The Turning-Values of Cubic and Quartic Functions and the Nature of the Roots of Cubic and Quartic Equations.

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THE CUBIC.

Let a cubic function be denoted by

$$z^3 + a_1 z^2 + b_1 z + c_1.$$

Put $z = x - \dfrac{a_1}{3}$ and the function takes the form

$$x^3 + 3qx + r.$$

Consider the graph of this function, represented by

$$y = x^3 + 3qx + r. \quad - \quad - \quad - \quad (1)$$

If we shift the origin to the point (h, k) the equation of the graph becomes

$$\eta + k = (\xi + h)^3 + 3q(\xi + h) + r$$
$$= \xi^3 + 3\xi^2 h + 3\xi(h^2 + q) + h^3 + 3qh + r.$$

The point (h, k) is a turning-point if

$$h^2 + q = 0 \quad - \quad - \quad - \quad - \quad - \quad (2)$$

and
$$k = h^3 + 3qh + r$$
$$= h(h^2 + q) + 2qh + r$$
$$= 2qh + r; \quad - \quad - \quad - \quad - \quad (3)$$

for the equation of the graph is then

$$\eta = \xi^3 + 3h\xi^2, \quad - \quad - \quad - \quad - \quad (4)$$

and, near the origin, the graph has approximately the same shape as the parabola $\eta = 3h\xi^2$,

so that, at the origin, there is a minimum turning-point if h is positive, and a maximum turning-point if h is negative.

From (2) and (3) $(k - r)^2 = 4q^2h^2 = -4q^3$;

$\therefore$ the turning-points are given by

$$h^2 + q = 0 \quad - \quad - \quad (2)$$

and
$$k^2 - 2rk + (r^2 + 4q^3) = 0. \quad - \quad - \quad (5)$$

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These are quadratic equations; therefore there are two, and only two turning-points on the graph of a cubic function :

(*a*) the two turning-points coincide if $q = 0$, when equation (4) becomes $\eta = \xi^3$, whose form is well known (Fig. 28 (*a*));

(*b*) the two turning-points are imaginary if q is positive, in which case it is easy to see that $y = 3qx + r$ is an inflexional tangent (Fig. 28 (*b*));

(*c*) there are two real turning-points if q is negative (Fig. 28 (*c*)).

Hence the x-axis may cut the graph in three coincident points in case (*a*); or in one real point and two imaginary points in cases (*a*), (*b*), (*c*); or in three real points in case (*c*), when two of the points may coincide.

Correspondingly, the roots of the equation

$$x^3 + 3qx + r = 0$$

will be all equal if $q = 0$ and $r = 0$;
one of the roots will be real and two imaginary,

if $q = 0$,

if q is positive, in case (*b*),

if the product of the roots of equation (5) is positive, in case (*c*),

i.e., if $r^2 + 4q^3$ is positive.

Hence $r^2 + 4q^3$ positive is a general criterion for one real and two imaginary roots.

The roots will be all real and different if the roots of equation (5) are unlike in sign, *i.e.*, if $r^2 + 4q^3$ is negative.

There will be two equal roots if a value of k as determined from equation (5) is zero, *i.e.*, if $r^2 + 4q^3$ is zero.

The discussion of the cubic function

$$Az^3 + a_1z^2 + b_1z + c_1$$

and of the corresponding equation is clearly contained in the above.

The Quartic.

Let a quartic function be denoted by

$$z^4 + a_1z^3 + b_1z^2 + c_1z + d_1.$$

Put $z = x - \frac{a_1}{4}$ and the function takes the form

$$x^4 + 6qx^2 + 4rx + s.$$

Consider the graph of the function represented by

$$y = x^4 + 6qx^2 + 4rx + s.$$

Shift the origin to the point (h, k) and the equation of the graph becomes

$$\eta + k = (\xi + h)^4 + 6q(\xi + h)^2 + 4r(\xi + h) + s$$
$$= \xi^4 + 4h\xi^3 + 6\xi^2(h^2 + q) + 4\xi(h^3 + 3qh + r) + (h^4 + 6qh^2 + 4rh + s).$$

