R_{n-1}/pR_{n-1})^{[F]})^{\oplus m_1} & \longrightarrow & R_\infty \otimes_{R_n} M_n^{[F]} & \longrightarrow & 0 \\
\Phi^{m_0} \downarrow & & \Phi^{m_1} \downarrow & & \downarrow & & \\
(R_\infty \otimes_{R_n} (R_n/pR_n)^{\oplus m_0})^{[F]} & \longrightarrow & (R_\infty \otimes_{R_n} (R_n/pR_n)^{\oplus m_1})^{[F]} & \longrightarrow & (R_\infty \otimes_{R_n} M_{n+1})^{[F]} & \longrightarrow & 0 \\
\wr \downarrow & & \wr \downarrow & & & & \\
((R_\infty/pR_\infty)^{\oplus m_0})^{[F]} & \longrightarrow & ((R_\infty/pR_\infty)^{\oplus m_1})^{[F]} & & & &
\end{array}
$$

in which both Φ^{m_0} and Φ^{m_1} are induced by the R_∞-module map

$$\Phi : R_\infty \otimes_{R_n} (R_{n-1}/pR_{n-1})^{[F]} \to (R_\infty/pR_\infty)^{[F]} : \quad r \otimes m \mapsto r^p m$$

and $(R_{n-1}/pR_{n-1})^{[F]}$ is viewed as an R_n-module via the Frobenius $F : R_n/pR_n \to R_{n-1}/pR_{n-1}$. Then Φ factors as

$$R_\infty \otimes_{R_n} (R_{n-1}/pR_{n-1})^{[F]} \to R_\infty \otimes_{R_n} R_n/p^{1/p}R_n \to R_\infty/p^{1/p}R_\infty \to (R_\infty/pR_\infty)^{[F]},$$

which is obviously an R_∞-algebra isomorphism. Hence,

$$R_\infty \otimes_{R_n} M_n^{[F]} \simeq (R_\infty \otimes_{R_n} M_{n+1})^{[F]}.$$

Since the R_∞-module $N := R_\infty \otimes_{R_n} M_{n+1}$ gets mapped onto M and N is finitely presented, it follows that

$$\lambda_\infty(M) \leqslant \lambda_\infty(N) = p^d \cdot \lambda_\infty(N^{[F]}) \leqslant p^d \cdot \lambda_\infty(N').$$

Therefore, we get the desired formula $\lambda(M) = p^d \cdot \lambda(M^{[F]})$ for the case of finitely generated modules.

Finally, let M be an arbitrary module in $\mathcal{M}$, and let $N \subset M$ be any finitely generated submodule. Then $N^{[F]}$ is an R_∞-submodule of $M^{[F]}$ and $N^{[F]}$ is finitely generated over R_∞ since the residue field of R_∞ is assumed to be perfect. We have

$$\lambda_\infty(N) = p^d \cdot \lambda_\infty(N^{[F]}) \leqslant p^d \cdot \lambda_\infty(M^{[F]}),$$

which gives $\lambda_\infty(M) \leqslant p^d \cdot \lambda_\infty(M^{[F]})$. Conversely, let $N' \subset M^{[F]}$ be a finitely generated submodule. Then we can find a set of generators $\{v_i\}_{i \in I}$ of M, a finite subset $I' \subset I$, and an R_∞-module N such that

$$M^{[F]} = \left(\sum_{i \in I} R_\infty v_i\right)^{[F]}, \quad N = \sum_{i \in I'} R_\infty v_i, \quad \text{and} \quad N' = N^{[F]}.$$

Then we deduce that $\lambda_\infty(M) \geqslant \lambda_\infty(N) = p^d \cdot \lambda_\infty(N^{[F]}) = p^d \cdot \lambda_\infty(N')$ and, hence, $\lambda_\infty(M) = p^d \cdot \lambda_\infty(M^{[F]})$, which completes the proof. $\qquad \square$

In the following, we write

$$(x^{1/\infty}) := \bigcup_{n \in \mathbb{N}} (x^{1/p^n}) R_\infty$$

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for an element $x \in R_\infty$ whenever $x^{1/p^n} \in R_\infty$ for all $n \in \mathbb{N}$. Note that this is a flat R_∞-module since it is an ascending union of principal ideals. As a corollary, one can prove the following.

