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RESEARCH ARTICLE

Categorical traces and a relative Lefschetz–Verdier formula

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Abstract

We prove a relative Lefschetz–Verdier theorem for locally acyclic objects over a Noetherian base scheme. This is done by studying duals and traces in the symmetric monoidal 2-category of cohomological correspondences. We show that local acyclicity is equivalent to dualisability and deduce that duality preserves local acyclicity. As another application of the category of cohomological correspondences, we show that the nearby cycle functor over a Henselian valuation ring preserves duals, generalising a theorem of Gabber.

Introduction

The notions of dual and trace in symmetric monoidal categories were introduced by Dold and Puppe [DP]. They have been extended to higher categories and have found important applications in algebraic geometry and other contexts (see [BZN] by Ben-Zvi and Nadler and the references therein).

The goal of the present article is to record several applications of the formalism of duals and traces to the symmetric monoidal 2-category of cohomological correspondences in étale cohomology. One of our main results is the following relative Lefschetz–Verdier theorem.

Theorem 0.1. *Let S be a Noetherian scheme and let Λ be a Noetherian commutative ring with $m\Lambda = 0$ for some m invertible on S. Let*

$$
\begin{array}{ccccccccc}
X & \xleftarrow{\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y & \xleftarrow{\overleftarrow{d}} & D & \xrightarrow{\overrightarrow{d}} & X \\
{\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\
X' & \longleftarrow & C' & \longrightarrow & Y' & \longleftarrow & D' & \longrightarrow & X'
\end{array}
$$

be a commutative diagram of schemes separated of finite type over S, with p and $D \to D' \times_{Y'} Y$ proper. Let $L \in D_{cft}(X, \Lambda)$ such that L and $f_! L$ are locally acyclic over S. Let $M \in D(Y, \Lambda)$, $u\colon \overleftarrow{c}^* L \to \overrightarrow{c}^! M$, $v\colon \overleftarrow{d}^* M \to \overrightarrow{d}^! L$. Then $s\colon C \times_{X \times_S Y} D \to C' \times_{X' \times_S Y'} D'$ is proper and

$$
s_* \langle u, v \rangle = \langle (f, p, g)_! u, (g, q, f)_! v \rangle.
$$

Here $D_{cft}(X, \Lambda) \subseteq D(X, \Lambda)$ denotes the full subcategory spanned by objects of finite tor-dimension and of constructible cohomology sheaves, and $\langle u, v \rangle$ is the relative Lefschetz–Verdier pairing.

Remark 0.2. In the case where S is the spectrum of a field, local acyclicity is trivial and the theorem generalises [SGA5, III Corollaire 4.5] and (the scheme case of) [V1, Proposition 1.2.5]. For S smooth over a perfect field and under additional assumptions of smoothness and transversality, Theorem 0.1 was proved by Yang and Zhao [YZ, Corollary 3.10]. The original proof in [SGA5] and its adaptation in [YZ] require the verification of a large amount of commutative diagrams. The categorical interpretation we adopt makes our proof arguably more conceptual.

It was observed by Lurie that Grothendieck's cohomological operations can be encoded by a (pseudo) functor $\mathscr{B} \to \mathscr{C}at$, where $\mathscr{B}$ denotes the category of correspondences and $\mathscr{C}at$ denotes the 2-category of categories. Contrary to the situation of [BZN, Definition 2.15], in the context of étale cohomology, the functor has a *right-lax* symmetric monoidal structure that is not expected to be symmetric monoidal even after enhancement to higher categories. Instead, we apply the formalism of traces to the corresponding cofibred category produced by the Grothendieck construction, which is the category $\mathscr{C}$ of cohomological correspondences. The relative Lefschetz–Verdier formula follows from the functoriality of traces for dualisable objects (X, L) of $\mathscr{C}$.

To complete the proof, we show that under the assumption $L \in D_{\mathrm{cft}}(X, \Lambda)$, dualisability is equivalent to local acyclicity (Theorem 2.16). As a byproduct of this equivalence, we deduce immediately that local acyclicity is preserved by duality (Corollary 2.18). Note that this last statement does not involve cohomological correspondences.

We also give applications to the nearby cycle functor Ψ over a Henselian valuation ring. The functor Ψ extends the usual nearby cycle functor over a Henselian *discrete* valuation ring and was studied by Huber [H, Section 4.2]. By studying specialisation of cohomological correspondences, we generalise Gabber's theorem that Ψ preserves duals and a fixed point theorem of Vidal to Henselian valuation rings (Corollaries 3.8 and 3.13). We hope that the latter can be used to study ramification over higher-dimensional bases.

Scholze remarked that our arguments also apply in the étale cohomology of diamonds and imply the equivalence between dualisability and universal local acyclicity in this situation. This fact and applications are discussed in his work with Fargues on the geometrisation of the Langlands correspondence [FS].

Let us briefly mention some other categorical approaches to Lefschetz type theorems. In [DP, Section 4], the Lefschetz fixed point theorem is deduced from the functoriality of traces by passing to suspension spectra. In [P], a categorical framework is set up for Lefschetz–Lunts type formulas. In May 2019, as a first draft of this article was being written, Varshavsky informed us that he had a different strategy to deduce the Lefschetz–Verdier formula, using categorical traces in $(\infty, 2)$-categories.

This article is organised as follows. In Section 1, we review duals and traces in symmetric monoidal 2-categories and the Grothendieck construction. In Section 2, we define the symmetric monoidal 2-category of cohomological correspondences and prove the relative Lefschetz–Verdier theorem. In Section 3, we discuss applications to the nearby cycle functor over a Henselian valuation ring.

1. Pairings in symmetric monoidal 2-categories

We review duals, traces and pairings in symmetric monoidal 2-categories. We give the definitions in Subsection 1.1 and discuss the functoriality of pairings in Subsection 1.2. These two subsections are mostly standard (see [BZN] and [HSS] for generalisations to higher categories). In Subsection 1.3 we review the Grothendieck construction in the symmetric monoidal context, which will be used to interpret the category of cohomological correspondences later.

By a 2-category, we mean a weak 2-category (also known as a bicategory in the literature).

1.1. Pairings

Let $(\mathscr{C}, \otimes, 1_{\mathscr{C}})$ be a symmetric monoidal 2-category.

Definition 1.1 (dual). An object X of $\mathscr{C}$ is *dualisable* if there exist an object $X^\vee$ of $\mathscr{C}$, called the *dual* of X, and morphisms $\mathrm{ev}_X : X^\vee \otimes X \to 1_\mathscr{C}$, $\mathrm{coev}_X : 1_\mathscr{C} \to X \otimes X^\vee$, called evaluation and coevaluation, respectively, such that the composites

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X, \quad X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{coev}_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}} X^\vee$$

are isomorphic to identities.

Remark 1.2. For X dualisable, $X^\vee$ is dualisable of dual X. For X and Y dualisable, $X \otimes Y$ is dualisable of dual $X^\vee \otimes Y^\vee$.

For X and Y in $\mathscr{C}$, we let $\mathscr{H}om(X, Y)$ denote the internal mapping object if it exists.

Remark 1.3. Assume that X is dualisable of dual $X^\vee$.

(a) The morphisms coev_X and ev_X exhibit $- \otimes X^\vee$ as right (and left) adjoint to $- \otimes X$. Thus, for every object Y, $\mathscr{H}om(X, Y)$ exists and is equivalent to $Y \otimes X^\vee$. In particular, $\mathscr{H}om(X, 1_\mathscr{C})$ exists and is equivalent to $X^\vee$.

(b) If, moreover, $\mathscr{H}om(Y, 1_\mathscr{C})$ exists, then we have equivalences

$$\mathscr{H}om(X \otimes Y, 1_\mathscr{C}) \simeq \mathscr{H}om(X, \mathscr{H}om(Y, 1_\mathscr{C})) \overset{(a)}{\simeq} \mathscr{H}om(Y, 1_\mathscr{C}) \otimes \mathscr{H}om(X, 1_\mathscr{C}),$$
$$\mathscr{H}om(Y, X) \simeq \mathscr{H}om(Y, \mathscr{H}om(X^\vee, 1_\mathscr{C})) \simeq \mathscr{H}om(X^\vee \otimes Y, 1_\mathscr{C})$$
$$\simeq \mathscr{H}om(Y, 1_\mathscr{C}) \otimes \mathscr{H}om(X^\vee, 1_\mathscr{C}) \simeq \mathscr{H}om(Y, 1_\mathscr{C}) \otimes X.$$

Lemma 1.4. *An object X is dualisable if and only if $\mathscr{H}om(X, 1_\mathscr{C})$ and $\mathscr{H}om(X, X)$ exist and the morphism $m \colon X \otimes \mathscr{H}om(X, 1_\mathscr{C}) \to \mathscr{H}om(X, X)$ adjoint to*

$$X \otimes \mathscr{H}om(X, 1_\mathscr{C}) \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

is a split epimorphism. Here $\mathrm{ev}_X \colon \mathscr{H}om(X, 1_\mathscr{C}) \otimes X \to 1_\mathscr{C}$ denotes the co-unit.

Proof. The 'only if' part is a special case of Remark 1.3. For the 'if' part, we define $\mathrm{coev}_X \colon 1_\mathscr{C} \to X \otimes \mathscr{H}om(X, 1_\mathscr{C})$ to be the composite of a section of m and the morphism $1_\mathscr{C} \to \mathscr{H}om(X, X)$ corresponding to id_X. It is easy to see that ev_X and coev_X exhibit $\mathscr{H}om(X, 1_\mathscr{C})$ as a dual of X. □

For X and Y dualisable, the dual of a morphism $u \colon X \to Y$ is the composite

$$u^\vee \colon Y^\vee \xrightarrow{\mathrm{id}_{Y^\vee} \otimes \mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id}_{Y^\vee} \otimes u \otimes \mathrm{id}_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^\vee}} X^\vee.$$

This construction gives rise to a functor $\mathrm{Hom}_\mathscr{C}(X, Y) \to \mathrm{Hom}_\mathscr{C}(Y^\vee, X^\vee)$. We have commutative squares with invertible 2-morphisms

$$\begin{array}{ccc} 1_\mathscr{C} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee & \qquad & X \otimes Y^\vee \xrightarrow{u \otimes \mathrm{id}} Y \otimes Y^\vee \\ {\scriptstyle \mathrm{coev}_Y} \downarrow \qquad \downarrow {\scriptstyle u \otimes \mathrm{id}} & & {\scriptstyle \mathrm{id} \otimes u^\vee} \downarrow \qquad \downarrow {\scriptstyle \mathrm{ev}_Y} \\ Y \otimes Y^\vee \xrightarrow{\mathrm{id} \otimes u^\vee} Y \otimes X^\vee & & X \otimes X^\vee \xrightarrow{\mathrm{ev}_X} 1_\mathscr{C}. \end{array} \qquad (1.1)$$

Moreover, for $X \xrightarrow{u} Y \xrightarrow{v} Z$ with X, Y, Z dualisable, we have $(vu)^\vee \simeq u^\vee v^\vee$.

Notation 1.5. We let $\Omega\mathscr{C}$ denote the category $\mathrm{End}(1_\mathscr{C})$.

Construction 1.6 (dimension, trace and pairing). Let X be a dualisable object of $\mathscr{C}$ and let $e\colon X \to X$ be an endomorphism. We define the trace $\mathrm{tr}(e)$ to be the object of $\Omega\mathscr{C}$ given by the composite

$$1_{\mathscr{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^{\vee} \xrightarrow{e \otimes \mathrm{id}_{X^{\vee}}} X \otimes X^{\vee} \xrightarrow{\mathrm{ev}_X} 1_{\mathscr{C}},$$

where in the last arrow we used the commutativity constraint.