Now choose $h^3 + 3qh + r = 0$ - - - - - (6)

and $k = h^4 + 6qh^2 + 4rh + s$

$$= h(h^3 + 3qh + r) + 3qh^2 + 3rh + s$$
$$= 3qh^2 + 3rh + s. \qquad (7)$$

The equation of the graph then becomes

$$\eta = \xi^4 + 4h\xi^3 + 6\xi^2(h^2 + q). \qquad (8)$$

The equation (6) has for its roots the abscissae of the turning-points of the graph, and (7) gives the corresponding values of the ordinates. The values of the ordinates are expressed as the roots of an equation in k by eliminating h between (6) and (7).

Multiplying (6) by $3q$ and (7) by h, and subtracting, we get

$$3rh^2 + (s - k - 9q^2)h - 3qr = 0. \qquad (9)$$

Solving equations (7) and (9) for h^2 and h we get

$$h^2 = \frac{9qr^2 + (s - k)(s - k - 9q^2)}{9r^2 - 3q(s - k - 9q^2)},$$

$$h = \frac{3r(s - k) + 9q^2r}{3q(s - k - 9q^2) - 9r^2},$$

whence, after reduction,

$$k^3 - 3k^2(s - 6q^2) + 3k(s^2 - 12sq^2 + 27q^4 + 18qr^2)$$
$$- (s^3 - 18q^2s^2 + 81sq^4 - 54q^3r^2 - 27r^4 + 54qr^2s) = 0. \qquad (10)$$

It is convenient to denote the absolute term by Δ.

(α) The three roots of the cubic equation

$$h^3 + 3qh + r = 0 \qquad (6)$$

may be all equal. Since the coefficient of h^2 in the equation is zero, each root is zero; $\therefore$ $q = 0$ and $r = 0$ and the equation of the graph becomes, by (8), $\eta = \xi^4.$

The graph is now easily traced. (Fig. 29 (a)).

(β) Two of the roots of the cubic equation

$$h^3 + 3qh + r = 0$$

may be equal. We know that in that case

$$r^2 + 4q^3 = 0 ;$$

$$\therefore \quad 4q^3h^3 - 3qr^2h - r^3 = 0 ;$$

$$\therefore \quad (4q^2h^2 + 4qrh + r^2)(qh - r) = 0 ;$$

$$\therefore \quad h = -\frac{r}{2q},\ -\frac{r}{2q},\ \text{or}\ \frac{r}{q}.$$

Taking $h = -\dfrac{r}{2q}$ and $k = 3qh^2 + 3rh + s = 3q^2 + s$,

equation (8) becomes

$$\eta = \xi^4 - \frac{2r}{q} \cdot \xi^3 .$$

Hence the ξ-axis is an inflexional tangent at the origin, and there is a turning-point between $\xi = 0$ and $\xi = \dfrac{2r}{q}$. To determine the turning-point we notice that η has a turning-value when

$$\xi \cdot \xi \cdot \xi \left(-3\xi + \frac{6r}{q} \right)$$

has a turning-value, which occurs when $\xi = -3\xi + \dfrac{6r}{q}$, since the sum of the factors is constant, *i.e.*, when $\xi = \dfrac{3r}{2q}$; and therefore when $\eta = -\dfrac{27}{16} \cdot \dfrac{r^4}{q^4}$. (Fig. 29 ($\beta$)).

(γ) There may be two imaginary roots of the equation

$$h^3 + 3qh + r = 0. \qquad - \quad - \quad - \quad (6)$$

Take h equal to the real root of this equation

and $$k = 3qh^2 + 3rh + s.$$

Then the equation of the graph by (8) is

$$\eta = \xi^2 \{ (\xi + 2h)^2 + 2(h^2 + 3q) \}.$$

Now $h^2 + 3q = -\dfrac{r}{h}$.