Corollary 2.13. *Let* $M \in \mathcal{M}$ *such that* $p^{1/p^n} \cdot M = 0$ *for all* $n > 0$. *Then* $\lambda_\infty(M) = 0$.

Proof. By the definition, we easily reduce to the case in which $M \in \mathcal{M}$ is finitely generated. Then by assumption, we must have $M \simeq M^{[F]}$ as the Frobenius map on $R_\infty/(p^{1/\infty})$ is a bijection. Then we have $\lambda_\infty(M) = p^d \cdot \lambda_\infty(M^{[F]}) = p^d \cdot \lambda_\infty(M)$ and thus $\lambda_\infty(M) = 0$ since $\lambda_\infty(M)$ is finite. $\qquad\square$

Finally, we discuss the relation of v-almost zero modules to the vanishing of the normalized length. Let us first recall the definition of an $\mathfrak{m}$-adic valuation on R.

Definition 2.14. Let K be the field of fractions of R. An $\mathfrak{m}$-*adic valuation* on R is a discrete valuation $v : K\backslash\{0\} \to \mathbb{Z}$ satisfying the property that for any non-zero $b \in R, v = v(b)$ is defined as the integer such that $b \in \mathfrak{m}^v$, but $b \notin \mathfrak{m}^{v+1}$.

If v denote the $\mathfrak{m}$-adic valuation on R, then since R_∞ is integral over R, it extends to a valuation on R_∞ with value group $\mathbb{Q}$. We denote by v this extended valuation for simplicity.

Proposition 2.15. *Suppose that* $\lambda_\infty(M) = 0$ *for* $M \in \mathcal{M}$. *Then* M *is* v-*almost zero*.

Proof. It will suffice to consider the case in which M is a cyclic R_∞-module. Then $R_\infty \cdot x \simeq M$ for some $x \in M$ and there is an isomorphism $R_\infty/\mathrm{Ann}_{R_\infty}(x) \simeq R_\infty \cdot x$. Now assume that there exists a non-zero $k \in \mathbb{Z}$ such that

$$\inf\{v(b) \mid b \in \mathrm{Ann}_{R_\infty}(R_\infty \cdot x)\} \geqslant p^{-k}.$$

Then for any $n \in \mathbb{N}$ with $n > k$, we deduce that

$$\ell(R_n \cdot x) \geqslant \#\left\{ p^{\epsilon_1} x_2^{\epsilon_2} \cdots x_d^{\epsilon_d} \ \middle| \ \sum_{i=1}^d \epsilon_i < p^{-k} \text{ and } p^{-n} \leqslant \epsilon_i \right\} = p^{d(n-k)},$$

where the middle term denotes the number of monomials $p^{\epsilon_1} x_2^{\epsilon_2} \cdots x_d^{\epsilon_d} \in R_n$ with specified conditions. Now it follows easily from Lemma 2.3 that

$$\lambda_\infty(R_\infty \cdot x) = \lim_{n\to\infty} \frac{\ell(R_n \cdot x)}{p^{dn}} \geqslant \frac{1}{p^{dk}} > 0,$$

which contradicts our assumption that $\lambda_\infty(M) = 0$. Hence, the proposition follows. $\qquad\square$

Remark 2.16. We do not know whether the converse of the above proposition holds. It seems that the answer to this question is yes even if the annihilator of the module is quite complicated. Obviously, it suffices to consider the case where M is a cyclic module.

