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COMPARISON AND POSITIVE SOLUTIONS FOR PROBLEMS WITH THE (p, q)-LAPLACIAN AND A CONVECTION TERM

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Abstract The aim of this paper is to prove the existence of a positive solution for a quasi-linear elliptic problem involving the (p, q)-Laplacian and a convection term, which means an expression that is not in the principal part and depends on the solution and its gradient. The solution is constructed through an approximating process based on gradient bounds and regularity up to the boundary. The positivity of the solution is shown by applying a new comparison principle, which is established here.

Keywords: quasi-linear elliptic equation; (p, q)-Laplacian; convection term; positive solution; approximation; comparison

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1. Introduction

In this paper we study the existence of (positive) solutions for the following quasi-linear elliptic equation with Dirichlet boundary condition:

$$\left.\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x, u, \nabla u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{P}$$

on a bounded domain Ω in $\mathbb{R}^N$ with a $C^{1,\alpha}$-boundary $\partial\Omega$, for some $0 < \alpha \leqslant 1$. On the left-hand side of the equation in (P) we have the p-Laplacian Δ_p and the q-Laplacian Δ_q with $1 < q < p < +\infty$, and a constant $\mu \geqslant 0$. The problem covers the corresponding statement with the p-Laplacian in the principal part, for which it is sufficient to take $\mu = 0$. Here $-\Delta_p$ is regarded as the operator $-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, where $1/p + 1/p' = 1$, defined by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

The right-hand side of the equation in (P) is in the form of a convection term, meaning a nonlinearity $f(x, u, \nabla u)$ that depends on the point x in the domain Ω, on the solution u and on its gradient ∇u. The essential feature of this paper is the dependence on the gradient ∇u, which prevents the use of variational methods.

We assume that $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the growth condition:

(F) $b_0|t|^{r_0} \leqslant f(x,t,\xi) \leqslant b_1(1+|t|^{r_1}+|\xi|^{r_2})$ for all $(x,t,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$, with constants $b_0, b_1 > 0$, $r_1, r_2 \in [0, p-1)$, $r_0 \in [0, p-1)$ if $\mu = 0$ and $r_0 \in [0, q-1)$ if $\mu > 0$.

Since we are looking for positive solutions of problem (P), without any loss of generality we will suppose in the following that $f(x,t,s) \equiv 0$ for all $t \leqslant 0$ and $(x,s) \in \bar{\Omega} \times \mathbb{R}^N$.

The (p,q)-Laplacian problems have received much interest due to their rich mathematical substance and various applications in quantum physics, biophysics, plasma physics and chemical reaction design (see, for example, [**1**, **7**]). One of the main goals in the study of such problems has been to show the existence of positive solutions. This has only been achieved for (p,q)-Laplacian problems in the semilinear case, that is, when the right-hand side of the equation in problem (P) does not depend on the gradient of the solution. In such a case, the approach has relied on variational methods using the variational structure of the semilinear equation (see, for example, [**3**, **7**, **16**]).

The existence of positive solutions for problems with the p-Laplacian (but not the (p,q)-Laplacian) and a convection term, that is, the case where $\mu = 0$ in (P), has been studied in [**15**,**17**]. In these works the imposed hypotheses are different from (F), assuming among other things that $r_0 > p-1$, which is a complementary condition to our requirement in (F). The approach developed in these works was through fixed-point theorems in cones, which is also different from ours.

To the best of our knowledge, this paper seems to be the first work dealing simultaneously with the (p,q)-Laplacian and the convection term. An important special case is when we only have the p-Laplacian in the principal part of the elliptic equation, which corresponds to the case $\mu = 0$ in problem (P). Our main result is the following existence theorem of positive solutions.

Theorem 1.1. *Under (F), problem (P) admits a (positive) solution* $u \in C_0^1(\bar{\Omega})$.

We recall that $C_0^1(\bar{\Omega})$ stands for the Banach space of functions $u \in C^1(\bar{\Omega})$ vanishing on $\partial\Omega$. The convection term means that problem (P) does not have a variational structure, so it is not applicable for the variational methods. Here the existence of a solution to problem (P) is established through an approximating process by means of a Schauder basis, *a priori* estimates and passing to the limit. It is worth mentioning that these steps in the proof are made in the presence of the (p,q)-Laplace operator and of another nonlinearity depending on the solution and its gradient. In our argument an essential part is played by the gradient bounds and the regularity of the solution up to the boundary, for which we refer the reader to [**9**, **10**].