Let $u\colon X \to Y$ and $v\colon Y \to X$ be morphisms with X dualisable. We define the pairing by $\langle u, v \rangle = \mathrm{tr}(v \circ u)$.

We define the *dimension* of a dualisable object X to be $\dim(X) := \langle \mathrm{id}_X, \mathrm{id}_X \rangle$, which is the composite $1_{\mathscr{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^{\vee} \xrightarrow{\mathrm{ev}_X} 1_{\mathscr{C}}$.

If X and Y are both dualisable, then $\langle u, v \rangle$ is isomorphic to the composite

$$1_{\mathscr{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^{\vee} \xrightarrow{u \otimes v^{\vee}} Y \otimes Y^{\vee} \xrightarrow{\mathrm{ev}_Y} 1_{\mathscr{C}}.$$

In this case, we have an isomorphism $\langle u, v \rangle \simeq \langle v, u \rangle$. In fact, by (1.1), we have commutative squares with invertible 2-morphisms
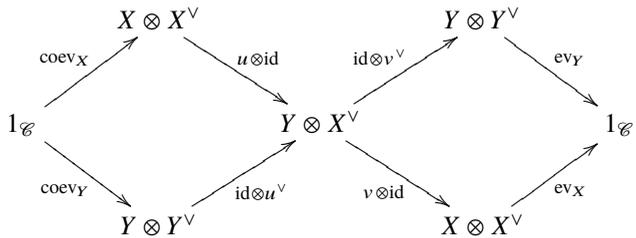
The definition and construction above hold in particular for symmetric monoidal 1-categories. In the next subsection, 2-morphisms will play an important role.

1.2. Functoriality of pairings

A morphism $f\colon X \to X'$ in a 2-category is said to be *right adjointable* if there exist a morphism $f^!\colon X' \to X$, called the right adjoint of f, and 2-morphisms $\eta\colon \mathrm{id}_X \to f^! \circ f$ and $\epsilon\colon f \circ f^! \to \mathrm{id}_{X'}$ such that the composites

$$f \xrightarrow{\mathrm{id} \circ \eta} f \circ f^! \circ f \xrightarrow{\epsilon \circ \mathrm{id}} f, \quad f^! \xrightarrow{\eta \circ \mathrm{id}} f^! \circ f \circ f^! \xrightarrow{\mathrm{id} \circ \epsilon} f^!$$

are identities.

Let $(\mathscr{C}, \otimes, 1_{\mathscr{C}})$ be a symmetric monoidal 2-category.

Construction 1.7. Consider a diagram in $\mathscr{C}$

$$\begin{array}{ccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & X \\
{\scriptstyle f}\downarrow & {\scriptstyle \Swarrow\alpha} & {\scriptstyle g}\downarrow & {\scriptstyle \Swarrow\beta} & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & X'
\end{array} \qquad (1.2)$$

with X and X' dualisable and f right adjointable. We will construct a morphism $\langle u, v \rangle \to \langle u', v' \rangle$ in $\Omega\mathscr{C}$.

In the case where Y and Y' are also dualisable and g is also right adjointable, we define $\langle u, v \rangle \to \langle u', v' \rangle$ by the diagram

$$
\begin{array}{ccccc}
1_{\mathscr{C}} & \xrightarrow{\ \text{coev}_X\ } & X \otimes X^{\vee} & \xrightarrow{\ u \otimes v^{\vee}\ } & Y \otimes Y^{\vee} \\
& {}_{\text{coev}_{X'}}\!\!\!\swarrow & \Big\downarrow{\scriptstyle f \otimes f^{!\vee}}\ \ {}^{\alpha \otimes \beta^{!\vee}}\!\!\!\!\!\!\!\Swarrow & & \Big\downarrow{\scriptstyle \text{ev}_Y} \\
& & X' \otimes X'^{\vee} & \xrightarrow[\ u' \otimes v'^{\vee}\ ]{} & Y' \otimes Y'^{\vee} \xrightarrow{\ \text{ev}_{Y'}\ } 1_{\mathscr{C}}
\end{array}
$$

where $\beta^!$ is the composite

$$
v \circ g^! \xrightarrow{\ \eta_f\ } f^! \circ f \circ v \circ g^! \xrightarrow{\ \text{id} \circ \beta \circ \text{id}\ } f^! \circ v' \circ g \circ g^! \xrightarrow{\ \epsilon_g\ } f^! \circ v'
$$

and the 2-morphisms in the triangles are

$$
(f \otimes f^{!\vee}) \circ \text{coev}_X \simeq ((f \circ f^!) \otimes \text{id}) \circ \text{coev}_{X'} \xrightarrow{\ \epsilon_f\ } \text{coev}_{X'}, \tag{1.3}
$$

$$
\text{ev}_Y \xrightarrow{\ \eta_g\ } \text{ev}_Y \circ ((g^! \circ g) \otimes \text{id}) \simeq \text{ev}_{Y'} \circ (g \otimes g^{!\vee}). \tag{1.4}
$$

In particular, a morphism $\text{tr}(e) \to \text{tr}(e')$ is defined for every diagram in $\mathscr{C}$ of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ e\ } & X \\
{\scriptstyle f}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle f} \\
X' & \xrightarrow[\ e'\ ]{} & X'
\end{array} \tag{1.5}
$$

with X and X' dualisable and f right adjointable.

In general, we define $\langle u, v \rangle \to \langle u', v' \rangle$ as the morphism $\text{tr}(v \circ u) \to \text{tr}(v' \circ u')$ associated to the composite down-square of (1.2).

Trace can be made into a functor $\text{End}(\mathscr{C}) \to \Omega\mathscr{C}$, where $\text{End}(\mathscr{C})$ is a $(2, 1)$-category whose objects are pairs $(X, e \colon X \to X)$ with X dualisable and morphisms are diagrams (1.5) with f right adjointable [HSS, Section 2.1]. Composition in $\text{End}(\mathscr{C})$ is given by vertical composition of diagrams.

For the case of Theorem 0.1 where f is not proper, we will need to relax the adjointability condition in Construction 1.7 as follows. In a 2-category, a *down-square equipped with a splitting* is a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ u\ } & Y \\
{\scriptstyle f}\Big\downarrow & {\scriptstyle\Downarrow}\!\!\nearrow\!\!{}_{w}\!\!{\scriptstyle\Downarrow} & \Big\downarrow{\scriptstyle g} \\
X' & \xrightarrow[\ u'\ ]{} & Y'.
\end{array} \tag{1.6}
$$

Note that the composition of (1.6) with a down-square on the left or on the right is a down-square equipped with a splitting. Moreover, a down-square with one vertical arrow f right adjointable is equipped with a splitting induced by the diagram

$$
\begin{array}{ccc}
X & =\!=\!= & X \\
{\scriptstyle f}\Big\downarrow & {}_{\eta\Downarrow}\!\!\nearrow\!\!{}^{f^!}\!\!\!\!{\scriptstyle\Downarrow\epsilon} & \Big\downarrow{\scriptstyle f} \\
X' & =\!=\!= & X'.
\end{array}
$$

Construction 1.8. Consider a diagram in $\mathscr{C}$

$$
\begin{array}{ccc}
X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } X \\
\end{array}
\tag{1.7}
$$

with X and X' dualisable. We will construct a morphism $\langle u, v \rangle \to \langle u', v' \rangle$ in $\Omega\mathscr{C}$.

In the case where Y is also dualisable, we decompose (1.7) into

and take the composite

$$
\langle u, v \rangle \simeq \langle v, u \rangle \to \langle fv, w \rangle \simeq \langle w, fv \rangle \to \langle u', v' \rangle.
$$

Here the two arrows are given by the case $f = \mathrm{id}$ of Construction 1.7. In particular, a morphism $\mathrm{tr}(e) \to \mathrm{tr}(e')$ is defined for every diagram in $\mathscr{C}$ of the form

$$
\begin{array}{ccc}
X \xrightarrow{\ e\ } X \\
\end{array}
$$

with X and X' dualisable.

In general, we define $\langle u, v \rangle \to \langle u', v' \rangle$ as the morphism $\mathrm{tr}(v \circ u) \to \mathrm{tr}(v' \circ u')$ associated to the horizontal composition of (1.7).

Remark 1.9. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric monoidal 2-categories and let $F \colon \mathscr{C} \to \mathscr{D}$ be a symmetric monoidal functor. Then F preserves duals, pairings and functoriality of pairings.

1.3. The Grothendieck construction

Given a category B and a (pseudo) functor $F \colon B \to \mathscr{C}at$, Grothendieck constructed a category cofibred over B whose strict fibre at an object X of B is $F(X)$ [SGA1, Exposé VI]. We review Grothendieck's construction in the context of symmetric monoidal 2-categories. Our convention on 2-morphisms is made with applications to categorical correspondences in mind.

Let $(\mathscr{B}, \otimes, 1_{\mathscr{B}})$ be a symmetric monoidal 2-category. We consider the symmetric monoidal 2-category $(\mathscr{C}at^{\mathrm{co}}, \times, *)$, where $\mathscr{C}at^{\mathrm{co}}$ denotes the 2-category obtained from the 2-category $\mathscr{C}at$ of categories by reversing the 2-morphisms, $\times$ denotes the strict product and $*$ denotes the category with a unique object and a unique morphism.

Construction 1.10. Let $F \colon (\mathscr{B}, \otimes, 1_{\mathscr{B}}) \to (\mathscr{C}at^{\mathrm{co}}, \times, *)$ be a right-lax symmetric monoidal functor. We have an object e_F of $F(1_{\mathscr{B}})$ and functors $F(X) \times F(X') \xrightarrow{\boxtimes} F(X \otimes X')$ for objects X and X' of $\mathscr{B}$.

Given morphisms $c\colon X \to Y$ and $c'\colon X' \to Y'$ in $\mathscr{B}$, we have a natural transformation

$$
\begin{array}{ccc}
F(X) \times F(X') & \xrightarrow{\boxtimes} & F(X \otimes X') \\
{\scriptstyle F(c) \times F(c')}\downarrow & \overset{F_{c,c'}}{\Swarrow} & \downarrow{\scriptstyle F(c \otimes c')} \\
F(Y) \times F(Y') & \xrightarrow{\boxtimes} & F(Y \otimes Y').
\end{array}
\tag{1.8}
$$

The Grothendieck construction provides a symmetric monoidal 2-category $(\mathscr{C}, \otimes, 1_{\mathscr{C}})$ as follows.

An object of $\mathscr{C} = \mathscr{C}_F$ is a pair (X, L), where $X \in \mathscr{B}$ and $L \in F(X)$. A morphism $(X, L) \to (Y, M)$ in $\mathscr{C}$ is a pair (c, u), where $c\colon X \to Y$ is a morphism in $\mathscr{B}$ and $u\colon F(c)(L) \to M$ is a morphism in $F(Y)$. A 2-morphism $(c, u) \to (d, v)$ is a 2-morphism $p\colon c \to d$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F(c)(L) & \xrightarrow{\;u\;} & M. \\
{\scriptstyle F(p)(L)}\uparrow & \nearrow{\scriptstyle v} & \\
F(d)(L) & &
\end{array}
$$

We take $1_{\mathscr{C}} = (1_{\mathscr{B}}, e_F)$. We put $(X, L) \otimes (X', L') := (X \otimes X', L \boxtimes L')$. For morphisms $(c, u)\colon (X, L) \to (Y, M)$ and $(c', u')\colon (X', L') \to (Y', M')$, we put $(c, u) \otimes (c', u') := (c \otimes c', v)$, where

$$
v\colon F(c \otimes c')(L \boxtimes L') \xrightarrow{F_{c,c'}} F(c)L \boxtimes F(c')L' \xrightarrow{u \boxtimes u'} M \boxtimes M'.
$$

In applications in later sections, $F_{c,c'}$ will be a natural isomorphism.