3. Applications to local cohomology modules

In this section, we start with a finite extension of local domains $R \to S$ and introduce a notion of a *semi-perfect algebra* S_∞ over S.

Definition 3.1. Let $R \to S$ be a ring extension of domains of mixed characteristic $p > 0$. Then an algebra S_∞ over S is called *semi-perfect* if:

(i) there exists a commutative square

$$\begin{CD} R_\infty @>>> S_\infty \\ @AAA @AAA \\ R @>>> S \end{CD}$$

such that every map is a ring extension of domains;

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(ii) S_∞ is an integrally closed domain in its fraction field;

(iii) the Frobenius endomorphism $F_{S_\infty} : S_\infty/pS_\infty \to S_\infty/pS_\infty$ is surjective.

In what follows, S_∞ shall denote some fixed semi-perfect algebra for S. Such a ring is usually obtained as a large ring extension of S.

Positive characteristic. Let S be a reduced ring of positive characteristic $p > 0$. Let S^∞ be a filtered direct limit of the system $S \hookrightarrow S^{1/p} \hookrightarrow \cdots \hookrightarrow S^{1/p^n} \hookrightarrow \cdots$. Then S^∞ is the minimal perfect S-algebra. In fact, the Frobenius map on it is a bijection.

Mixed characteristic. In this case, it is easy to see that the Frobenius map on R_∞/pR_∞ is surjective. However, since R_∞/pR_∞ is not reduced, the Frobenius map is not injective. Let R^+ be the *absolute integral closure* of R (see [Art71]); that is, it is the integral closure of R in the algebraic closure of the field of fractions of R. Then R^+ is obviously a semi-perfect ring over any domain S that is integral over R. In the mixed characteristic case, the notion of semi-perfect rings is quite subtle since there are not many known examples.

LEMMA 3.2. *Let S_∞ be a semi-perfect S-algebra. Then there exists a ring isomorphism:*

$$\bar{F}_{S_\infty} : S_\infty/p^{1/p}S_\infty \simeq S_\infty/pS_\infty,$$

which is induced by the Frobenius endomorphism $F_{S_\infty} : S_\infty/pS_\infty \to S_\infty/pS_\infty$.

Proof. Let $\bar{K}$ denote the algebraic closure of the fraction field of S_∞. Then it is easy to see that $F_{S_\infty}(\bar{x}) = 0$ with $x \in S_\infty \iff x^p = p \cdot \theta$ for some $\theta \in S_\infty$. Taking the pth root of $x^p = p \cdot \theta$ in $\bar{K}$, we have $x = p^{1/p} \cdot \tilde{\theta}$, where $\tilde{\theta}$ is a root to the equation $t^p - \theta = 0$. Hence, $\tilde{\theta} \in S_\infty[1/p]$, which gives $\tilde{\theta} \in S_\infty$, since S_∞ is assumed integrally closed. This proves the desired claim. $\qquad\square$

In what follows, we let v be a valuation on R_∞ that extends an $\mathfrak{m}$-adic valuation of R, and let $d = \dim R$. A ring isomorphism $\bar{F}_{S_\infty} : S_\infty/p^{1/p}S_\infty \simeq S_\infty/pS_\infty$ yields a well-defined homomorphism on Čech complexes:

$$\bar{F}_{S_\infty *} : C^\bullet(S_\infty/p^{1/p}S_\infty) \simeq C^\bullet(S_\infty/pS_\infty)^{[F]}$$

and the induced map on local cohomology modules:

$$\bar{F}_{S_\infty *} : H^i_{\mathfrak{m}}(S_\infty/p^{1/p}S_\infty) \simeq H^i_{\mathfrak{m}}(S_\infty/pS_\infty)^{[F]}$$

for $i \in \mathbb{N}$. Now let us assume that the following condition holds. For $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$, there exists an element $r \in R_\infty$ such that

$$\lambda_\infty(r \cdot H^i_{\mathfrak{m}}(S_\infty)) < \infty \qquad\qquad (***)$$

with $v(r) < \epsilon$ for some $i < d$. Recall that the condition $(***)$ simply says that the local cohomology $H^i_{\mathfrak{m}}(S_\infty)$ has almost finite length with respect to the valuation v. In order to prove the main theorem, we need the following lemma taken from [Gab04].