The fact that the solution obtained is positive is proven on the basis of a generalized version of the strong maximum principle (see [**14**]). In this respect, we present here a new comparison principle stated as Theorem 2.2, which is the second main result of the paper. This theorem is of independent interest, allowing comparison of a subsolution and a supersolution for nonlinear elliptic problems whose principal part is $-\Delta_p u - \mu\Delta_q u$ with $\mu \geqslant 0$. In turn, our comparison principle is obtained from a slightly extended version of [**6**, Lemma 2] (see also [**5**]). In order to show that the solution constructed in the proof of Theorem 1.1 is positive, we apply our comparison principle to a suitable auxiliary problem related to the growth condition in (F). Here we show that the assumptions of Theorem 2.2 are satisfied with the aid of the Hopf boundary point lemma in the strong maximum principle adapted to the operator $-\Delta_p u - \mu\Delta_q u$.

The rest of the paper is organized as follows. Section 2 presents our comparison principle. Section 3 is devoted to the construction of approximate solutions. Section 4 contains the proof of Theorem 1.1.

2. Comparison principle

The Sobolev space $W_0^{1,p}(\Omega)$ with $1 < p < \infty$ is endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

Throughout the paper, the solutions of the elliptic boundary-value problems are in the weak sense.

The following result is a slightly extended version of [**6**, Lemma 2].

Lemma 2.1. *Let $w_1, w_2 \in L^\infty(\Omega)$ satisfy $w_i \geqslant 0$ almost everywhere (a.e.) in Ω, $w_i^{1/q} \in W^{1,p}(\Omega)$, $\Delta_p w_i^{1/q} \in L^\infty(\Omega)$ for $i = 1, 2$ and $w_1 = w_2$ on $\partial\Omega$, where $1 < q < p < +\infty$. If $w_1/w_2, w_2/w_1 \in L^\infty(\Omega)$, then it holds that*

$$\int_\Omega \left(-\frac{\Delta_p w_1^{1/q} + \mu\Delta_q w_1^{1/q}}{w_1^{(q-1)/q}} + \frac{\Delta_p w_2^{1/q} + \mu\Delta_q w_2^{1/q}}{w_2^{(q-1)/q}}\right)(w_1 - w_2)\,\mathrm{d}x \geqslant 0.$$

Proof. We introduce the functional $J\colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u^{1/q}|^p \,\mathrm{d}x & \text{if } u \geqslant 0 \text{ and } u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that J is convex. To this end, let $w_1, w_2 \in L^1(\Omega)$ satisfy $w_1, w_2 \geqslant 0$ and $w_1^{1/q}, w_2^{1/q} \in W^{1,p}(\Omega)$. Hence, we have $w_1^{1/q}, w_2^{1/q} \in W^{1,q}(\Omega)$. It is shown in the proof of [**6**, Lemma 1] that

$$|\nabla(tw_1 + (1-t)w_2)^{1/q}|^q \leqslant t|\nabla w_1^{1/q}|^q + (1-t)|\nabla w_2^{1/q}|^q \quad \text{for all } t \in [0, 1].$$

Since $1 < q \leqslant p < +\infty$, the function $s \mapsto s^{p/q}$ is convex on $[0, +\infty)$. Then the above inequality implies that

$$|\nabla(tw_1 + (1-t)w_2)^{1/q}|^p \leqslant t|\nabla w_1^{1/q}|^p + (1-t)|\nabla w_2^{1/q}|^p \quad \text{for all } t \in [0, 1].$$

Therefore, the function J is convex. Now the monotonicity of the differential of J on its domain, in conjunction with [**6**, Lemma 2] (note that $\mu \geqslant 0$), ensures the desired conclusion. $\square$

Lemma 2.1 enables us to establish a comparison principle for a subsolution and a supersolution of the Dirichlet problem

$$\left.\begin{aligned} -\Delta_p u - \mu\Delta_q u = g(u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where $1 < q < p < +\infty$, $\mu \geqslant 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

We recall that $u_1 \in W^{1,p}(\Omega)$ is a subsolution of problem (2.1) if $u_1 \leqslant 0$ a.e. on $\partial\Omega$ and

$$\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi + \mu|\nabla u_1|^{q-2}\nabla u_1\nabla\varphi)\,\mathrm{d}x \leqslant \int_\Omega g(u_1)\varphi\,\mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω, while $u_2 \in W^{1,p}(\Omega)$ is a supersolution of (2.1) if the reversed inequalities are satisfied with u_2 in place of u_1 for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω.