Given a morphism $f\colon X \to X'$ in $\mathscr{B}$ and an object L of $F(X)$, we write $f_{\natural} = (f, \mathrm{id}_{F(f)L})\colon (X, L) \to (X', F(f)L)$.

Lemma 1.11. *Given a 2-morphism*

$$
\begin{array}{ccc}
X & \xrightarrow{\;c\;} & Y \\
{\scriptstyle f}\downarrow & \Swarrow{\scriptstyle p} & \downarrow{\scriptstyle g} \\
X' & \xrightarrow{\;c'\;} & Y'
\end{array}
\tag{1.9}
$$

in $\mathscr{B}$ and a morphism $(c, u)\colon (X, L) \to (Y, M)$ in $\mathscr{C}$ above c, there exists a unique morphism $(c', u')\colon (X', F(f)L) \to (Y', F(g)M)$ in $\mathscr{C}$ above c' such that p defines a 2-morphism in $\mathscr{C}$:

$$
\begin{array}{ccc}
(X, L) & \xrightarrow{\;(c,u)\;} & (Y, M) \\
{\scriptstyle f_{\natural}}\downarrow & \Swarrow{\scriptstyle p} & \downarrow{\scriptstyle g_{\natural}} \\
(X', F(f)L) & \xrightarrow{\;(c',u')\;} & (Y', F(g)M).
\end{array}
$$

Proof. By definition, u' is the morphism $F(c')F(f)L \simeq F(c'f)L \xrightarrow{F(p)} F(gc)L \simeq F(g)F(c)L \xrightarrow{u} F(g)M$. $\square$

Remark 1.12. Let $f\colon X \to X'$ be a morphism in $\mathscr{B}$ admitting a right adjoint $f^!\colon X' \to X$. Let $\eta\colon \mathrm{id}_X \to f^! \circ f$ and $\epsilon\colon f \circ f^! \to \mathrm{id}_{X'}$ denote the unit and the co-unit. Let L be an object of $F(X)$.

(a) $f_{\natural}\colon (X, L) \to (X', F(f)L)$ admits the right adjoint

$$
f^{\natural} = (f^!, F(\eta)(L))\colon (X', F(f)L) \to (X, L),
$$

with unit and co-unit given by η and ϵ.

(b) Assume that $F_{c,c'}$ is an isomorphism for all c and c', (X, L) is dualisable in $\mathscr{C}$ of dual $(X^\vee, L^\vee)$ and X' is dualisable in $\mathscr{B}$ of dual $X'^\vee$. Then $(X', F(f)(L))$ is dualisable in $\mathscr{C}$ of dual $(X'^\vee, F(f^{!\vee})(L^\vee))$. The coevaluation and evaluation are given by

$$F(\operatorname{coev}_{X'})(e_F) \xrightarrow{F(\bar{\epsilon})} F(f \otimes f^{!\vee})F(\operatorname{coev}_X)(e_F) \xrightarrow{\operatorname{coev}_L} F(f \otimes f^{!\vee})(L \boxtimes L^\vee)$$

$$\xrightarrow{F_{f,f^{!\vee}}} F(f)(L) \boxtimes F(f^{!\vee})(L^\vee),$$

$$F(\operatorname{ev}_{X'})(F(f^{!\vee})(L^\vee) \boxtimes F(f)(L)) \xrightarrow{F^{-1}_{f^{!\vee},f}} F(\operatorname{ev}_{X'})F(f^{!\vee} \otimes f)(L^\vee \boxtimes L)$$

$$\xrightarrow{F(\bar{\eta})} F(\operatorname{ev}_X)(L^\vee \boxtimes L) \xrightarrow{\operatorname{ev}_L} e_F,$$

where $\bar{\epsilon}$ is (1.3), $\bar{\eta}$ is (1.4) (with $g = f$) and coev_L and ev_L denote the second components of $\operatorname{coev}_{(X,L)}$ and $\operatorname{ev}_{(X,L)}$, respectively.

Construction 1.13. Let $F, G \colon (\mathscr{B}, \otimes, 1_{\mathscr{B}}) \to (\mathscr{C}at^{\operatorname{co}}, \times, *)$ be right-lax symmetric monoidal functors. Let $\alpha \colon F \to G$ be a right-lax symmetric monoidal natural transformation, which consists of the following data:

○ for every object X of $\mathscr{B}$, a functor $\alpha_X \colon F(X) \to G(X)$;
○ for every morphism $c \colon X \to Y$, a natural transformation

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(c)} & F(Y) \\
\alpha_X \downarrow & \overset{\alpha_c}{\nearrow} & \downarrow \alpha_Y \\
G(X) & \xrightarrow{G(c)} & G(Y);
\end{array}
$$

○ a morphism $e_\alpha \colon e_G \to \alpha_{1_{\mathscr{B}}}(e_F)$ in $F(1_{\mathscr{B}})$;
○ for objects X and X' of $\mathscr{B}$, a natural transformation

$$
\begin{array}{ccc}
F(X) \times F(X') & \xrightarrow{\boxtimes} & F(X \otimes X') \\
\alpha_X \times \alpha_{X'} \downarrow & \overset{\alpha_{X,X'}}{\nearrow} & \downarrow \alpha_{X \otimes X'} \\
G(X) \times G(X') & \xrightarrow{\boxtimes} & G(X \otimes X');
\end{array}
$$

subject to various compatibilities. We construct a right-lax symmetric monoidal functor $\psi \colon (\mathscr{C}_F, \otimes, 1) \to (\mathscr{C}_G, \otimes, 1)$ as follows.

We take $\psi(X, L) = (X, \alpha_X(L))$ and $\psi(c, u) = (c, \psi u)$, where

$$\psi u \colon G(c)(\alpha_X(L)) \xrightarrow{\alpha_c} \alpha_Y(F(c)L) \xrightarrow{u} \alpha_Y(M)$$

for $(c, u) \colon (X, L) \to (Y, M)$. We let ψ send every 2-morphism p to p. The right-lax symmetric monoidal structure on ψ is given by

$$(\mathrm{id}, e_\alpha)\colon (1_{\mathscr{B}}, e_G) \to (1_{\mathscr{B}}, \alpha_{1_{\mathscr{B}}}(e_F)) = \psi(1_{\mathscr{B}}, e_F),$$

$$\psi(X, L) \otimes \psi(X', L') = (X \otimes X', \alpha_X(L) \boxtimes \alpha_{X'}(L'))$$

$$\xrightarrow{(\mathrm{id}, \alpha_{X,X'})} (X \otimes X', \alpha_{X \otimes X'}(L \boxtimes L')) = \psi((X, L) \otimes (X', L')),$$

$$\psi(X, L) \otimes \psi(X', L') \xrightarrow{(\mathrm{id}, \alpha_{X,X'})} \psi((X, L) \otimes (X', L'))$$

$$\psi(c,u) \otimes \psi(c',u') \Big\downarrow \qquad = \qquad \Big\downarrow \psi((c,u) \otimes (c',u'))$$

$$\psi(Y, M) \otimes \psi(Y', M') \xrightarrow{(\mathrm{id}, \alpha_{Y,Y'})} \psi((Y, M) \otimes (Y', M')).$$

This is a symmetric monoidal structure if e_α and $\alpha_{X,X'}$ are isomorphisms (which will be the case in our applications).

Lemma 1.14. *Consider a 2-morphism (1.9) in $\mathscr{B}$ and a morphism $(c, u)\colon (X, L) \to (Y, M)$ in $\mathscr{C}$ above c. Let $(c', u')\colon (X', F(f)L) \to (Y', G(g)M)$ be the morphism associated to (c, u) and let $(c', (\psi u)')\colon (X', G(f)\alpha_X L) \to (Y', G(g)\alpha_Y M)$ be the morphism associated to $(c, \psi u)$. Then the following square commutes:*

$$\begin{array}{ccc} G(c')G(f)\alpha_X L & \xrightarrow{(\psi u)'} & G(g)\alpha_Y M \\ \alpha_f \Big\downarrow & & \Big\downarrow \alpha_g \\ G(c')\alpha_{X'}F(f)L & \xrightarrow{\psi u'} & \alpha_Y F(g)M. \end{array}$$

Proof. The square decomposes into

$$\begin{array}{ccccc} G(c')G(f)\alpha_X L & \xrightarrow{G(p)} & G(g)G(c)\alpha_X L & & \\ \alpha_f \Big\downarrow & & \Big\downarrow \alpha_c & & \\ G(c')\alpha_{X'}F(f)L & & G(g)\alpha_Y F(c)L & \xrightarrow{u} & G(g)\alpha_Y M \\ \alpha_{c'} \Big\downarrow & & \Big\downarrow \alpha_g & & \Big\downarrow \alpha_g \\ \alpha_{Y'}F(c')F(f)L & \xrightarrow{F(p)} & \alpha_{Y'}F(g)F(c)L & \xrightarrow{u} & \alpha_Y F(g)M \end{array}$$

where the inner cells commute. □

Construction 1.15. Let $(\mathscr{B}, \otimes, 1_{\mathscr{B}}) \xrightarrow{H} (\mathscr{B}', \otimes, 1_{\mathscr{B}'}) \xrightarrow{G} (\mathscr{C}at^{\mathrm{co}}, \times, *)$ be right-lax symmetric monoidal functors. Then we have an obvious right-lax symmetric monoidal functor $\mathscr{C}_{GH} \to \mathscr{C}_G$ sending (X, L) to (HX, L), (c, u) to (Hc, u) and every 2-morphism p to Hp. This is a symmetric monoidal functor if H is.

Construction 1.16. Let

$$\begin{array}{ccc} & (\mathscr{B}, \otimes, 1_{\mathscr{B}}) & \\ H \Big\downarrow & \overset{\textstyle F}{\underset{\alpha}{\searrow}} & \\ (\mathscr{B}', \otimes, 1_{\mathscr{B}'}) & \xrightarrow{G} & (\mathscr{C}at^{\mathrm{co}}, \times, *) \end{array}$$

be a diagram of right-lax symmetric monoidal functors and right-lax symmetric monoidal transformation. Combining the two preceding constructions, we obtain right-lax symmetric monoidal functors $\mathscr{C}_F \to \mathscr{C}_{GH} \to \mathscr{C}_G$.

2. A relative Lefschetz–Verdier formula

We apply the formalism of duals and pairings to the symmetric monoidal 2-category of cohomological correspondences, which we define in Subsection 2.2. We prove relative Künneth formulas in Subsection 2.1 and use them to show the equivalence of dualisability and local acyclicity (Theorem 2.16) in Subsection 2.3. We prove the relative Lefschetz–Verdier theorem for dualisable objects (Theorem 2.21) in Subsection 2.4. Together, the two theorems imply Theorem 0.1. In Subsection 2.5, we prove that base change preserves duals (Proposition 2.26).

We will often drop the letters L and R from the notation of derived functors.