LEMMA 3.3. *Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence in the category $\mathcal{M}$ and let $a, b \in R_\infty$. Then the following hold:*

(1) $\lambda_\infty(abN) \leqslant \lambda_\infty(aN') + \lambda_\infty(bN'')$;

(2) *suppose that $\lambda_\infty(N') = 0$ (respectively, $\lambda_\infty(N'') = 0$), then $\lambda_\infty(aN) = \lambda_\infty(aN'')$ (respectively, $\lambda_\infty(aN) = \lambda_\infty(aN')$).*

Proof. (1) First of all, we note that there is a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & bN \cap N' & \longrightarrow & bN & \longrightarrow & bN'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & bN \cap N' & \longrightarrow & bN & \longrightarrow & bN'' & \longrightarrow & 0
\end{array}
$$

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where the horizontal sequences are short exact. It is easy to show that there is a surjection $bN''/L \twoheadrightarrow abN''$ where L denotes the image of the kernel of the multiplication map $a : bN \to bN$ under the surjection $bN \twoheadrightarrow bN''$. Hence, there follows an exact sequence

$$0 \to a(bN \cap N') \to abN \to bN''/L \to 0.$$

So $\lambda_\infty(abN) = \lambda_\infty(a(bN \cap N')) + \lambda_\infty(bN''/L) \leqslant \lambda_\infty(aN') + \lambda_\infty(bN'')$. Hence, part (1) is proved.

(2) This follows easily from the short exact sequence

$$0 \to aN \cap N' \to aN \to aN'' \to 0 \quad (\text{respectively, } 0 \to aN' \to aN \to P \to 0)$$

where P is a quotient of N'', and the fact that $\lambda_\infty(aN \cap N') = 0$ (respectively, $\lambda_\infty(P) = 0$). $\qquad\square$

Now we are ready to prove the almost vanishing theorem for local cohomology modules. Let M be a module over a ring A and let $a \in A$. Then we use the following notation: $M_a := \{m \in M \mid am = 0\}$.

THEOREM 3.4. *Let $R \to S$ be a module-finite extension of domains and we fix $k \in \mathbb{N}$ with $0 < k < d$. Assume the following conditions:*

(1) *S_∞ is a semi-perfect algebra over S;*

(2) *$\lambda_\infty(H_{\mathfrak{m}}^{k-1}(S_\infty)) = 0$;*

(3) *$H_{\mathfrak{m}}^k(S_\infty)$ has almost finite length.*

Then the local cohomology module $H_{\mathfrak{m}}^k(S_\infty)$ is v-almost zero.

Proof. Note that by assumption we have a well-defined R_∞-isomorphism:

$$\bar{F}_{S_\infty *} : H_{\mathfrak{m}}^i(S_\infty/p^{1/p}S_\infty) \simeq H_{\mathfrak{m}}^i(S_\infty/pS_\infty)^{[F]}.$$

Note that since S_∞ is a domain, $H_{\mathfrak{m}}^0(S_\infty) = 0$. The following diagram of standard short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_\infty & \xrightarrow{p^{1/p}} & S_\infty & \longrightarrow & S_\infty/p^{1/p}S_\infty & \longrightarrow & 0 \\
 & & & & & & \wr \downarrow \bar{F}_{S_\infty} & & \\
0 & \longrightarrow & S_\infty & \xrightarrow{p} & S_\infty & \longrightarrow & S_\infty/pS_\infty & \longrightarrow & 0
\end{array}$$