Theorem 2.2. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $t^{1-q}g(t)$ is decreasing for $t > 0$ if $\mu > 0$, and $t^{1-p}g(t)$ is decreasing for $t > 0$ if $\mu = 0$. Assume that $u_1 \in W_0^{1,p}(\Omega)$ and $u_2 \in W_0^{1,p}(\Omega)$ are a positive subsolution and a positive supersolution of problem (2.1), respectively. If $u_i \in L^\infty(\Omega) \cap C^{1,\alpha}(\Omega)$, $\Delta_p u_i \in L^\infty(\Omega)$, $u_i/u_j \in L^\infty(\Omega)$ for $i, j = 1, 2$, then $u_2 \geqslant u_1$ in Ω.*

Proof. We only give the proof for $\mu > 0$, because the case $\mu = 0$ can be handled in the same way and is in fact simpler. Arguing by contradiction, suppose that the set $\Omega_0 = \{x \in \Omega\colon u_1(x) > u_2(x)\}$ is non-empty. Let U be an open connected subset of Ω_0 such that $u_1 = u_2$ on ∂U. Using the fact that u_1 and u_2 are a subsolution and a supersolution of problem (2.1), respectively, we derive that

$$\begin{aligned} u_2^{q-1}(\Delta_p u_1 + \mu\Delta_q u_1) - u_1^{q-1}(\Delta_p u_2 + \mu\Delta_q u_2) &\geqslant g(u_2)u_1^{q-1} - g(u_1)u_2^{q-1} \\ &= u_1^{q-1}u_2^{q-1}\left(\frac{g(u_2)}{u_2^{q-1}} - \frac{g(u_1)}{u_1^{q-1}}\right) \end{aligned} \tag{2.2}$$

in Ω. Then the monotonicity assumption on g enables us to obtain

$$-\frac{\Delta_p u_2 + \mu\Delta_q u_2}{u_2^{q-1}} + \frac{\Delta_p u_1 + \mu\Delta_q u_1}{u_1^{q-1}} \geqslant \frac{g(u_2)}{u_2^{q-1}} - \frac{g(u_1)}{u_1^{q-1}} > 0 \quad \text{in } U. \tag{2.3}$$

Since $u_1 > u_2$ in U, from (2.3) we infer that

$$\int_U \left(-\frac{\Delta_p u_2 + \mu\Delta_q u_2}{u_2^{q-1}} + \frac{\Delta_p u_1 + \mu\Delta_q u_1}{u_1^{q-1}}\right)(u_1^q - u_2^q)\,\mathrm{d}x > 0. \tag{2.4}$$

On the other hand, note that we can apply Lemma 2.1 with $w_i = u_i^q$, $i = 1, 2$, and U in place of Ω. The conclusion obtained from the application of Lemma 2.1 with these choices contradicts (2.4) in the case where U is non-empty. Consequently, $\Omega_0 = \emptyset$. This completes the proof. $\square$

3. Approximate solutions

In order to prove Theorem 1.1, we approximate problem (P) with problems possessing positive solutions that are uniformly bounded from below away from 0. Towards this aim, for every $\varepsilon > 0$, we associate to (P) the following Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x, u+\varepsilon, \nabla u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{P$_\varepsilon$}$$

In problem (P$_\varepsilon$) the data Ω, p, q, μ and f satisfy the same conditions as in problem (P).