2.1. Relative Künneth formulas

We extend some Künneth formulas over fields [SGA5, III 1.6, Proposition 1.7.4, (3.1.1)] to Noetherian base schemes under the assumption of universal local acyclicity. Some special cases over a smooth scheme over a perfect field were previously known [YZ, Corollary 3.3, Proposition 3.5].

Let S be a coherent scheme and let Λ be a torsion commutative ring. Let X be a scheme over S. We let $D(X, \Lambda)$ denote the unbounded derived category of the category of étale sheaves of Λ-modules on X. Following [D, Th. finitude, Définition 2.12], we say that $L \in D(X, \Lambda)$ is *locally acyclic* over S if the canonical map $L_x \to R\Gamma(X_{(x)t}, L)$ is an isomorphism for every geometric point $x \to X$ and every algebraic geometric point $t \to S_{(x)}$. Here $X_{(x)t} := X_{(x)} \times_{S_{(x)}} t$ denotes the Milnor fibre. For X of finite type over S, local acyclicity coincides with strong local acyclicity [LZ, Lemma 4.7].

Notation 2.1. For $a_X \colon X \to S$ separated of finite type, we write $K_{X/S} = a_X^! \Lambda_S$ and $D_{X/S} = R\mathscr{H}om(-, K_X)$. Note that $K_{S/S} = \Lambda_S$ is in general not an (absolute) dualising complex.

Assume in the rest of Subsection 2.1 that S and Λ are Noetherian. We let $D_{\mathrm{ft}}(X, \Lambda)$ denote the full subcategory of $D(X, \Lambda)$ consisting of complexes of finite tor-amplitude.

Proposition 2.2. *Let X', X, Y be schemes of finite type over S and let $f \colon X \to X'$ be a morphism over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D^+(X, \Lambda)$. Then the canonical morphism $f_*L \boxtimes_S M \to (f \times_S \mathrm{id}_Y)_*(L \boxtimes_S M)$ is an isomorphism.*

This follows from [F, Theorem 7.6.9]. We recall the proof for completeness.

Proof. By cohomological descent for a Zariski open cover, we may assume f separated. By Nagata compactification, we are reduced to two cases: either f is proper, in which case we apply proper base change, or f is an open immersion, in which case we apply [D, Th. finitude, App., Proposition 2.10] (with $i = \mathrm{id}_{X'}$). $\qquad\square$

In the rest of Subsection 2.1, assume that $m\Lambda = 0$ for some integer m invertible on S.

Proposition 2.3. *Let X', X, Y be schemes of finite type over S and let $f \colon X \to X'$ be a separated morphism over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D^+(X', \Lambda)$. Then the canonical morphism $f^! L \boxtimes_S M \to (f \times_S \mathrm{id}_Y)^!(L \boxtimes_S M)$ is an isomorphism.*

The morphism is adjoint to

$$(f \times_S \mathrm{id}_Y)_!(f^! L \boxtimes_S M) \simeq f_! f^! L \boxtimes_S M \xrightarrow{\mathrm{adj}\boxtimes_S \mathrm{id}_M} L \boxtimes_S M,$$

where $\mathrm{adj} \colon f_! f^! L \to L$ denotes the adjunction.

Proof. We may assume that f is smooth or a closed immersion. For f smooth of dimension d, $f^*(d)[2d] \simeq f^!$ and the assertion is clear. Assume that f is a closed immersion and let j be the complementary open immersion. Let $f_Y = f \times_S \mathrm{id}_Y$ and $j_Y = j \times_S \mathrm{id}_Y$. Then we have a morphism of distinguished triangles

$$f^!L \boxtimes_S M \longrightarrow f^*L \boxtimes_S M \longrightarrow f^*j_*j^*L \boxtimes_S M \longrightarrow$$

$$\alpha \downarrow \qquad\qquad \simeq \downarrow \qquad\qquad \beta \downarrow$$

$$f^!_Y(L \boxtimes_S M) \longrightarrow f^*_Y(L \boxtimes_S M) \longrightarrow f^*_Y j_{Y*} j^*_Y(L \boxtimes_S M) \longrightarrow,$$

where β is an isomorphism by Proposition 2.2. It follows that α is an isomorphism. □

The following is a variant of [S, Corollary 8.10] and [LZ, Theorem 6.8]. Here we do not require smoothness or regularity.

Corollary 2.4. *Let X and Y be schemes of finite type over S, with X separated over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S. Then the canonical morphism $K_{X/S} \boxtimes_S M \to p^!_Y M$ is an isomorphism, where $p_Y : X \times_S Y \to Y$ is the projection.*

Proof. This is Proposition 2.3 applied to $X' = S$ and $L = \Lambda_S$. □

Proposition 2.5. *Let X and Y be schemes of finite type over S, with X separated over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D^-_c(X, \Lambda)$. Then the canonical morphism $D_{X/S}L \boxtimes_S M \to R\mathcal{H}om(p^*_X L, p^!_Y M)$ is an isomorphism. Here $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ are the projections.*

The morphism is adjoint to $(D_{X/S}L \otimes L) \boxtimes_S M \to K_{X/S} \boxtimes_S M \to p^!_Y M$.

Proof. By [SGA4, IX Proposition 2.7], we may assume $L = j_! \Lambda$ for $j: U \to X$ étale with U affine. Then the morphism can be identified with

$$j_* D_{U/S} \Lambda_U \boxtimes_S M \to j_{Y*}(D_{U/S}\Lambda_U \boxtimes M) \to j_{Y*} R\mathcal{H}om(\Lambda_{U \times_S Y}, j^!_Y p^!_Y M)$$

$$\simeq R\mathcal{H}om(j_{Y!}\Lambda_{U \times_S Y}, p^!_Y M),$$

where $j_Y = j \times_S \mathrm{id}_Y : U \times_S Y \to X \times_S Y$. The first arrow is an isomorphism by Proposition 2.2. The second arrow is an isomorphism by Corollary 2.4. □

2.2. The category of cohomological correspondences

Let S be a coherent scheme and let Λ be a torsion commutative ring.

Construction 2.6. We define the 2-category of cohomological correspondences $\mathscr{C} = \mathscr{C}_{S,\Lambda}$ as follows. An object of $\mathscr{C}$ is a pair (X, L), where X is a scheme separated of finite type over S and $L \in D(X, \Lambda)$. A correspondence over S is a pair of morphisms $X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y$ of schemes over S, where X, Y and C are separated and of finite type over S. A morphism $(X, L) \to (Y, M)$ in $\mathscr{C}$ is a cohomological correspondence over S, namely, a pair (c, u), where $c = (\overleftarrow{c}, \overrightarrow{c})$ is a correspondence over S and $u: \overleftarrow{c}^* L \to \overrightarrow{c}^! M$ is a morphism in $D(C, \Lambda)$. Given cohomological correspondences $(X, L) \xrightarrow{(c,u)} (Y, M) \xrightarrow{(d,v)} (Z, N)$, the composite is (e, w), where e is the composite correspondence given by the diagram

$$\begin{array}{c}(2.1)\end{array}$$

and w is given by the composite

$$\overleftarrow{d}' * \overleftarrow{c}^* L \xrightarrow{u} \overleftarrow{d}' * \overrightarrow{c}^! M \xrightarrow{\alpha} \overrightarrow{c}'! \overleftarrow{d}^* M \xrightarrow{v} \overrightarrow{c}'! \overrightarrow{d}^! N,$$

where α is adjoint to the base change isomorphism $\overrightarrow{c}'_! \overleftarrow{d}'_* \simeq \overleftarrow{d}^* \overrightarrow{c}_!$. Given (c, u) and (d, v) from (X, L) to (Y, M), a 2-morphism $(c, u) \to (d, v)$ is a *proper* morphism of schemes $p \colon C \to D$ satisfying $\overleftarrow{d} p = \overleftarrow{c}$ and $\overrightarrow{d} p = \overrightarrow{c}$ and such that v is equal to

$$\overleftarrow{d}^* L \xrightarrow{\mathrm{adj}} p_* p^* \overleftarrow{d}^* L \simeq p_! \overleftarrow{c}^* L \xrightarrow{u} p_! \overrightarrow{c}^! M \simeq p_! p^! \overrightarrow{d}^! M \xrightarrow{\mathrm{adj}} \overrightarrow{d}^! M.$$

Here we used the canonical isomorphism $p_! \simeq p_*$. Composition of 2-morphisms is given by composition of morphisms of schemes.

The 2-category admits a symmetric monoidal structure. We put

$$(X, L) \otimes (X', L') := (X \times_S X', L \boxtimes_S L').$$

Given $(c, u) \colon (X, L) \to (Y, M)$ and $(c', u') \colon (X', L') \to (Y', M')$, we define $(c, u) \otimes (c', u')$ to be (d, v), where $d = (\overleftarrow{c} \times_S \overleftarrow{c'}, \overrightarrow{c} \times_S \overrightarrow{c'})$ and v is the composite

$$\overleftarrow{d}^* (L \boxtimes_S L') \simeq \overleftarrow{c}^* L \boxtimes_S \overleftarrow{c'}^* L' \xrightarrow{u \boxtimes_S u'} \overrightarrow{c}^! M \boxtimes_S \overrightarrow{c'}^! M' \xrightarrow{\alpha} \overrightarrow{d}^! (M \boxtimes_S M'),$$

where α is adjoint to the Künneth formula $\overrightarrow{d}_! (- \boxtimes_S -) \simeq \overrightarrow{c}_! - \boxtimes_S \overrightarrow{c'}_! -$. Tensor product of 2-morphisms is given by product of morphisms of schemes over S. The monoidal unit of $\mathscr{C}$ is (S, Λ_S).

Remark 2.7. Let $\mathscr{B}_S$ be the symmetric monoidal 2-category of correspondences obtained by omitting L from the above construction. The symmetric monoidal structure on $\mathscr{B}_S$ is given by fibre product of schemes over S (which is not the product in $\mathscr{B}_S$ for S nonempty). Consider the functor $F \colon \mathscr{B}_S \to \mathscr{C}at^{\mathrm{co}}$ carrying X to $D(X, \Lambda)$ and $c = (\overleftarrow{c}, \overrightarrow{c})$ to $\overrightarrow{c}_! \overleftarrow{c}^*$ and a 2-morphism $p \colon c \to d$ to the natural transformation $\overrightarrow{d}_! \overleftarrow{d}^* \xrightarrow{\mathrm{adj}} \overrightarrow{d}_! p_* p^* \overleftarrow{d}^* \simeq \overrightarrow{c}_! \overleftarrow{c}^*$. The compatibility of F with composition (2.1) is given by the base change isomorphism $\overleftarrow{d}^* \overrightarrow{c}_! \simeq \overrightarrow{c}'_! \overleftarrow{d}'_*$. The functor F admits a right-lax symmetric monoidal structure given by $e_F = \Lambda_S$ and $\boxtimes_S$, with Künneth formula for !-pushforward providing a natural isomorphism $F_{c,c'}$ (1.8). The Grothendieck construction (Construction 1.10) then produces $\mathscr{C}_{S,\Lambda}$.

The category $\Omega\mathscr{C}$ consists of pairs (X, α), where X is a scheme separated of finite type over S and $\alpha \in H^0(X, K_{X/S})$. A morphism $(X, \alpha) \to (Y, \beta)$ is a proper morphism $X \to Y$ of schemes over S such that $\beta = p_* \alpha$, where

$$p_* \colon H^0(X, K_{X/S}) \to H^0(Y, K_{Y/S}) \tag{2.2}$$

is given by adjunction $p_* p^! \simeq p_! p^! \to \mathrm{id}$.