yields a diagram of local cohomology modules:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{\mathfrak{m}}^{k-1}(S_\infty/p^{1/p}S_\infty) & \longrightarrow & H_{\mathfrak{m}}^k(S_\infty) & \xrightarrow{p^{1/p}} & H_{\mathfrak{m}}^k(S_\infty) & \longrightarrow & \cdots \\
 & & \wr \downarrow \bar{F}_{S_\infty *} & & & & & & \\
\cdots & \longrightarrow & H_{\mathfrak{m}}^{k-1}(S_\infty/pS_\infty) & \longrightarrow & H_{\mathfrak{m}}^k(S_\infty) & \xrightarrow{p} & H_{\mathfrak{m}}^k(S_\infty) & \longrightarrow & \cdots
\end{array}$$

Let us fix a positive $\epsilon \in \mathbb{Q}$. Then there exists $r \in R_\infty$ with $v(r) < \epsilon$ such that $\lambda_\infty(r \cdot H_{\mathfrak{m}}^k(S_\infty)) < \infty$. From the above exact sequence there follows a surjection:

$$H_{\mathfrak{m}}^{k-1}(S_\infty/p^{1/p}S_\infty) \twoheadrightarrow H_{\mathfrak{m}}^k(S_\infty)_{p^{1/p}} \quad (\text{respectively, } H_{\mathfrak{m}}^{k-1}(S_\infty/pS_\infty) \twoheadrightarrow H_{\mathfrak{m}}^k(S_\infty)_p).$$

Then by the Frobenius pull-back formula, we deduce

$$\lambda_\infty(r \cdot H_{\mathfrak{m}}^{k-1}(S_\infty/p^{1/p}S_\infty)) = \lambda_\infty(r \cdot (H_{\mathfrak{m}}^{k-1}(S_\infty/pS_\infty))^{[F]}) = \frac{1}{p^d} \cdot \lambda_\infty(r^p \cdot H_{\mathfrak{m}}^{k-1}(S_\infty/pS_\infty)) < \infty,$$

which, together with Lemma 3.3(2), yields the equality:

$$\lambda_\infty(r \cdot H_{\mathfrak{m}}^k(S_\infty)_{p^{1/p}}) = \frac{1}{p^d} \cdot \lambda_\infty(r^p \cdot H_{\mathfrak{m}}^k(S_\infty)_p). \tag{1}$$

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To ease the notation, we set $N := H_{\mathfrak{m}}^k(S_\infty)$. Let us consider the filtration of N_p as follows:

$$N_{p^{1/p}} \subset N_{p^{2/p}} \subset \cdots \subset N_{p^{t/p}} \subset \cdots \subset N_p.$$

Then since $p^{(t-1)/p} N_{p^{t/p}} \subset N_{p^{1/p}}$, the multiplication map $p^{(t-1)/p} : N_{p^{t/p}} \to N_{p^{1/p}}$ induces an injective map $N_{p^{t/p}}/N_{p^{(t-1)/p}} \hookrightarrow N_{p^{1/p}}$ and we have

$$\lambda_\infty(N_{p^{t/p}}/N_{p^{(t-1)/p}}) \leqslant \lambda_\infty(N_{p^{1/p}}). \tag{2}$$

Let us next consider the short exact sequence

$$0 \to N_{p^{(t-1)/p}} \to N_{p^{t/p}} \to N_{p^{t/p}}/N_{p^{(t-1)/p}} \to 0.$$

Then from Lemma 3.3 we have

$$\lambda_\infty(r^t(N_{p^{t/p}})) \leqslant \lambda_\infty(r^{t-1}(N_{p^{(t-1)/p}})) + \lambda_\infty(r(N_{p^{t/p}}/N_{p^{(t-1)/p}}))$$

for $t = 2, \ldots, p$. An induction on t yields

$$\lambda_\infty(r^p(N_p)) \leqslant \sum_{t=1}^p \lambda_\infty(r(N_{p^{t/p}}/N_{p^{(t-1)/p}})), \tag{3}$$