Theorem 3.1. *Assume that condition (F) holds true. Then, for every $\varepsilon > 0$, the approximate problem (P$_\varepsilon$) has at least a (positive) solution $u \in C_0^1(\bar{\Omega})$.*

Proof. Fix $\varepsilon > 0$ and let $\{e_1, \dots, e_m, \dots\}$ be a Schauder basis of $W_0^{1,p}(\Omega)$ (see [**2**, p. 146] and [**4**]). For each positive integer m, we set

$$V_m = \operatorname{span}\{e_1, \dots, e_m\}.$$

This is an m-dimensional vector subspace of $W_0^{1,p}(\Omega)$ that we endow with the norm given by

$$\|v\|_m = \sum_{j=1}^{m} |\xi_j|, \quad \text{where } v = \sum_{j=1}^{m} \xi_j e_j \in V_m.$$

Since the norms $\|\cdot\|_m$ and $\|\cdot\|$ on V_m are equivalent, there exist positive constants $c(m)$ and $k(m)$ such that

$$c(m)\|v\|_m \leqslant \|v\| \leqslant k(m)\|v\|_m \quad \text{for all } v \in V_m.$$

We identify the normed spaces $(V_m, \|\cdot\|_m)$ and $(\mathbb{R}^m, |\cdot|_s)$, where $|\xi|_s = \sum_{j=1}^{m} |\xi_j|$ for $\xi = (\xi_1, \dots, \xi_m)$, by the isometric linear isomorphism

$$v = \sum_{j=1}^{m} \xi_j e_j \in V_m \mapsto (\xi_1, \dots, \xi_m) \in \mathbb{R}^m.$$

Via this identification, we define the map $T = (T_1, \dots, T_m)\colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_j(\xi) = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla e_j \,\mathrm{d}x + \mu \int_\Omega |\nabla v|^{q-2}\nabla v \nabla e_j \,\mathrm{d}x - \int_\Omega f(x, |v|+\varepsilon, \nabla v) e_j \,\mathrm{d}x, \quad j = 1, \dots, m. \tag{3.1}$$

We see that

$$\langle T(\xi), \xi\rangle = \int_\Omega (|\nabla v|^p + \mu|\nabla v|^q)\,\mathrm{d}x - \int_\Omega f(x, |v|+\varepsilon, \nabla v) v \,\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^m. \tag{3.2}$$

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Using assumption (F), we obtain

$$\int_\Omega f(x, |v| + \varepsilon, \nabla v)v \,\mathrm{d}x \leqslant b_1 \int_\Omega (1 + (|v| + \varepsilon)^{r_1} + |\nabla v|^{r_2})|v| \,\mathrm{d}x. \tag{3.3}$$

Let us estimate the terms on the right-hand side of (3.3). The Sobolev embedding theorem yields

$$\int_\Omega |v| \,\mathrm{d}x \leqslant c\|v\|$$

and

$$\int_\Omega (|v| + \varepsilon)^{r_1} |v| \,\mathrm{d}x \leqslant c(\|v\|^{r_1+1} + \|v\|),$$

with a constant $c > 0$. By the Hölder inequality we have

$$\int_\Omega |\nabla v|^{r_2} v \,\mathrm{d}x \leqslant \left(\int_\Omega |\nabla v|^p \,\mathrm{d}x\right)^{r_2/p} \left(\int_\Omega |v|^{p/(p-r_2)} \,\mathrm{d}x\right)^{(p-r_2)/p}.$$

We note that $p/(p - r_2) < p$. Then the Sobolev embedding theorem ensures that

$$\int_\Omega |\nabla v|^{r_2} v \,\mathrm{d}x \leqslant c\|v\|^{r_2+1},$$

with a constant $c > 0$. Gathering the above inequalities leads to

$$\int_\Omega f(x, |v| + \varepsilon, \nabla v)v \,\mathrm{d}x \leqslant c(\|v\| + \|v\|^{r_1+1} + \|v\|^{r_2+1}). \tag{3.4}$$