Lemma 2.8. *The symmetric monoidal structure $\otimes$ on $\mathscr{C}$ is closed, with internal mapping object $\mathscr{H}om((X, L), (Y, M)) = (X \times_S Y, R\mathscr{H}om(p_X^* L, p_Y^! M))$.*

Proof. We construct an isomorphism of categories

$$F \colon \mathrm{Hom}((X, L) \otimes (Y, M), (Z, N)) \simeq \mathrm{Hom}((X, L), \mathscr{H}om((Y, M), (Z, N)))$$

as follows. An object of the source (respectively target) is a pair $(C \xrightarrow{c} X \times_S Y \times_S Z, u)$, where u belongs to $H^0(C, c^! -)$ applied to the left-hand (respectively right-hand) side of the isomorphism

$$\alpha \colon R\mathscr{H}om(p_X^* L \otimes p_Y^* M, p_Z^! N) \simeq R\mathscr{H}om(p_X^* L, R\mathscr{H}om(p_Y^* M, p_Z^! N)).$$

Here p_X, p_Y, p_Z denote the projections from $X \times_S Y \times_S Z$. We define F by $F(c, u) = (c, u')$, where u' is the image of u under the map induced by α and $F(p) = p$ for every morphism p in the source of F. □

For an object (X, L) of $\mathscr{C}$ and a morphism $f \colon X \to X'$ of schemes separated of finite type over S, we let

$$f_\natural = (\mathrm{id}_X, f)_\natural = ((\mathrm{id}_X, f), L \xrightarrow{\mathrm{adj}} f^! f_! L) \colon (X, L) \to (X', f_! L).$$

Lemma 2.9. *Let (X, L) be an object of $\mathscr{C}$ and let $f \colon X \to X'$ be a proper morphism of schemes separated of finite type over S. Then $f_\natural \colon (X, L) \to (X', f_* L)$ admits the right adjoint*

$$f^\natural = ((f, \mathrm{id}_X), f^* f_* L \xrightarrow{\mathrm{adj}} L) \colon (X', f_* L) \to (X, L).$$

Proof. The co-unit $f_\natural f^\natural \to \mathrm{id}_{(X', f_* L)}$ is given by f and the unit $\mathrm{id}_{(X, L)} \to f^\natural f_\natural$ is given by the diagonal $X \to X \times_{X'} X$. (This is an example of Remark 1.12 (a).) □

Construction 2.10 (!-pushforward). Consider a commutative diagram of schemes separated of finite type over S

$$\begin{array}{ccccc}
X & \xleftarrow{\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\
X' & \xleftarrow{\overleftarrow{c'}} & C' & \xrightarrow{\overrightarrow{c'}} & Y'
\end{array} \tag{2.3}$$

such that $q \colon C \to X \times_{X'} C'$ is proper. Let $(c, u) \colon (X, L) \to (Y, M)$ be a cohomological correspondence above c. Let $p^\sharp = (f, p, g)$. By Lemma 1.11, we have a unique cohomological correspondence $(c', p^\sharp_! u) \colon (X', f_! L') \to (Y', g_! M')$ above c' such that q defines a 2-morphism in $\mathscr{C}$:

$$\begin{array}{ccc}
(X, L) & \xrightarrow{(c, u)} & (Y, M) \\
\downarrow{\scriptstyle f_\natural} & \Swarrow{\scriptstyle q} & \downarrow{\scriptstyle g_\natural} \\
(X', f_! L) & \xrightarrow[(c', p^\sharp_! u)]{} & (Y', g_! M).
\end{array}$$

For a more explicit construction of $p^\sharp_! u$, see [Z, Construction 7.16]. We will often be interested in the case where f, g and p are proper. In this case, we write $p^\sharp_* u$ for $p^\sharp_! u$.

This construction is compatible with horizontal and vertical compositions.

2.3. Dualisable objects

Let S and Λ be as in Subsection 2.2. Next we study dualisable objects of $\mathscr{C} = \mathscr{C}_{S,\Lambda}$.

Proposition 2.11. *Let (X, L) be a dualisable object of $\mathscr{C}$.*

(a) *The dual of (X, L) is $(X, D_{X/S} L)$ and the biduality morphism $L \to D_{X/S} D_{X/S} L$ is an isomorphism. Moreover, for any object (Y, M) of $\mathscr{C}$, the canonical morphisms*

$$D_{X/S} L \boxtimes_S M \to R\mathscr{H}om(p_X^* L, p_Y^! M), \tag{2.4}$$
$$L \boxtimes_S D_{Y/S} M \to R\mathscr{H}om(p_Y^* M, p_X^! L),$$
$$D_{X/S} L \boxtimes_S D_{Y/S} M \to D_{X \times_S Y/S}(L \boxtimes_S M)$$

are isomorphisms. Here $p_X \colon X \times_X Y \to X$ and $p_Y \colon X \times_S Y \to Y$ are the projections.

(b) *For every morphism of schemes* $g\colon Y \to Y'$ *separated of finite type over* S *and all* $M \in D(Y,\Lambda)$, $M' \in D(Y',\Lambda)$, *the canonical morphisms*

$$L \boxtimes_S g_*M \to (\mathrm{id}_X \times_S g)_*(L \boxtimes_S M),$$
$$L \boxtimes_S g^!M' \to (\mathrm{id}_X \times_S g)^!(L \boxtimes_S M')$$

are isomorphisms. Moreover, for morphisms of schemes $f\colon X \to X'$ *and* $f'\colon X'' \to X$ *separated of finite type over* S *such that* $(X', f_!D_{X/S}L)$ *and* $(X'', f'*D_{X/S}L)$ *are dualisable and* $M \in D(Y,\Lambda)$, *the canonical morphisms*

$$f_*L \boxtimes_S M \to (f \times_S \mathrm{id}_Y)_*(L \boxtimes_S M),$$
$$f'^!L \boxtimes_S M \to (f' \times_S \mathrm{id}_Y)^!(L \boxtimes_S M)$$

are isomorphisms.

(c) *If* $L \in D^+(X,\Lambda)$, *then* L *is locally acyclic over* S.

(d) *If* $R\Delta^!$ *commutes with small direct sums and* U *has finite* Λ-*cohomological dimension for every affine scheme* U *étale over* X, *then* L *is c-perfect. Here* $\Delta\colon X \to X \times_S X$ *is the diagonal.*

Following [ILO, XVII Définition 7.7.1] we say $L \in D(X,\Lambda)$ is *c-perfect* if there exists a finite stratification (X_i) of X by constructible subschemes such that for each i, $L|_{X_i} \in D(X_i,\Lambda)$ is locally constant of perfect values. For Λ Noetherian, 'c-perfect' is equivalent to '$\in D_{\mathrm{cft}}$'.

The condition that $R\Delta^!$ commutes with small direct sums is satisfied if

(∗) S is Noetherian finite-dimensional and $m\Lambda = 0$ with m invertible on S,

by Lemma 2.13 and [ILO, XVIII$_A$ Corollary 1.4]. Moreover, the proof below shows that the assumption $L \in D^+(X,\Lambda)$ in (c) can be removed under condition (∗).

Proof. (a) follows from Remarks 1.2, 1.3 and the identification of internal mapping objects (Lemma 2.8). Via biduality and (2.4), the morphisms in (b) can be identified with the isomorphisms

$$R\mathscr{H}om(p_X'^*L^\vee, p_{Y'}^!g_*M) \simeq R\mathscr{H}om(p_X'^*L^\vee, g_{X*}p_Y^!M) \simeq g_{X*}R\mathscr{H}om(p_X^*L^\vee, p_Y^!M),$$
$$R\mathscr{H}om(p_X^*L^\vee, p_Y^!g^!M') \simeq R\mathscr{H}om(g_X^*p_X'^*L^\vee, g_X^!p_{Y'}^!M') \simeq g_X^!R\mathscr{H}om(p_X'^*L^\vee, p_{Y'}^!M'),$$
$$R\mathscr{H}om(p_{X'}^*, f_!L^\vee, p_Y'^!M) \simeq R\mathscr{H}om(f_{Y!}p_X^*L^\vee, p_Y'^!M) \simeq f_{Y*}R\mathscr{H}om(p_X^*L^\vee, p_Y^!M),$$
$$R\mathscr{H}om(p_{X''}^*f'^*L^\vee, p_Y''^!M') \simeq R\mathscr{H}om(f_Y'^*p_X^*L^\vee, f_Y'^!p_Y^!M) \simeq f_Y'^!R\mathscr{H}om(p_X^*L^\vee, p_Y^!M),$$

where $L^\vee = D_{X/S}L$, $g_X = \mathrm{id}_X \times_S g$, $f_Y = f \times_S \mathrm{id}_Y$, $f_Y' = f' \times_S \mathrm{id}_Y$ and $p_X'\colon X \times_S Y' \to X$, $p_X'\colon X' \times_S Y \to X'$, $p_Y''\colon X'' \times_S Y \to X''$ are the projection. (c) follows from the first isomorphism in (b) and Lemma 2.12. For (d), note that for $M \in D(X,\Lambda)$, $\mathrm{Hom}(\Lambda_X, \Delta^!(D_{X/S}L \boxtimes_S M)) \simeq \mathrm{Hom}(L, M)$ by (2.4). Since $\Delta^!$ commutes with small direct sums and Λ_X is a compact object of $D(X,\Lambda)$, it follows that L is a compact object, which is equivalent to being c-perfect by [BS, Proposition 6.4.8]. □

The following is a variant of [F, Theorem 7.6.9] and [S, Proposition 8.11].

Lemma 2.12. *Let* $X \to S$ *be a morphism of coherent schemes and let* $L \in D(X,\Lambda)$. *Assume that for every quasi-finite morphism* $g\colon Y \to Y'$ *of affine schemes with* Y' *étale over* S, *the canonical morphism* $L \boxtimes_S g_*\Lambda_Y \to (\mathrm{id}_X \times_S g)_*(L \boxtimes_S \Lambda_Y)$ *is an isomorphism. Assume either* $L \in D^+(X,\Lambda)$ *or that* $(\mathrm{id}_X \times_S g)_*$ *has bounded* Λ-*cohomological dimension. Then* L *is locally acyclic over* S.

Proof. Let $s \to S$ be a geometric point and let $g \colon t \to S_{(s)}$ be an algebraic geometric point. Consider the diagram

$$
\begin{array}{ccccc}
X_t & \xrightarrow{g_X} & X_{(s)} & \xleftarrow{i_X} & X_s \\
\downarrow & & \downarrow & & \downarrow \\
t & \xrightarrow{g} & S_{(s)} & \xleftarrow{i} & s
\end{array}
$$

obtained by base change. By the assumption and passing to the limit, the morphism $L|_{X_s} \to i_X^* g_{X*}(L|_{X_t})$ can be identified with $L \boxtimes_S -$ applied to $\Lambda_s \to i^* g_* \Lambda_t$, which is an isomorphism. $\qquad\square$

Lemma 2.13. *Let $i \colon Y \to X$ be a closed immersion of finite presentation. Assume that $i^!$ has finite Λ-cohomological dimension; then $Ri^!$ commutes with small direct sums.*

Proof. Let j be the complementary open immersion. It suffices to show that Rj_* commutes with small direct sums under the condition that j_* has finite Λ-cohomological dimension. This is standard. See, for example, [LZ, Lemma 1.10]. $\qquad\square$

Lemma 2.14. *An object (X, L) of $\mathscr{C}$ is dualisable if and only if the canonical morphism $L \boxtimes_S D_{X/S} L \to R\mathscr{H}om(p_2^* L, p_1^! L)$ is an isomorphism. Here p_1 and p_2 are the projections $X \times_S X \to X$.*

Proof. The 'only if' part is a special case of Proposition 2.11 (a). The 'if' part follows from Lemma 1.4 and the identification of the internal mapping objects (Lemma 2.8). $\qquad\square$

Remark 2.15. The evaluation and coevaluation maps for a dualisable object (X, L) of $\mathscr{C}$ can be given explicitly as follows. The evaluation map $(X \times_S X, D_{X/S} L \boxtimes_S L) \to (S, \Lambda)$ is given by $X \times_S X \xleftarrow{\Delta} X \to S$ and the usual evaluation map

$$
\Delta^*(D_{X/S} L \boxtimes_S L) \simeq D_{X/S} L \otimes L \to K_{X/S},
$$

where Δ denotes the diagonal. The coevaluation map $(S, \Lambda) \to (X \times_S X, L \boxtimes_S D_{X/S} L)$ is given by $S \leftarrow X \xrightarrow{\Delta} X \times_S X$ and id_L considered as a morphism

$$
\Lambda_X \to R\mathscr{H}om(L, L) \simeq \Delta^! R\mathscr{H}om(p_2^* L, p_1^! L) \simeq \Delta^!(L \boxtimes_S D_{X/S} L).
$$

We can identify dualisable objects of $\mathscr{C}$ under mild assumptions.