and it follows easily from (2) and (3) that

$$\lambda_\infty(r^p(N_p)) \leqslant p \cdot \lambda_\infty(r(N_{p^{1/p}})).$$

We can now deduce from this inequality and (1):

$$\frac{\lambda_\infty(r^p(N_p))}{p^d} = \lambda_\infty(r(N_{p^{1/p}})) \geqslant \frac{\lambda_\infty(r^p(N_p))}{p},$$

which is possible only when $\lambda_\infty(r(N_{p^{1/p}})) = 0$ (or, equivalently, $\lambda_\infty(r^p(N_p)) = 0$). For simplicity, we denote r^p by r, so that $\lambda_\infty(r(N_p)) = 0$. Now let $\eta \in N$. Since N is p-torsion, we have $\eta \in N_{p^n}$ for sufficiently large $n \in \mathbb{N}$. Then we may use the fact that $p \cdot N_{p^t} \subset N_{p^{t-1}}$, Lemma 3.3 together with the following short exact sequence inductively

$$0 \longrightarrow N_p \longrightarrow N_{p^t} \overset{p}{\longrightarrow} p \cdot N_{p^t} \longrightarrow 0$$

to deduce that

$$\lambda_\infty(r^n(N_{p^n})) = 0.$$

In light of Proposition 2.15, for $\epsilon \in \mathbb{Q}$ as above, there exists an element $s \in R_\infty$ such that $(sr^n)\eta = 0$ with $v(sr^n) = v(s) + v(r^n) < \epsilon + n\epsilon = \epsilon(n+1)$. If we have chosen ϵ sufficiently small, we can make $\epsilon(n+1)$ also sufficiently small, as $n \in \mathbb{N}$ depends on the choice of $\eta \in N$. Hence, this proves the desired claim. $\square$

Remark 3.5. An R_∞-module M with $\lambda_\infty(M) = 0$ is v-almost zero. However, it might be the case that the annihilator of every element of M is very complicated and M is not finitely generated. It looks reasonable to ask whether the local cohomology module of any semi-perfect ring is v-almost zero in light of the above theorem. For some relevant results of the annihilator, see also [Rob76, Sch82].

Now we remind the reader that R_n and $d = \dim R$ are defined as previously. Let $R_n \to S_n$ be a module-finite extension of domains with $p^{1/p} \in S_n$. Let S_∞ be any semi-perfect algebra over S_n and let $\phi : H_{\mathfrak{m}}^k(S_n) \to H_{\mathfrak{m}}^k(S_\infty)$ be the map induced by the inclusion $S_n \to S_\infty$. Then define the map

$$\Phi : H_{\mathfrak{m}}^k(S_n) \otimes_{R_n} R_\infty \to H_{\mathfrak{m}}^k(S_\infty)$$

by $\Phi(a \otimes b) := \phi(a)b$ for $a \in H_{\mathfrak{m}}^k(S_n)$ and $b \in R_\infty$. We also denote by

$$\Phi_{p^{1/p}} : H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \otimes_{R_n} R_\infty \to H_{\mathfrak{m}}^k(S_\infty)_{p^{1/p}}$$

the restriction map of Φ to the submodule annihilated by $p^{1/p}$. Alternatively, one may regard the map Φ to be induced by the natural map $S_n \otimes_{R_n} R_\infty \to S_\infty$ as R_∞ is flat over R_n.