Now, combining (3.2) and (3.4) entails

$$\begin{aligned}\langle T(\xi), \xi\rangle &\geqslant \|v\|^p - c(\|v\| + \|v\|^{r_1+1} + \|v\|^{r_2+1}) \\ &\geqslant C_1(m)|\xi|_s^p - C_2(m)(|\xi|_s + |\xi|_s^{r_1+1} + |\xi|_s^{r_2+1}),\end{aligned}$$

with constants $C_1(m), C_2(m) > 0$. Then, we derive that for every $r > 0$ it holds that

$$\langle T(\xi), \xi\rangle \geqslant r \quad \text{whenever } |\xi|_s = \rho_m,$$

provided $\rho_m > 0$ is sufficiently large. This is true because $r_1 + 1, r_2 + 1 < p$, as supposed in (F). By a well-known consequence of Brouwer's fixed-point theorem, it follows that there exists $\xi_m \in \mathbb{R}^m$ such that

$$T(\xi_m) = 0 \quad \text{and} \quad |\xi_m|_s \leqslant \rho_m.$$

Consequently, through the isometric identification between $(\mathbb{R}^m, |\cdot|_s)$ and $(V_m, \|\cdot\|_m)$ and also invoking (3.1), we can find $u_m \in V_m$, with $\|u_m\|_m \leqslant \rho_m$, satisfying

$$\int_\Omega (|\nabla u_m|^{p-2}\nabla u_m + \mu|\nabla u_m|^{q-2}\nabla u_m)\nabla v \,\mathrm{d}x = \int_\Omega f(x, |u_m| + \varepsilon, \nabla u_m)v \,\mathrm{d}x \tag{3.5}$$

for all $v \in V_m$.

The main step in the proof is to show that a subsequence of $\{u_m\}$ is strongly convergent in $W_0^{1,p}(\Omega)$. To this end, substituting $v = u_m$ into (3.5) and using (3.4) yields

$$\|u_m\|^p \leqslant c(\|u_m\| + \|u_m\|^{r_1+1} + \|u_m\|^{r_2+1}), \tag{3.6}$$

where the constant $c > 0$ is independent of m. Since $r_1 + 1, r_2 + 1 < p$, it turns out from (3.6) that the sequence $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, along a relabelled subsequence, we can assume that

$$u_m \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_m(x) \to u(x) \quad \text{for a.e. in } \Omega \text{ as } m \to \infty, \tag{3.7}$$

with some $u \in W_0^{1,p}(\Omega)$.

We claim that

$$u_m \to u \text{ in } W_0^{1,p}(\Omega) \quad \text{as } m \to \infty. \tag{3.8}$$

Using the fact that $\{e_1, \dots, e_m, \dots\}$ is a Schauder basis of $W_0^{1,p}(\Omega)$, u can be uniquely expressed as the sum of a series $\sum_{n\geqslant 1} \alpha_n e_n$ in $W_0^{1,p}(\Omega)$, with a sequence $\{\alpha_n\}_{n\geqslant 1}$ in $\mathbb{R}$, so

$$w_m := \sum_{j=1}^{m} \alpha_j e_i \to u \text{ in } W_0^{1,p}(\Omega) \quad \text{as } m \to \infty. \tag{3.9}$$

Since $u_m, w_m \in V_m$, we can substitute $v = u_m - w_m$ into (3.5), which gives

$$\int_\Omega (|\nabla u_m|^{p-2}\nabla u_m + \mu|\nabla u_m|^{q-2}\nabla u_m)\nabla(u_m - w_m)\,\mathrm{d}x = \int_\Omega f(x, |u_m| + \varepsilon, \nabla u_m)(u_m - w_m)\,\mathrm{d}x. \tag{3.10}$$

By (3.7), (3.9) and hypothesis (F), the Lebesgue dominated convergence theorem leads to

$$\lim_{m\to\infty} \int_\Omega f(x, |u_m| + \varepsilon, \nabla u_m)(u_m - w_m)\,\mathrm{d}x = 0 \tag{3.11}$$

and

$$\lim_{m\to\infty} \int_\Omega (|\nabla u_m|^{p-2}\nabla u_m + \mu|\nabla u_m|^{q-2}\nabla u_m)\nabla(u - w_m)\,\mathrm{d}x = 0. \tag{3.12}$$

Then (3.9)–(3.12) imply that

$$\lim_{m\to\infty} \int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla(u_m - u)\,\mathrm{d}x = 0.$$

Now it is sufficient to apply the (S_+)-property of $-\Delta_p$ (see, for example, [**13**, Proposition 3.5]) to obtain (3.8).