But the product of the three roots of equation (6) is $-r$ and the product of the two imaginary roots is positive ;

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$\therefore\quad h$ and r have opposite signs;

$\therefore\quad h^2+3q$ is positive, $=\dfrac{m^2}{2}$, say,

and the equation of the graph can be written in the form

$$\eta=\xi^2\{(\xi+2h)^2+m^2\},$$

from which the form of the graph is easily seen (Fig. 29 (γ)).

(δ) The roots of the equation

$$h^3+3qh+r=0$$

may be all real.

The sum of these roots is zero; therefore there is a positive root and a negative root. Taking h to be a root of the same sign as r, if $r\neq 0$, so that $h^2+3q=-\dfrac{r}{h}=-\dfrac{n^2}{2}$, say, the equation of the graph takes the form

$$\eta=\xi^2\{(\xi+2h)^2-n^2\},$$

from which the form of the graph is easily seen (Fig. 29 (δ)).

If $r=0$, take $h=0$ and the equation becomes $\eta=\xi^2\{\xi^2+6q\}$; where, of course, q is negative since r^2+4q^3 is negative. (Fig. 29 (δ')).

Now consider the equation

$$x^4+6qx^2+4rx+s=0.$$

1. This equation will have four equal roots if

$$\eta=-k \text{ and } \eta=\xi^4$$

intersect in four coincident points.

$$\therefore\quad q=0 \text{ and } r=0.$$

But $k=0$ also, $\therefore\ s=0$.

Hence the criterion is $\quad q=0,\ r=0,\ s=0.$

2. The equation will have three equal roots if

$$\eta=-k \text{ and } \eta=\xi^4-\frac{2r}{q}\,.\,\xi^3$$

intersect in three coincident points, and a fourth different point.

$$\therefore\quad r^2+4q^3=0, \text{ and } k=0.$$

But it has been shown that $k=s+3q^2$;

$\therefore$ the conditions are that

$$r^2+4q^3=0 \text{ and } s+3q^2=0.$$

3. The equation has two pairs of roots equal when the graph is of the form shown in Fig. 29 (δ') and when $\eta = -k$ is a tangent at two of the turning-points of
$$\eta = \xi^2\{\xi^2 + 6q\}.$$
Hence $r = 0$ and two of the roots of equation (10) vanish;
$$\therefore \quad s^2 - 12sq^2 + 27q^4 = 0$$
$$\text{and} \quad s^3 - 18q^2s^2 + 81sq^4 = 0;$$
$$\therefore \quad s - 9q^2 = 0 \text{ is both necessary and sufficient.}$$
$\therefore$ the conditions are
$$r = 0 \text{ and } s = 9q^2.$$

4. The equation has two roots equal if one root of equation (10) is zero;
$$\therefore \quad \Delta = 0.$$

5. The equation has two real and two imaginary roots if the product of the roots of equation (10) is negative;
$$\therefore \quad \Delta \text{ is negative.}$$

6. The equation has its roots all real and different if the product of the roots of equation (10) is positive;
$$\therefore \quad \Delta \text{ is positive.}$$

7. The equation has its roots all imaginary, if the product of the roots of equation (10) is positive;
$$\therefore \quad \Delta \text{ is positive.}$$

It remains to distinguish the last two cases.

In case 6, $r^2 + 4q^3$ is negative as only form Fig. 29 (δ) or (δ') of the graph is possible. In case 7, $r^2 + 4q^3$ may be either negative or positive, or zero, since the graph may have any of the possible forms. If, then, q is positive and Δ is positive all the roots are imaginary; but, if q is negative, we observe that the roots of equation (10) are all positive if the roots under discussion are all imaginary, and the roots of equation (10) are two negative and one positive if the roots under discussion are all real; therefore, only if these roots are all imaginary, can the order of the signs of the terms of equation (10) be $+ - + -$. Hence if q is positive or zero and Δ positive, the roots are all imaginary; if q is negative, Δ positive, $s - 6q^2$ positive, and $s^2 - 12sq^2 + 27q^4 + 18qr^2$ positive, the roots are all imaginary; otherwise, if Δ is positive, the roots are all real.

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