THEOREM 3.6. *Keeping the notation and hypothesis as above, assume that we have* $H_{\mathfrak{m}}^{k-1}(S_n) = 0$ *and* $\ell(H_{\mathfrak{m}}^k(S_n)) < \infty$ *for* $0 < k < d$. *Then*

$$\lambda_\infty(\mathrm{Im}(\Phi_{p^{1/p}})) \leqslant \frac{1}{p^{d(n+1)-1}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}}).$$

In particular, Φ *is not injective if* $H_{\mathfrak{m}}^k(S_n)$ *is non-zero.*

Proof. First, we note that $H_{\mathfrak{m}}^k(S_n) \neq 0 \iff H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \neq 0$. Since S_n is a domain, one has the following short exact sequence:

$$0 \longrightarrow S_n \xrightarrow{b} S_n \longrightarrow S_n/bS_n \longrightarrow 0$$

for any non-zero $b \in S_n$. Then, by assumption, the associated long exact sequence

$$0 \longrightarrow H_{\mathfrak{m}}^{k-1}(S_n/bS_n) \longrightarrow H_{\mathfrak{m}}^k(S_n) \xrightarrow{b} H_{\mathfrak{m}}^k(S_n) \longrightarrow \cdots$$

yields an isomorphism

$$H_{\mathfrak{m}}^k(S_n)_b \simeq H_{\mathfrak{m}}^{k-1}(S_n/bS_n). \tag{1'}$$

Let us consider the following commutative square of R_∞-modules:

$$
\begin{array}{ccc}
S_n/p^{1/p}S_n \otimes_{R_n} R_\infty & \longrightarrow & S_\infty/p^{1/p}S_\infty \\
\bar{F}_{S_n} \downarrow & & \bar{F}_{S_\infty} \downarrow \wr \\
(S_n/pS_n \otimes_{R_n} R_\infty)^{[F]} & \longrightarrow & (S_\infty/pS_\infty)^{[F]}
\end{array}
$$

in which both of the horizontal maps are induced by the natural map $S_n/p^{1/p}S_n \to S_\infty/p^{1/p}S_\infty$ (respectively, $S_n/pS_n \to S_\infty/pS_\infty$) and the vertical maps are induced by the Frobenius map. Then by use of the isomorphism (1') together with the flatness of R_∞ over R_n, we have the induced map on local cohomology modules:

$$
\begin{array}{ccc}
H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \otimes_{R_n} R_\infty & \xrightarrow{\widetilde{\Phi}_{p^{1/p}}} & H_{\mathfrak{m}}^{k-1}(S_\infty/p^{1/p}S_\infty) \\
\bar{F}_{S_n*} \downarrow & & \bar{F}_{S_\infty*} \downarrow \wr \\
(H_{\mathfrak{m}}^k(S_n)_p \otimes_{R_n} R_\infty)^{[F]} & \xrightarrow{\widetilde{\Phi}_p} & H_{\mathfrak{m}}^{k-1}(S_\infty/pS_\infty)^{[F]}
\end{array}
\tag{2'}
$$

By assumption, the module $H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \otimes_{R_n} R_\infty$ is finitely presented in $\mathcal{M}$, and we get

$$\lambda_\infty(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \otimes_{R_n} R_\infty) = \frac{1}{p^{dn}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}})$$

and similarly

$$\lambda_\infty(H_{\mathfrak{m}}^k(S_n)_p \otimes_{R_n} R_\infty) = \frac{1}{p^{dn}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_p).$$

On the other hand, an easy calculation shows that $\ell(H_{\mathfrak{m}}^k(S_n)_p) \leqslant p \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}})$. Hence, it follows from the diagram (2') and the Frobenius pull-back formula that

$$\lambda_\infty(\mathrm{Im}(\widetilde{\Phi}_{p^{1/p}})) \leqslant \lambda_\infty((\mathrm{Im}(\widetilde{\Phi}_p))^{[F]}) = \frac{1}{p^d} \cdot \lambda_\infty(\mathrm{Im}(\widetilde{\Phi}_p)) \leqslant \frac{1}{p^{d(n+1)}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_p)$$

$$\leqslant \frac{p}{p^{d(n+1)}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}}) = \frac{1}{p^{d(n+1)-1}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}}).$$