Fix $k \geqslant 1$ and $v \in V_k$. For each $m \geqslant k$, we know from (3.5) that

$$\int_\Omega (|\nabla u_m|^{p-2}\nabla u_m + \mu|\nabla u_m|^{q-2}\nabla u_m)\nabla v\,\mathrm{d}x = \int_\Omega f(x, |u_m| + \varepsilon, \nabla u_m)v\,\mathrm{d}x.$$

Letting $m \to \infty$, on account of (3.8) we arrive at

$$\int_{\Omega} (|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u)\nabla v \, dx = \int_{\Omega} f(x, |u| + \varepsilon, \nabla u)v \, dx. \tag{3.13}$$

As k is arbitrary, we see that (3.13) is valid for every $v \in \bigcup_{k\geqslant 1} V_k$. Actually, (3.13) holds true for every $v \in W_0^{1,p}(\Omega)$ because $\bigcup_{k\geqslant 1} V_k$ is dense in $W_0^{1,p}(\Omega)$. Then we may test (3.13) with $v = -u^- = -\max\{-u, 0\}$, which yields

$$\int_{\Omega} (|\nabla u^-|^p + \mu|\nabla u^-|^q) \, dx = -\int_{\Omega} f(x, |u| + \varepsilon, \nabla u)u^- \, dx. \tag{3.14}$$

By assumption (F), we see that $f \geqslant 0$. Hence, (3.14) shows that $u \geqslant 0$ in Ω. This and (3.13) show that u is a weak solution of the equation in (P_ε).

The first inequality in hypothesis (F) and the equation in (P_ε) guarantee that $u \neq 0$. Here the presence of $\varepsilon > 0$ is needed. Next, we observe that hypothesis (F) allows us to refer to [**8**, Theorem 7.1], from which we infer that $u \in L^\infty(\Omega)$. Furthermore, the regularity result up to the boundary in [**9**, Theorem 1] and [**10**, p. 320] ensures that $u \in C^{1,\beta}(\bar{\Omega})$ with some $\beta \in (0, 1)$. We also note that we may apply the strong maximum principle in [**14**, Theorem 5.4.1] (see also [**12**, Theorem B]) by taking therein the function $A(t) = t^{p-2} + \mu t^{q-2}$ for $t > 0$. Indeed, we find that

$$c := \lim_{t\to 0^+} \frac{tA'(t)}{A(t)} = \begin{cases} q-2 & \text{if } \mu > 0, \\ p-2 & \text{if } \mu = 0 \end{cases} \tag{3.15}$$

is strictly bigger than -1 because $p \geqslant q > 1$. Thus hypothesis (5.4.3) in [**14**, Theorem 5.4.1] is satisfied. In addition, with the constant c in (3.15), we obtain

$$\frac{2 + c + 2\sqrt{1+c}}{|c|} = \begin{cases} \dfrac{q + 2\sqrt{q-1}}{|q-2|} & \text{if } \mu > 0, \ q \neq 2, \\[2ex] \dfrac{p + 2\sqrt{p-1}}{|p-2|} & \text{if } \mu = 0, \ p \neq 2, \end{cases}$$

which is strictly bigger than 1. Therefore, hypothesis (5.4.4) in [**14**, Theorem 5.4.1] is also satisfied because, using the notation therein, in our setting we have $\Lambda(t) = \lambda(t) \equiv 1$ for all $t \geqslant 0$. We are thus in a position to apply [**14**, Theorem 5.4.1], concluding that $u > 0$ in Ω because we know that $u \geqslant 0$ but $u \neq 0$; thus, u is a solution of problem (P_ε). This completes the proof. □

4. Proof of Theorem 1.1

We start with an auxiliary result that is useful in conjunction with our comparison principle in Theorem 2.2 and hypothesis (F).

Lemma 4.1. *Let $1 < q < p < +\infty$ and $\mu \geqslant 0$. For any constants $b > 0$ and $0 < r < p-1$, with $0 < r < q-1$ if $\mu > 0$, the problem*

$$\left.\begin{aligned} -\Delta_p u - \mu\Delta_q u &= bu^r \quad \text{in } \Omega, \\ u &> 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.1}$$

admits a solution $u_0 \in C_0^1(\bar{\Omega})$.