Theorem 2.16. *Let S be a Noetherian scheme and Λ a Noetherian commutative ring with $m\Lambda = 0$ for m invertible on S. Let X be a scheme separated of finite type over S, $L \in D_{c\mathrm{ft}}(X, \Lambda)$. Then (X, L) is a dualisable object of $\mathscr{C}$ if and only if L is locally acyclic over S. In this case, the dual of (X, L) is $(X, D_{X/S} L)$.*

We will use Gabber's theorem that for X of finite type over S, $L \in D_c^b(X, \Lambda)$ is locally acyclic if and only if it is universally locally acyclic [LZ, Corollary 6.6].

Proof. We have already seen the last assertion and the 'only if' part of the first assertion in Parts (a) and (c) of Proposition 2.11. The 'if' part of the first assertion follows from Lemma 2.14, Proposition 2.5 and Gabber's theorem. $\qquad\square$

Remark 2.17. Without invoking Gabber's theorem, our proof and Proposition 2.26 show that for $L \in D_{c\mathrm{ft}}(X, L)$, (X, L) is dualisable if and only if L is universally locally acyclic over S.

Corollary 2.18. *For S, Λ and X as in Theorem 2.16 and $L \in D_{cft}(X, \Lambda)$ locally acyclic over S, $D_{X/S}L$ is locally acyclic over S.*

This was known under the additional assumption that S is regular (and excellent) [LZ, Corollary 5.13] (see also [BG, Section B.6 2)] for S smooth over a field). Our proof here is different from the one in [LZ]. In fact, without invoking Gabber's theorem, our proof here shows that $D_{X/S}$ preserves universal local acyclicity and makes no use of oriented topoi.

Proof. By Theorem 2.16 and Remark 1.2, $(X, D_{X/S}L)$ is dualisable. We conclude by Proposition 2.11 (c). □

Corollary 2.19. *Let S be an Artinian scheme, Λ and X as in Theorem 2.16, and $L \in D(X, \Lambda)$. Then (X, L) is a dualisable object of $\mathscr{C}$ if and only if $L \in D_{cft}(X, \Lambda)$.*

Proof. For $L \in D_{cft}(X, \Lambda)$, L is locally acyclic over S by [D, Th. finitude, Corollaire 2.16] and thus (X, L) is dualisable by the theorem. (Alternatively, one can apply Lemma 2.14 and [SGA5, III Formule (3.1.1)].) For the converse, we may assume that S is the spectrum of a separably closed field by Proposition 2.26. In this case, Proposition 2.11 (d) applies. □

2.4. The relative Lefschetz–Verdier pairing

Let S be a coherent scheme and Λ a torsion commutative ring.

Notation 2.20. For objects (X, L) and (Y, M) of $\mathscr{C}$ with (X, L) dualisable and morphisms $(c, u) \colon (X, L) \to (Y, M)$ and $(d, v) \colon (Y, M) \to (X, L)$, we write the pairing $\langle (c, u), (d, v) \rangle \in \Omega\mathscr{C}$ in Construction 1.6 as $(F, \langle u, v \rangle)$, where $F = C \times_{X \times_S Y} D$. We call $\langle u, v \rangle \in H^0(F, K_{F/S})$ the relative Lefschetz–Verdier pairing. The pairing is symmetric: $\langle u, v \rangle$ can be identified with $\langle v, u \rangle$ via the canonical isomorphism $\langle c, d \rangle \simeq \langle d, c \rangle$.

For an endomorphism (e, w) of a dualisable object (X, L) of $\mathscr{C}$, we write $\mathrm{tr}(e, w) = (X^e, \mathrm{tr}(w))$, where $X^e = E \times_{e, X \times_S X, \Delta} X$ and $\mathrm{tr}(w) = \langle w, \mathrm{id}_L \rangle \in H^0(X^e, K_{X^e/S})$. We define the *characteristic class* $\mathrm{cc}_{X/S}(L)$ to be $\mathrm{tr}(\mathrm{id}_L) = \langle \mathrm{id}_L, \mathrm{id}_L \rangle \in H^0(X, K_{X/S})$. In other words, $\dim(X, L) = (X, \mathrm{cc}_{X/S}(L))$.

Theorem 2.21 (Relative Lefschetz–Verdier). *Let*

$$
\begin{array}{ccccccccc}
X & \xleftarrow{\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y & \xleftarrow{\overleftarrow{d}} & D & \xrightarrow{\overrightarrow{d}} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\
X' & \xleftarrow{\overleftarrow{c'}} & C' & \xrightarrow{\overrightarrow{c'}} & Y' & \xleftarrow{\overleftarrow{d'}} & D' & \xrightarrow{\overrightarrow{d'}} & X'
\end{array}
\tag{2.5}
$$

be a commutative diagram of schemes separated of finite type over S, with p and $D \to D' \times_{Y'} Y$ proper. Let $L \in D(X, \Lambda)$ such that (X, L) and $(X', f_!L)$ are dualisable objects of $\mathscr{C}$. Let $M \in D(Y, \Lambda)$, $u \colon \overleftarrow{c}^*L \to \overrightarrow{c}^!M$, $v \colon \overleftarrow{d}^*M \to \overrightarrow{d}^!L$. Then $s \colon C \times_{X \times_S Y} D \to C' \times_{X' \times_S Y'} D'$ is proper and

$$
s_* \langle u, v \rangle = \langle p_!^\sharp u, q_!^\sharp v \rangle.
$$

Combining this with Theorem 2.16, we obtain Theorem 0.1.

Proof. By Construction 2.10 applied to the right half of (2.5) and to the decomposition (which was used in the proof of [Z, Proposition 8.11])

$$
\begin{array}{ccccc}
X & \xleftarrow{\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y \\
\downarrow{\scriptstyle f} & & \| & & \| \\
X' & \xleftarrow{f\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y \\
\| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\
X' & \xleftarrow{\overleftarrow{c'}} & C' & \xrightarrow{\overrightarrow{c'}} & Y'
\end{array}
$$

of the left half of (2.5), we get a diagram in $\mathscr{C}$

$$
\begin{array}{ccc}
(X, L) & \xrightarrow{(c,u)} (Y, M) \xrightarrow{(d,v)} & (X, L) \\
\downarrow{\scriptstyle f_\natural} \quad \underset{(e,w)}{\overset{\Downarrow}{\nearrow}} \quad \downarrow{\scriptstyle g_\natural} \quad \Downarrow & & \downarrow{\scriptstyle f_\natural} \\
(X', f_! L) \xrightarrow[(c', p_!^\sharp u)]{\Downarrow} & (Y', g_! M) \xrightarrow{(d', q_!^\sharp v)} & (X', f_! L)
\end{array}
$$

where $e = (f\overleftarrow{c}, \overrightarrow{c})$ and $w = (f, \mathrm{id}_C, \mathrm{id}_Y)_! u$. By Construction 1.8, we then get a morphism $(F, \langle u, v \rangle) \to$ $(F', \langle p_!^\sharp u, q_!^\sharp v \rangle)$ in $\Omega\mathscr{C}$ given by $s \colon F \to F'$. □

In the case where f is proper, the dualisability of $(X', f_* L)$ follows from that of (X, L) by Proposition 2.23. Moreover, in this case, by Lemma 2.9, $f_\natural$ is right adjointable and it suffices in the above proof to apply the more direct Construction 1.7 in place of Construction 1.8.

Corollary 2.22. *Let* $f \colon X \to X'$ *be a proper morphism of schemes separated of finite type over S and let $L \in D(X, \Lambda)$ such that (X, L) is a dualisable object of $\mathscr{C}$. Then $f_* \mathrm{cc}_{X/S}(L) = \mathrm{cc}_{X'/S}(f_* L)$.*

Proof. This follows from Theorem 2.21 applied to $c = d = (\mathrm{id}_X, \mathrm{id}_X)$, $c' = d' = (\mathrm{id}_{X'}, \mathrm{id}_{X'})$ and $u = v = \mathrm{id}_L$. □

Proposition 2.23. *Let* $f \colon X \to Y$ *be a proper morphism of schemes separated of finite type over S. Let (X, L) be a dualisable object of $\mathscr{C}$. Then $(Y, f_* L)$ is dualisable.*

Proof. This follows formally from Remark 1.12 (b). We can also argue using internal mapping objects as follows. By Proposition 2.11 and Lemma 2.14, the canonical morphism

$$
\alpha \colon D_{X/S} L \boxtimes_S M \to R\mathscr{H}om(p_X^* L, p_Z^! M)
$$

is an isomorphism for every object (Z, M) of $\mathscr{C}$, and it suffices to show that the canonical morphism

$$
\beta \colon D_{Y/S} f_* L \boxtimes_S M \to R\mathscr{H}om(q_Y^* f_* L, q_Z^! M)
$$

is an isomorphism. Here p_X, p_Z, q_Y, q_Z are the projections as shown in the commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{p_X} & X \times_S Z \\
\downarrow{\scriptstyle f} & \scriptstyle{f \times_S \mathrm{id}_Z}\downarrow & \searrow{\scriptstyle p_Z} \\
Y & \xleftarrow{q_Y} & Y \times_S Z \xrightarrow{q_Z} Z.
\end{array}
$$

Via the isomorphisms $D_{Y/S} f_* L \boxtimes_S M \simeq (f \times_S \mathrm{id}_Z)_* (D_{X/S} L \boxtimes_S M)$ and

$$
R\mathscr{H}om(q_Y^* f_* L, q_Z^! M) \simeq R\mathscr{H}om((f \times_S \mathrm{id}_Z)_* p_X^* L, q_Z^! M) \simeq (f \times_S \mathrm{id}_Z)_* R\mathscr{H}om(p_X^* L, p_Z^! M),
$$

β can be identified with $(f \times_S \mathrm{id}_Z)_* \alpha$. □

Remark 2.24. The relative Lefschetz–Verdier formula and the proof given above hold for Artin stacks of finite type over an Artin stack S, with proper morphisms replaced by a suitable class of morphisms equipped with canonical isomorphisms $f_! \simeq f_*$ (such as proper representable morphisms). The characteristic class lives in $H^0(I_{X/S}, K_{I_{X/S}/S})$, where $I_{X/S} = X \times_{\Delta, X \times_S X, \Delta} X$ is the inertia stack of X over S.