Let $j_b : H_{\mathfrak{m}}^{k-1}(S_\infty/bS_\infty) \twoheadrightarrow H_{\mathfrak{m}}^k(S_\infty)_b$ denote the surjection map induced by the cohomology exact sequence associated to

$$0 \longrightarrow S_\infty \xrightarrow{b} S_\infty \longrightarrow S_\infty/bS_\infty \longrightarrow 0$$

for any non-zero $b \in S_\infty$. Then, by assumption, it is easy to see that $\Phi_b = j_b \circ \widetilde{\Phi}_b$, from which $\lambda_\infty(\operatorname{Im}(\widetilde{\Phi}_{p^{1/p}})) \geqslant \lambda_\infty(\operatorname{Im}(\Phi_{p^{1/p}}))$. Hence, the theorem follows easily from this inequality.

The non-injectivity of the map Φ is due to the following equality:

$$\lambda_\infty(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}} \otimes_{R_n} R_\infty) = \frac{1}{p^{dn}} \cdot \ell(H_{\mathfrak{m}}^k(S_n)_{p^{1/p}}). \qquad \square$$

The next corollary is a weak version of Heitmann's direct summand theorem.

COROLLARY 3.7. *Let $R_n \to S_n$ be a module-finite extension of normal domains and $p^{1/p} \in S_n$, and let $d > 2$. Then the map*

$$\Phi : H_{\mathfrak{m}}^2(S_n) \otimes_{R_n} R_\infty \to H_{\mathfrak{m}}^2(S^+)$$

is not injective if $H_{\mathfrak{m}}^2(S_n)$ is non-zero.

Proof. By Serre's normality criterion, it follows that $H_{\mathfrak{m}}^1(S_n) = 0$ and $\ell(H_{\mathfrak{m}}^2(S_n)) < \infty$ by local duality. Then the corollary follows from Theorem 3.6. $\qquad \square$

There is another implication for splinters studied by Singh [Sin99]. Recall that a Noetherian domain A is a *splinter* if A is a direct summand, as an A-module, of every module-finite extension domain. It is easy to see that a splinter is a normal domain.

COROLLARY 3.8. *In addition to the hypothesis of Theorem 3.6, assume that $S \otimes_R R_\infty$ is a domain and $H_{\mathfrak{m}}^k(S)$ is non-zero. Then there exists a flat module-finite extension $S \to T$ such that T is not a splinter.*

Proof. Since $\Phi : H_{\mathfrak{m}}^k(S) \otimes_R R_\infty \to H_{\mathfrak{m}}^k(S^+)$ is not injective and $S \otimes_R R_\infty = \bigcup_{n \geqslant 0} S \otimes_R R_n$, the map $H_{\mathfrak{m}}^k(S) \otimes_R R_n \to H_{\mathfrak{m}}^k(S^+)$ is not injective for some $n \geqslant 0$. Now it is easy to see that the domain $T := S \otimes_R R_n$ is flat over S and is not a splinter. $\qquad \square$

Finally, we end this section with an example of a normal domain that is not a splinter, due to Singh (unpublished).

Example 3.9. Let

$$S := \frac{\mathbb{Z}_2[[x,y]]}{(x^k - y^l - 2^m)}$$

where $\mathbb{Z}_2$ is the ring of 2-adic integers and $(k,l,m) > (2,2,2)$. Then S is shown to be a normal domain. Now take $u := \sqrt{x^{k-2}y^{l-2}} \notin S$ and $v := 2^{-1}(x^{k-1} - uy) \notin S$. Then u is integral over S and it is easy to verify that v is subject to the following equation:

$$v^2 + uyv - x^{k-2}2^{m-2} = 0.$$

Hence, $S \to T := S[u,v]$ is module-finite. Then since $x^{k-1} = uy + v2$, if S were a direct summand of T, we would have that $x \in (y,2)T$, while $x \notin (y,2)S = (y,2)T \cap S$, which is a contradiction.

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