Proof. Given the constants $b > 0$ and $0 < r < q-1$, we define the functional $I\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{\mu}{q}\int_\Omega |\nabla u|^q \,\mathrm{d}x - \frac{b}{r+1}\int_\Omega (u^+)^{r+1}\,\mathrm{d}x \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $u^+ = \max\{0, u\}$. Note that I is of class C^1. By using the Sobolev embedding theorem, we have the estimate

$$I(u) \geqslant \frac{1}{p}\|u\|^p - c\|u\|^{r+1} \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

with a constant $c > 0$. Since $p > r+1$, I is bounded from below and coercive. Taking into account that the first two terms in the expression of I are convex and continuous on $W_0^{1,p}(\Omega)$ and that the embedding of $W_0^{1,p}(\Omega)$ into $L^{r+1}(\Omega)$ is compact, we infer that I is sequentially weakly lower semicontinuous. Therefore, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$I(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} I(u)$$

(see, for example, [**11**, Theorems 1.1, 1.2]). Hence, u_0 is a critical point of I that reads as

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \nabla v \,\mathrm{d}x + \mu\int_\Omega |\nabla u_0|^{q-2}\nabla u_0\nabla v\,\mathrm{d}x = b\int_\Omega (u_0^+)^r v\,\mathrm{d}x \tag{4.2}$$

for all $v \in W_0^{1,p}(\Omega)$. Applying to (4.2) the regularity up to the boundary in [**9**, Theorem 1] and [**10**, p. 320] shows that $u_0 \in C^{1,\beta}(\bar{\Omega})$ with some $\beta \in (0,1)$.

It remains to justify that $u_0 > 0$. Inserting $v = -u_0^- = -\max\{0, -u_0\}$ into (4.2) leads to $u_0^- = 0$, so $u_0 \geqslant 0$ in Ω. We observe that the condition $0 < r < q-1$ ensures that $I(tu) < 0$, provided $u \neq 0$ and $t > 0$ is sufficiently small, which implies that $u_0 \neq 0$. Finally, as in the proof of Theorem 2.2, we can verify that the strong maximum principle in [**14**, Theorem 5.4.1] applies in the case of equation (4.2). At this point we need to have $0 < r < q-1$ if $\mu > 0$. We conclude that $u_0 > 0$ in Ω, so u_0 is a solution of problem (4.1) and belongs to $C_0^1(\bar{\Omega})$. The case where $\mu = 0$ can be handled in the same way. □

We proceed with the proof of Theorem 1.1. For every $\varepsilon \in (0,1)$, Theorem 3.1 provides a solution $u_\varepsilon \in C_0^1(\bar{\Omega})$ of problem (P_ε). This allows us to obtain

$$\int_\Omega (|\nabla u_\varepsilon|^p + \mu|\nabla u_\varepsilon|^q)\,\mathrm{d}x = \int_\Omega f(x, u_\varepsilon + \varepsilon, \nabla u_\varepsilon)u_\varepsilon\,\mathrm{d}x.$$

Then, through hypothesis (F) and reasoning as in the proof of Theorem 3.1, we derive the estimate

$$\|u_\varepsilon\|^p \leqslant C(\|u_\varepsilon\| + \|u_\varepsilon\|^{r_1+1} + \|u_\varepsilon\|^{r_2+1}),$$

with a constant $C > 0$ that is independent of ε. The latter is the consequence of the fact that we argue with ε in a bounded set, namely $\varepsilon \in (0, 1)$. Since $1, r_1 + 1, r_2 + 1 < p$, we infer that

$$\|u_\varepsilon\| \leqslant C_0 \tag{4.3}$$

for a constant $C_0 > 0$ independent of ε.

In view of (4.3), we can argue as for (3.8), to find a sequence $\varepsilon_n \to 0^+$ such that the corresponding sequence $\{u_n = u_{\varepsilon_n}\}$ is strongly convergent:

$$u_n \to u \text{ in } W_0^{1,p}(\Omega) \quad \text{as } n \to \infty, \tag{4.4}$$

with some $u \in W_0^{1,p}(\Omega)$. Then, from (4.4) and the fact that u_n solves (P_{ε_n}), it is straightforward to infer that u is a solution of the equation

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad\;\; \text{on } \partial\Omega. \end{aligned}$$

The regularity up to the boundary in [**9**, Theorem 1] and [**10**, p. 320] ensures that $u \in C^{1,\beta}(\bar{\Omega})$ with some $\beta \in (0, 1)$. In order to complete the proof, we have to prove that $u > 0$ in Ω. Since at this point we cannot guarantee that u is non-trivial, we develop a comparison argument.