Theorem 2.21 does not cover the twisted Lefschetz–Verdier formula in [XZ, Section A.2.19].

Remark 2.25. Scholze remarked that arguments of this article also apply in the étale cohomology of diamonds and imply the equivalence between dualisability and universal local acyclicity in this situation. This fact and applications are discussed in his work with Fargues on the geometrisation of the Langlands correspondence [FS]. In [HKW, Section 4], Hansen, Kaletha and Weinstein adapt our formalism and prove a Lefschetz–Verdier formula for diamonds and v-stacks.

2.5. Base change and duals

We conclude this section with a result on the preservation of duals by base change.

Let $g\colon S \to T$ be a morphism of coherent schemes and let Λ be a torsion commutative ring.

Proposition 2.26. *Let (Y, M) be a dualisable object of $\mathscr{C}_{T,\Lambda}$. Then $(Y_S, g_Y^* M)$ is a dualisable object of $\mathscr{C}_{S,\Lambda}$ and the canonical morphism $g_Y^* D_{Y/T} M \to D_{Y_S/S} g_Y^* M$ is an isomorphism. Here $Y_S = Y \times_T S$ and $g_Y\colon Y_S \to Y$ is the projection.*

We prove the proposition by constructing a symmetric monoidal functor $g^*\colon \mathscr{C}_{T,\Lambda} \to \mathscr{C}_{S,\Lambda}$ as follows. We take $g^*(Y, M) = (Y_S, g_Y^* M)$. For $(d, v)\colon (Y, M) \to (Z, N)$, we take $g^*(d, v) = (d_S, v_S)$, where d_S is the base change of d by g and v_S is the composite

$$\overleftarrow{d}_S^* g_Y^* M \simeq g_D^* \overleftarrow{d}^* M \xrightarrow{g_D^* v} g_D^* \overrightarrow{d}^! M \to \overrightarrow{d}_S^! g_Z^* M,$$

where D is the source of $\overleftarrow{d}$ and $\overrightarrow{d}$, g_D and g_Z are defined similar to g_Y. For every 2-morphism p of $\mathscr{C}_{T,\Lambda}$, we take $g^*(p) = p \times_T S$. The symmetric monoidal structure on g^* is obvious. Proposition 2.26 then follows from the fact that $g^*\colon \mathscr{C}_{T,\Lambda} \to \mathscr{C}_{S,\Lambda}$ preserves duals (Remark 1.9).

The construction above is a special case of Construction 1.16 (applied to $H\colon \mathscr{B}_T \to \mathscr{B}_S$ given by base change by g and α_Y given by g_Y^*).

Corollary 2.27. *Let $g\colon S \to T$ be a morphism of coherent schemes with T Noetherian and let Λ be a Noetherian commutative ring with $m\Lambda = 0$ for m invertible on T. Then for any scheme Y separated of finite type over T and any $M \in D_{\mathrm{cft}}(Y, \Lambda)$ locally acyclic over T, the canonical morphism $g_Y^* D_{Y/T} M \to D_{Y_S/S} g_Y^* M$ is an isomorphism. Here $Y_S = Y \times_T S$ and $g_Y\colon Y_S \to Y$ is the projection.*

Note that the statement does not involve cohomological correspondences.

Proof. This follows from Proposition 2.26 and Theorem 2.16. □

3. Nearby cycles over Henselian valuation rings

Let R be a Henselian valuation ring and let $S = \mathrm{Spec}(R)$. We do not assume that the valuation is discrete. In other words, we do not assume S Noetherian. Let η be the generic point and let s be the closed point. Let X be a scheme of finite type over S. Let $X_\eta = X \times_S \eta$, $X_s = X \times_S s$. We consider the morphisms of topoi

$$X_\eta \xrightarrow{\overleftarrow{\Psi}_X} X \overleftarrow{\times}_S \eta \xleftarrow{i_X} X_s \overleftarrow{\times}_S \eta \simeq X_s \bar{\times}_s \eta,$$

where $\overleftarrow{\times}$ denotes the oriented product of topoi [ILO, Exposé XI] and $\bar{\times}$ denotes the fibre product of topoi. Let Λ be a commutative ring such that $m\Lambda = 0$ for some m invertible on S. We will study the

composite functor

$$\Psi_X \colon D(X_\eta, \Lambda) \xrightarrow{\overleftarrow{\Psi}_X} D(X \overleftarrow{\times}_S \eta, \Lambda) \xrightarrow{i_X^*} D(X_s \overleftarrow{\times}_s \eta, \Lambda).$$

Let $\bar{s}$ be an algebraic geometric point above s and let $\bar{\eta} \to S_{(\bar{s})}$ be an algebraic geometric point above η. The restriction of $\Psi_X L$ to $X_{\bar{s}} \simeq X_{\bar{s}} \overleftarrow{\times}_{\bar{s}} \bar{\eta}$ can be identified with $(j_* L)|_{X_{\bar{s}}}$, where $j \colon X_{\bar{\eta}} \to X_{(\bar{s})}$ and was studied by Huber [H, Section 4.2]. We do not need Huber's results in this article.

In Subsection 3.1, we study the symmetric monoidal functor given by Ψ and cohomological correspondences. We deduce that Ψ commutes with duals (Corollary 3.8), generalising a theorem of Gabber. We also obtain a new proof of the theorems of Deligne and Huber that Ψ preserves constructibility (Corollary 3.9). In Subsection 3.2, extending results of Vidal, we use the compatibility of specialisation with proper pushforward to deduce a fixed point result.

3.1. Künneth formulas and duals

Proposition 3.1 (Künneth formulas). *Let X and Y be schemes of finite type over S and let $L \in D(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$; then the canonical morphisms*

$$\overleftarrow{\Psi}_X L \boxtimes \overleftarrow{\Psi}_Y M \to \overleftarrow{\Psi}_{X \times_S Y}(L \boxtimes M), \quad \Psi_X L \boxtimes \Psi_Y M \to \Psi_{X \times_S Y}(L \boxtimes M),$$

are isomorphisms.

The Künneth formula for Ψ over a Henselian *discrete* valuation ring is a theorem of Gabber ([I1, Théorème 4.7], [BB, Lemma 5.1.1]).

Proof. It suffices to show that the first morphism is an isomorphism. By passing to the limit and the finiteness of cohomological dimensions, it suffices to show that $\Psi_{X,U/S} \colon X_U \to X \overleftarrow{\times}_S U$ satisfies Künneth formula for each open subscheme $U \subseteq S$. We then reduce to the case $U = S$, where the Künneth formula is [I2, Theorem A.3]. The Ψ-goodness is satisfied by Orgogozo's theorem ([O, Théorème 2.1], [LZ, Example 4.26 (2)]). □

Construction 3.2. Let $f \colon X \to Y$ be a separated morphism of schemes of finite type over S. Then we have canonical natural transformations

$$f_s^* \Psi_Y \to \Psi_X f_\eta^*, \tag{3.1}$$

$$\Psi_Y f_{\eta*} \to f_{s*} \Psi_X, \tag{3.2}$$

$$f_{s!} \Psi_X \to \Psi_Y f_{\eta!}, \tag{3.3}$$

$$\Psi_Y f_\eta^! \to f_s^! \Psi_Y. \tag{3.4}$$

Here we denoted $f_s \overleftarrow{\times}_s \eta$ by f_s. (3.1) is the base change

$$f_s^* i_Y^* \overleftarrow{\Psi}_Y \simeq i_X^* (f \overleftarrow{\times}_S \mathrm{id}_\eta)^* \overleftarrow{\Psi}_Y \to i_X^* \overleftarrow{\Psi}_X f_\eta^*$$

and (3.4) is defined similar to [LZ, Formula (4.9)] as

$$i_X^* \overleftarrow{\Psi}_X f_\eta^! \simeq i_X^* (f \overleftarrow{\times}_S \mathrm{id}_\eta)^! \overleftarrow{\Psi}_Y \to f_s^! i_Y^* \overleftarrow{\Psi}_Y.$$

(3.1) and (3.2) correspond to each other by adjunction. The same holds for (3.3) and (3.4). For f proper, (3.2) and (3.3) are inverse to each other.

Construction 3.3. We construct symmetric monoidal 2-categories $\mathscr{C}_1$ and $\mathscr{C}_2$ and a symmetric monoidal functor $\psi\colon \mathscr{C}_1 \to \mathscr{C}_2$ as follows.

The construction of $\mathscr{C}_1$ is identical to that of $\mathscr{C}_{S,\Lambda}$ (Construction 2.6) except that we replace the derived category $D(-,\Lambda)$ by $D((-)_\eta,\Lambda)$. Thus, an object of $\mathscr{C}_1$ is a pair (X,L), where X is a scheme separated of finite type over S and $L \in D(X_\eta,\Lambda)$. A morphism $(X,L) \to (Y,M)$ is a pair (c,u), where $c\colon X \to Y$ is a correspondence over S and (c_η,u) is a cohomological correspondence over η. A 2-morphism $(c,u) \to (d,v)$ is a 2-morphism $p\colon c \to d$ such that p_η is a 2-morphism $(c_\eta,u) \to (d_\eta,v)$. We have $(X,L) \boxtimes (Y,M) = (X \times_S Y, L \boxtimes_\eta M)$. The monoidal unit is (S,Λ_η).

The construction of $\mathscr{C}_2$ is identical to that of $\mathscr{C}_{s,\Lambda}$ except that we replace the derived category $D(-,\Lambda)$ by $D((-)\bar{\times}_s \eta,\Lambda)$. Thus, an object of $\mathscr{C}_2$ is a pair (X,L), where X is a scheme separated of finite type over s and $L \in D(X \bar{\times}_s \eta,\Lambda)$. The monoidal unit is (s,Λ_η).

We define ψ by $\psi(X,L) = (X_s, \Psi_X L)$, $\psi(c,u) = (c_s, \psi u)$, where ψu is specialisation of u defined as the composite

$$\overleftarrow{c}_s^* \Psi_X L \xrightarrow{(3.1)} \Psi_C \overleftarrow{c}_\eta^* L \xrightarrow{\Psi_C(u)} \Psi_C \overrightarrow{c}_\eta^! M \xrightarrow{(3.4)} \overrightarrow{c}_s^! \Psi_Y M.$$

For every 2-morphism p, $\psi p = p_s$. The symmetric monoidal structure is given by the Künneth formula (Proposition 3.1) and the canonical isomorphism $\Psi_S \Lambda_S \simeq \Lambda_\eta$.

Remark 3.4. The symmetric monoidal 2-category $\mathscr{C}_1$ (respectively $\mathscr{C}_2$) is obtained via the Grothendieck construction (Construction 1.10) from the right-lax symmetric monoidal functor $\mathscr{B}_S \to \mathscr{C}at^{\mathrm{co}}$ (respectively $\mathscr{B}_s \to \mathscr{C}at^{\mathrm{co}}$) carrying X to $D(X_\eta,\Lambda)$ (respectively $D(X \bar{\times}_s \eta,\Lambda)$). The symmetric monoidal functor ψ is a special case of Construction 1.16 (with $H\colon \mathscr{B}_S \to \mathscr{B}_s$ given by taking special fibre). More explicitly, if $\mathscr{C}_2'$ denotes the symmetric monoidal 2-category obtained from the right-lax symmetric monoidal functor $\mathscr{B}_S \to \mathscr{C}at^{\mathrm{co}}$ carrying X to $D(X_s \bar{\times}_s \eta,\Lambda)$, then ψ decomposes into $\mathscr{C}_1 \xrightarrow{\psi_1} \mathscr{C}_2' \xrightarrow{\psi_2} \mathscr{C}_2$, where ψ_1 carries (X,L) to $(X,\Psi_X L)$ and ψ_2 carries (X,L) to (X_s,L).