Lemma 4.1 with $b = b_0$ and $r = r_0$ ensures the existence of a solution $\underline{u} \in C_0^1(\bar{\Omega})$ to the problem

$$\left.\begin{aligned} -\Delta_p u - \mu\Delta_q u &= b_0 u^{r_0} \quad \text{in } \Omega, \\ u &> 0 \qquad\quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.5}$$

The positive constants b_0 and r_0 are those in hypothesis (F). In the following, the solution $\underline{u}$ of problem (4.5) will be regarded as a subsolution of (4.5). Let us observe that hypothesis (F) implies that u_ε is a supersolution of problem (4.5) for each $\varepsilon \in (0, 1)$.

Now we apply the comparison principle in Theorem 2.2 to problem (4.5), that is, with the function $g(t) = b_0 t^{r_0}$ for $t > 0$, and by taking the subsolution $u_1 = \underline{u}$ and the supersolution $u_2 = u_\varepsilon$. Complying with assumption (F), if $\mu > 0$, the function $t^{1-q} g(t)$ is decreasing for $t > 0$ because $r_0 < q - 1$, and if $\mu = 0$, the function $t^{1-p} g(t)$ is decreasing for $t > 0$ because $r_0 < p - 1$. We emphasize that in order to apply Theorem 2.2 the information from Theorem 3.1 that $u_\varepsilon > 0$ in Ω is essential.

In order to apply Theorem 2.2 we also need to check that

$$\frac{u_\varepsilon}{\underline{u}}, \frac{\underline{u}}{u_\varepsilon} \in L^\infty(\Omega).$$

To this end it suffices to show that whenever $x \to x_0 \in \partial\Omega$ with $x \in \Omega$, one has

$$\max\left\{\limsup_{x\to x_0} \frac{\underline{u}(x)}{u_\varepsilon(x)}, \limsup_{x\to x_0} \frac{u_\varepsilon(x)}{\underline{u}(x)}\right\} < +\infty. \tag{4.6}$$

The property stated in (4.6) will be established on the basis of the Hopf boundary point lemma in the strong maximum principle for both Dirichlet problems (4.5) and (P_ε) with corresponding solutions $\underline{u}$ and u_ε, which means having

$$\frac{\partial \underline{u}}{\partial \nu}(x_0) < 0, \quad \frac{\partial u_\varepsilon}{\partial \nu}(x_0) < 0 \quad \text{for all } x_0 \in \partial\Omega, \tag{4.7}$$

where ν denotes the exterior normal unit vector to $\partial\Omega$. The Hopf boundary point lemma holds true for problems (4.5) and (P_ε) by virtue of [**14**, Theorem 5.5.1], where all the required conditions are satisfied. In this respect, conditions (5.4.3) and (5.4.4) in [**14**, Theorem 5.5.1] need to be satisfied. In the proof of Theorem 3.1 these conditions have already been shown to be true by arguing with the function $A(t) = t^{p-1} + \mu t^{q-1}$, $t > 0$. At this point, recalling that $u_\varepsilon, \underline{u} \in C^1(\bar{\Omega})$, it is clear from l'Hôpital's theorem and (4.7) that the property in (4.6) is satisfied. Therefore, Theorem 2.2 allows the comparison of the solution $\underline{u}$ (regarded as a subsolution) of (4.5) with the supersolution u_ε of (4.5), implying that

$$u_\varepsilon(x) \geqslant \underline{u}(x) > 0 \quad \text{for all } x \in \Omega \text{ and } \varepsilon \in (0, 1). \tag{4.8}$$

Using (4.4), we can pass to the limit in (4.8) along a sequence $\varepsilon_n \to 0$. This leads to $u(x) \geqslant \underline{u}(x) > 0$ for all $x \in \Omega$, so u is a solution of problem (P). The proof is thus complete.

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