The proof of the following lemma is identical to that of Lemma 2.8.

Lemma 3.5. *The symmetric monoidal structures $\otimes$ on $\mathscr{C}_1$ (respectively $\mathscr{C}_2$) are closed, with mapping object*

$$\mathscr{H}om((X,L),(Y,M)) = (X \times_S Y, R\mathscr{H}om(p_{X_\eta}^* L, p_{Y_\eta}^! M))$$

$$(\textit{respectively } \mathscr{H}om((X,L),(Y,M)) = (X \times_s Y, R\mathscr{H}om(p_X^* L, p_Y^! M))).$$

Remark 3.6. It follows from Remark 1.3 and Lemma 3.5 that the dual of a dualisable object (X,L) in $\mathscr{C}_1$ (respectively $\mathscr{C}_2$) is $(X, D_{X_\eta} L)$ (respectively $(X, D_{X \bar{\times}_s \eta} L)$). Here, for $a\colon U \to \eta$ and $b\colon V \to s$ separated of finite type, we write $K_U = K_{U/\eta}$, $D_U = D_{U/\eta}$ and $K_{V \bar{\times}_s \eta} = (b \bar{\times}_s \eta)^! \Lambda_\eta$, $D_{V \bar{\times}_s \eta} = R\mathscr{H}om(-, K_{V \bar{\times}_s \eta})$.

In the rest of Subsection 3.1, we assume that Λ is Noetherian.

Proposition 3.7. *An object (X,L) in $\mathscr{C}_1$ or $\mathscr{C}_2$ is dualisable if and only if $L \in D_{\mathrm{cft}}$.*

Proof. By Lemma 1.4 and the identification of internal mapping objects (Lemmas 2.8 and 3.5), (X,L) in $\mathscr{C}_1$ is dualisable if and only if (X_η, L) in $\mathscr{C}_\eta$ is dualisable. The latter condition is equivalent to $L \in D_{\mathrm{cft}}$ by Corollary 2.19.

Similarly, (X,L) in $\mathscr{C}_2$ is dualisable if and only if $(X_{\bar{s}}, L|_{X_{\bar{s}}})$ in $\mathscr{C}_{\bar{s}}$ is dualisable, by [LZ, Lemma 1.29]. The latter condition is equivalent to $L|_{X_{\bar{s}}} \in D_{\mathrm{cft}}$, which is in turn equivalent to $L \in D_{\mathrm{cft}}$. $\square$

Corollary 3.8. *Let X be a scheme separated of finite type over S and let $L \in D_c^-(X_\eta,\Lambda)$. The canonical morphism $\Psi_X D_{X_\eta} L \to D_{X_s \bar{\times}_s \eta} \Psi_X L$ is an isomorphism in $D(X_s \bar{\times}_s \eta,\Lambda)$.*

This generalises a theorem of Gabber for Henselian *discrete* valuation rings [I1, Théorème 4.2]. Our proof is different from that of Gabber. One can also deduce Corollary 3.8 from the commutation of duality with sliced nearby cycles over general bases [LZ, Theorem 0.1].

Proof. The cohomological dimension of Ψ_X is bounded by $\dim(X_\eta)$. Thus, we may assume that L is of the form $u_!\Lambda_U$, where $u\colon U \to X_\eta$ is an étale morphism of finite type. In particular, we may assume $L \in D_{\mathrm{cft}}(X_\eta, \Lambda)$. In this case, (X, L) is dualisable by Proposition 3.7. We conclude by the fact that ψ preserve duals (Remark 1.9) and the identification of duals (Remark 3.6). □

We also deduce a new proof of the following finiteness theorem of Deligne (for Henselian *discrete* valuation rings) [D, Th. finitude, Théorème 3.2] and Huber [H, Proposition 4.2.5]. Our proof relies on Deligne's theorem on local acyclicity over a field [D, Th. finitude, Corollaire 2.16].

Corollary 3.9. *Let X be a scheme of finite type over S. Then* Ψ_X *preserves* D_c^b *and* D_{cft}.

Proof. We may assume that X is separated. As in the proof of Corollary 3.8, we are reduced to the case of D_{cft}. This case follows from Proposition 3.7 and the fact that ψ preserves dualisable objects (Remark 1.9). □

By Remark 1.9, ψ also preserves pairings, and we obtain the following generalisation of [V1, Proposition 1.3.5].

Corollary 3.10. *Consider morphisms of schemes separated of finite type over S:*

$$X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y \xleftarrow{\overleftarrow{d}} D \xrightarrow{\overrightarrow{d}} X.$$

Let $L \in D_{\mathrm{cft}}(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$, $u\colon \overleftarrow{c}_\eta^* L \to \overrightarrow{c}_\eta^! M$, $v\colon \overleftarrow{d}_\eta^* M \to \overrightarrow{d}_\eta^! L$. *Then* $\mathrm{sp}\langle u, v\rangle = \langle \psi u, \psi v\rangle$, *where* sp *is the composition*

$$H^0(F_\eta, K_{F_\eta}) \to H^0(F_s \bar{\times}_s \eta, \Psi_F K_{F_\eta}) \to H^0(F_s \bar{\times}_s \eta, K_{F_s \bar{\times}_s \eta})$$

and $F = C \times_{X \times Y} D$.

3.2. Pushforward and fixed points

Construction 3.11 (!-Pushforward in $\mathscr{C}_2$). Consider a commutative diagram (2.3) in $\mathscr{B}_s$ such that $q\colon C \to X \times_{X'} C'$ is proper. Let $(c, u)\colon (X, L) \to (Y, M)$ be a morphism in $\mathscr{C}_2$ above c. By Lemma 1.11, we have a unique morphism $(c', p_!^\sharp u)\colon (X', f_! L') \to (Y', g_! M')$ in $\mathscr{C}_2$ above c' such that q defines a 2-morphism in $\mathscr{C}_2$:

$$
\begin{array}{ccc}
(X, L) & \xrightarrow{(c,u)} & (Y, M) \\
f_\natural \downarrow & \underset{(c', p_!^\sharp u)}{\overset{\psi q}{\Longrightarrow}} & \downarrow g_\natural \\
(X', f_! L) & \xrightarrow{} & (Y', g_! M).
\end{array}
$$

For f, g, p proper, we write $p_*^\sharp u$ for $p_!^\sharp u$.

Applying Lemma 1.14 to the functor ψ_1 in Remark 3.4, we obtain the following.

Proposition 3.12. *Consider a commutative diagram of schemes separated of finite type over S*

$$
\begin{array}{ccccc}
X & \xleftarrow{\overleftarrow{c}} & C & \xrightarrow{\overrightarrow{c}} & Y \\
{\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
X' & \xleftarrow{\overleftarrow{c'}} & C' & \xrightarrow{\overrightarrow{c'}} & Y'
\end{array}
$$

such that $C \to X \times_{X'} C'$ is proper. Let $L \in D(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$, $u\colon \overleftarrow{c}_\eta^* L \to \overrightarrow{c}_\eta^! M$. Then the square

$$
\begin{array}{ccc}
\overleftarrow{c'}_s^* f_{s!} \Psi_X L & \xrightarrow{p_{s!}^\sharp \psi u} & \overrightarrow{c'}_s^! g_{s!} \Psi_Y M \\
\downarrow & & \downarrow \\
\overleftarrow{c'}_s^* \Psi_{X'} f_{\eta!} L & \xrightarrow{\psi p_{\eta!}^\sharp u} & \overrightarrow{c'}_s^! \Psi_{Y'} g_{\eta!} M
\end{array}
$$

commutes. Here the vertical arrows are given by (3.3). In particular, in the case where f, g, p are proper, $p_{s*}^\sharp \psi u$ can be identified with $\psi p_{\eta*}^\sharp u$ via the isomorphisms $f_{s*} \Psi_X \simeq \Psi_{X'} f_{\eta*}$ and $g_{s*} \Psi_Y \simeq \Psi_{Y'} g_{\eta*}$.

This generalises a result of Vidal [V2, Théorème 7.5.1] for certain Henselian valuation rings of rank 1. As in [V2, Sections 7.5, 7.6], Proposition 3.12 implies the following fixed point result, generalising [V2, Proposition 5.1, Corollaire 7.5.3].

Corollary 3.13. *Assume that η is separably closed. Consider a commutative diagram of schemes*

$$
\begin{array}{ccc}
X & \xrightarrow{f} & S \\
{\scriptstyle g}\downarrow & {\scriptstyle \sigma}\downarrow & \simeq \\
X & \xrightarrow{f} & S
\end{array}
$$

with f proper and σ fixing s. Assume that g_s does not fix any point of X_s. Then $\operatorname{tr}(g, R\Gamma(X_\eta, \Lambda)) = 0$. If, moreover, g is an isomorphism and $U \subseteq X_\eta$ is an open subscheme such that $g(U) = U$, then $\operatorname{tr}(g, R\Gamma_c(U, \Lambda)) = 0$.

Proof. For completeness, we recall the arguments of [V2, Corollaire 7.5.2]. We may assume $\Lambda = \mathbb{Z}/m\mathbb{Z}$. We decompose the commutative diagram into

$$
\begin{array}{ccccc}
X & \xrightarrow{\gamma} & \sigma^* X & \longrightarrow & S \\
 & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
 & & X & \xrightarrow{f} & S.
\end{array}
$$

Consider the cohomological correspondences $((\operatorname{id}_{X_s}, \operatorname{id}_{X_s}), \sigma)\colon (X_s, \Psi_X \Lambda) \to (X_s, \Psi_{\sigma^* X} \Lambda)$ and $(c_\eta, u)\colon (\sigma^* X_\eta, \Lambda) \to (X_\eta, \Lambda)$, where $c = (\gamma, \operatorname{id}_X)$ and $u = \operatorname{id}_{\Lambda_{X_\eta}}$. We have a commutative diagram

$$
\begin{array}{ccccc}
R\Gamma(X_\eta, \Lambda) & \xrightarrow{\sigma} & R\Gamma(\sigma^* X_\eta, \Lambda) & \xrightarrow{f_{\eta*}^\sharp u} & R\Gamma(X_\eta, \Lambda) \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
R\Gamma(X_s, \Psi_X \Lambda) & \xrightarrow{\sigma} & R\Gamma(X_s, \Psi_{\sigma^* X} \Lambda) & \xrightarrow{f_{s*}^\sharp \psi u} & R\Gamma(X_s, \Psi_X \Lambda)
\end{array}
$$

where the square on the right commutes by Proposition 3.12. The composite of the upper horizontal arrows is the action of g. Thus, by the Lefschetz–Verdier formula over s, we have

$$\text{tr}(g, R\Gamma(X_\eta, \Lambda)) = \int_{X_s^{gs}} \langle \sigma, \psi u \rangle = 0,$$

where $\int_F \colon H^0(F, K_F) \to \Lambda$ denotes the trace map. For the last assertion of the corollary, it suffices to note that

$$\text{tr}(g, R\Gamma_c(U, \Lambda)) = \text{tr}(g, R\Gamma(X_\eta, \Lambda)) - \text{tr}(g, R\Gamma(Z_\eta, \Lambda)) = 0,$$

where Z is the closure of $X_\eta \backslash U$ in X, equipped with the reduced subscheme structure. □

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